

# The Surface Area Deviation of the Euclidean Ball and a Polytope \*

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## Abstract

While there is extensive literature on approximation of convex bodies by inscribed or circumscribed polytopes, much less is known in the case of generally positioned polytopes. Here we give upper and lower bounds for approximation of convex bodies by arbitrarily positioned polytopes with a fixed number of vertices in the symmetric surface area deviation.

## 1 Introduction and main results

How well can a convex body be approximated by a polytope? This is a fundamental question not only in convex geometry, but also in view of applications in stochastic geometry, complexity, geometric algorithms and many more (e.g., [7, 8, 10, 11, 12, 16, 17, 27, 29]).

Accuracy of approximation is often measured in the symmetric difference metric, which reflects the volume deviation of the approximating and approximated objects. Approximation of a convex body  $K$  by inscribed or circumscribed polytopes with respect to this metric has been studied extensively and many of the major questions have been resolved. We refer to, e.g., the surveys and books by Gruber [15, 18, 19] and the references there and to, e.g., [1, 2, 5, 13, 20, 28, 30, 32, 34].

Sometimes it is more advantageous to consider the surface area deviation  $\Delta_s$  [4, 5, 14] instead of the volume deviation  $\Delta_v$ . It is especially desirable because if best approximation of convex bodies is replaced by random approximation, then we have essentially the same amount of information for volume, surface area, and mean width ([5],[6]), which are three of the quermassintegrals of a convex body (see, e.g., [10, 31]).

For convex bodies  $K$  and  $L$  in  $\mathbb{R}^n$  with boundaries  $\partial K$  and  $\partial L$ , the symmetric surface area deviation is defined as

$$\Delta_s(K, L) = \text{vol}_{n-1}(\partial(K \cup L)) - \text{vol}_{n-1}(\partial(K \cap L)). \quad (1)$$

Typically, approximation by polytopes often involves side conditions, like a prescribed number of vertices, or, more generally,  $k$ -dimensional faces [2]. Most often in the literature, it is required that the body contains the approximating polytope or vice versa. This is too restrictive as a requirement and it needs to be dropped. Here, we do exactly that and prove upper and lower bounds for approximation of convex bodies by arbitrarily positioned polytopes in the symmetric surface area deviation. This addresses questions asked by Gruber [19].

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**Theorem 1.** *There exists an absolute constant  $c > 0$  such that for every integer  $n \geq 3$ , there is an  $N_n$  such that for every  $N \geq N_n$  there is a polytope  $P_N$  in  $\mathbb{R}^n$  with  $N$  vertices such that*

$$\Delta_s(B_2^n, P_N) \leq c \frac{\text{vol}_{n-1}(\partial B_2^n)}{N^{\frac{2}{n-1}}}.$$

Here,  $B_2^n$  is the  $n$ -dimensional Euclidean unit ball with boundary  $S^{n-1} = \partial B_2^n$ . Moreover, throughout the paper  $a, b, c, c_1, c_2$  will denote positive absolute constants that may change from line to line.

The proof of Theorem 1 is based on a random construction. A crucial step in its proof is a result by J. Müller [26] on the surface deviation of a polytope *contained* in the unit ball. It describes the asymptotic behavior of the surface deviation of a random polytope  $P_N$ , the convex hull of  $N$  randomly (with respect to the uniform measure) and independently chosen points on the boundary of the unit ball as the number of vertices increases. It says that

$$\lim_{N \rightarrow \infty} \frac{\text{vol}_{n-1}(S^{n-1}) - \mathbb{E} \text{vol}_{n-1}(\partial P_N)}{N^{-\frac{2}{n-1}}} = \frac{n-1}{n+1} \frac{\Gamma\left(n + \frac{2}{n-1}\right)}{2(n-2)!} \frac{(\text{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}. \quad (2)$$

The right hand side of (2) is of the order  $cn \text{vol}_{n-1}(\partial B_2^n)$ . Thus, dropping the restriction that  $P_N$  is contained in  $B_2^n$  improves the estimate by a factor of dimension. The same phenomenon was observed for the volume deviation in [21].

For the facets, we obtain the following lower bound for a polytope in arbitrary position.

**Theorem 2.** *There is a constant  $c > 0$  and  $M_0 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $M \in \mathbb{N}$  with  $M \geq M_0$  and all polytopes  $P_M$  in  $\mathbb{R}^n$  with no more than  $M$  facets*

$$\Delta_s(B_2^n, P_M) \geq c \frac{\text{vol}_{n-1}(\partial B_2^n)}{M^{\frac{2}{n-1}}}.$$

Again, we gain by a factor of dimension if we drop the requirement that the polytope contains  $B_2^n$ . Indeed, it follows from [15, 24] that the order of best approximation  $\Delta_v(B_2^n, P_M^{\text{best}})$  with  $B_2^n \subset P_M$  behaves asymptotically, for  $M \rightarrow \infty$ , like  $\text{vol}_{n-1}(\partial B_2^n)$ . Now observe that when  $B_2^n \subset P_M$ ,  $n \Delta_v(B_2^n, P_M) = \Delta_s(B_2^n, P_M)$ .

As a corollary to Theorem 2, we deduce a lower bound in the case of simple polytopes with at most  $N$  vertices. A polytope in  $\mathbb{R}^n$  is called simple if at every vertex exactly  $n$  facets meet.

**Corollary 3.** *There is a constant  $c > 0$  and  $N_0 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $N \in \mathbb{N}$  with  $N \geq N_0$  and all simple polytopes  $P_N$  in  $\mathbb{R}^n$  with no more than  $N$  vertices*

$$\Delta_s(B_2^n, P_N) \geq c \frac{\text{vol}_{n-1}(\partial B_2^n)}{N^{\frac{2}{n-1}}}.$$

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## 2 Notation and auxiliary lemmas

For a convex body  $K$  in  $\mathbb{R}^n$ , we denote by  $\text{int}(K)$  its interior. Its  $n$ -dimensional volume is  $\text{vol}_n(K)$  and the surface area of its boundary  $\partial K$  is  $\text{vol}_{n-1}(\partial K)$ . The usual surface area measure on  $\partial K$  is denoted by  $\mu_{\partial K}$ . The convex hull of points  $x_1, \dots, x_m$  is  $[x_1, \dots, x_m]$ .

The affine hyperplane in  $\mathbb{R}^n$  through the point  $x$  and orthogonal to the vector  $\xi$  is denoted by  $H(x, \xi)$ .

For any further notions related to convexity, we refer to the books by e.g., Gruber [19] and Schneider [31].

We start with several lemmas needed for the proof of Theorem 1. The first lemma says that almost all random polytopes of points chosen from a convex body are simplicial. Intuitively this is obvious: If we have chosen  $x_1, \dots, x_n$  and we want to choose  $x_{n+1}$  so that it is an element of the hyperplane spanned by  $x_1, \dots, x_n$ , then we are choosing  $x_{n+1}$  from a nullset. We refer to, e.g., [34] for the details.

**Lemma 4.** *Almost all random polytopes of points chosen from the boundary of the Euclidean ball with respect to the normalized surface measure are simplicial.*

We also need the following two lemmata due to Miles [25].

**Lemma 5.** [25]

$$\begin{aligned} & d\mu_{\partial B_2^n}(x_1) \cdots d\mu_{\partial B_2^n}(x_n) \\ &= (n-1)! \frac{\text{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{\frac{n}{2}}} d\mu_{\partial B_2^n \cap H}(x_1) \cdots d\mu_{\partial B_2^n \cap H}(x_n) dp d\mu_{\partial B_2^n}(\xi), \end{aligned}$$

where  $\xi$  is the normal to the hyperplane  $H$  through  $x_1, \dots, x_n$  and  $p$  is the distance of the hyperplane  $H$  to the origin.

**Lemma 6.** [25]

$$\begin{aligned} & \int_{\partial B_2^n(0,r)} \cdots \int_{\partial B_2^n(0,r)} (\text{vol}_n([x_1, \dots, x_{n+1}]))^2 d\mu_{\partial B_2^n(0,r)}(x_1) \cdots d\mu_{\partial B_2^n(0,r)}(x_{n+1}) \\ &= \frac{(n+1)r^{2n}}{n!n^n} (\text{vol}_{n-1}(\partial B_2^n(r)))^{n+1} = \frac{(n+1)r^{n^2+2n-1}}{n!n^n} (\text{vol}_{n-1}(\partial B_2^n))^n. \end{aligned}$$

A cap  $C$  of the Euclidean ball  $B_2^n$  is the intersection of a half space  $H^-$  with  $B_2^n$ . The radius of such a cap is the radius of the  $(n-1)$ -dimensional ball  $B_2^n \cap H$ .

The next two ingredients needed are from [34].

**Lemma 7.** [34] *Let  $H$  be a hyperplane,  $p$  its distance from the origin and  $s$  the surface area of the cap  $B_2^n \cap H^-$ , i.e.,*

$$s = \text{vol}_{n-1}(\partial B_2^n \cap H^-).$$

Then

$$\frac{dp}{ds} = -\frac{1}{(1-p^2)^{\frac{n-3}{2}} \text{vol}_{n-2}(\partial B_2^{n-1})}.$$

The following lemma is Lemma 3.13 from [34].

**Lemma 8.** [34] *Let  $C$  be a cap of the Euclidean unit ball. Let  $s$  be the surface area of this cap and  $r$  its radius. Then we have*

$$\begin{aligned} & \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)} \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{3}{n-1}} - c \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{5}{n-1}} \\ & \leq r(s) \\ & \leq \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)} \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{3}{n-1}} + c \left( \frac{s}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{5}{n-1}}, \end{aligned}$$

where  $c$  is a numerical constant.

### 3 Proof of Theorem 1

To prove Theorem 1, we use a probabilistic argument. We follow the strategy given in [21]. Instead of volume deviation, we now have to compute the expected surface area deviation between  $B_2^n$  and a random polytope  $[x_1, \dots, x_N]$  whose vertices are chosen randomly and independently from the boundary of a Euclidean ball with slightly bigger radius. For technical reasons, we choose the points from the boundary of  $B_2^n$  and we approximate  $(1 - \gamma)B_2^n$ . It will turn out that  $\gamma$  is of the order  $N^{-\frac{2}{n-1}}$ .

The expected surface area difference between  $(1 - \gamma)B_2^n$  and a random polytope  $P_N$  is

$$\begin{aligned} & \mathbb{E} [\Delta_s((1 - \gamma)B_2^n, P_N)] = \\ & \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left[ \text{vol}_{n-1} [\partial(P_N \cup (1 - \gamma)B_2^n)] - \text{vol}_{n-1} [\partial(P_N \cap (1 - \gamma)B_2^n)] \right] d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N), \end{aligned}$$

where  $\mathbb{P} = \frac{\mu_{\partial B_2^n}}{\text{vol}_{n-1}(\partial B_2^n)}$  is the uniform probability measure on  $\partial B_2^n$ . For a given  $N$ , we choose  $\gamma$  such that

$$\text{vol}_{n-1} \left( \partial((1 - \gamma)B_2^n) \right) = (1 - \gamma)^{n-1} \text{vol}_{n-1}(\partial B_2^n) = \mathbb{E} \text{vol}_{n-1}(\partial P_N). \quad (3)$$

From (2) we see that for large  $N$ ,  $(1 - \gamma)^{n-1}$  is asymptotically equal to

$$1 - N^{-\frac{2}{n-1}} \frac{n-1}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n + \frac{2}{n-1})}{2(n-2)!}.$$

As  $(1 - \gamma)^{n-1} \geq 1 - (n-1)\gamma$ , we get for large enough  $N$  that

$$\gamma \geq \frac{N^{-\frac{2}{n-1}}}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n + \frac{2}{n-1})}{2(n-2)!}. \quad (4)$$

For  $\gamma$  small enough,  $(1 - \gamma)^{n-1} \leq 1 - (1 - \frac{1}{n})(n-1)\gamma$ . Hence we get for small enough  $\gamma$  and large enough  $N$  that

$$\gamma \leq \frac{n}{n-1} \frac{N^{-\frac{2}{n-1}}}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n + \frac{2}{n-1})}{2(n-2)!}. \quad (5)$$

Therefore, for  $N$  large enough, there are absolute constants  $a$  and  $b$  such that

$$a N^{-\frac{2}{n-1}} \leq \gamma \leq b N^{-\frac{2}{n-1}}. \quad (6)$$

We continue the computation of the expected surface area deviation. Since

$$\text{vol}_{n-1} [\partial((1-\gamma)B_2^n)] = \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N] + \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c]$$

and

$$\mathbb{E} \text{vol}_{n-1} (\partial P_N) = \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap (1-\gamma)B_2^n) + \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap [(1-\gamma)B_2^n]^c),$$

our choice of  $\gamma$  means that

$$\begin{aligned} & \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N] + \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] \\ &= \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap (1-\gamma)B_2^n) + \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap [(1-\gamma)B_2^n]^c). \end{aligned} \quad (7)$$

Thus,

$$\begin{aligned} & \mathbb{E} [\Delta_s((1-\gamma)B_2^n, P_N)] \\ &= \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] + \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap [(1-\gamma)B_2^n]^c) \\ &\quad - \mathbb{E} \text{vol}_{n-1} (\partial P_N \cap (1-\gamma)B_2^n) - \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N] \\ &= 2 \left( \mathbb{E} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] - \mathbb{E} \text{vol}_{n-1} [(1-\gamma)B_2^n \cap \partial P_N] \right), \end{aligned}$$

where the last equality follows from equation (7). Hence,

$$\begin{aligned} & \mathbb{E} [\Delta_s((1-\gamma)B_2^n, P_N)] = \\ & 2 \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left\{ \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] - \text{vol}_{n-1} [(1-\gamma)B_2^n \cap \partial P_N] \right\} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N). \end{aligned}$$

We first consider

$$\begin{aligned} I_1 &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] \mathbb{1}_{\{0 \in \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &\quad + \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] \mathbb{1}_{\{0 \notin \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &\leq \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] \mathbb{1}_{\{0 \in \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &\quad + \text{vol}_{n-1}(\partial B_2^n) \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathbb{1}_{\{0 \notin \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N). \end{aligned}$$

By a result of [35] the second summand equals

$$\text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k} \leq \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n.$$

Therefore,

$$I_1 \leq \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1} [\partial((1-\gamma)B_2^n) \cap P_N^c] \mathbb{1}_{\{0 \in \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n. \quad (8)$$

We introduce functions  $\phi_{j_1 \dots j_n} : \prod_{i=1}^N \partial B_2^n \rightarrow \mathbb{R}$  defined by

$$\phi_{j_1 \dots j_n}(x_1, \dots, x_N) = \begin{cases} 0, & \text{if } [x_{j_1}, \dots, x_{j_n}] \text{ is not an } (n-1)\text{-dimensional face of } [x_1, \dots, x_N] \\ 0, & \text{if } 0 \notin \text{int}([x_1, \dots, x_N]) \\ \text{vol}_{n-1}((1-\gamma)S^{n-1} \cap P_N^c \cap \text{cone}(x_{j_1}, \dots, x_{j_n})), & \text{otherwise.} \end{cases}$$

For vectors  $y_1, \dots, y_k$  in  $\mathbb{R}^n$ ,

$$\text{cone}(y_1, \dots, y_k) = \left\{ \sum_{i=1}^k a_i y_i \mid \forall i : a_i \geq 0 \right\}$$

is the cone spanned by  $y_1, \dots, y_k$ . From (8) we get

$$I_1 \leq \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \sum_{\{j_1, \dots, j_n\} \subset \{1, \dots, N\}} \phi_{j_1, \dots, j_n}(x_1, \dots, x_N) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n. \quad (9)$$

Inequality (9) holds since  $0 \in \text{int}(P_N)$  and  $\mathbb{R}^n = \bigcup_{[x_{j_1}, \dots, x_{j_n}] \text{ is a facet of } P_N} \text{cone}(x_{j_1}, \dots, x_{j_n})$ . By

Lemma 4,  $P_N = [x_1, \dots, x_N]$  is simplicial with probability 1. Thus, the previous expression equals

$$\binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \phi_{1 \dots n}(x_1, \dots, x_N) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n.$$

Let  $H$  be the hyperplane containing the points  $x_1, \dots, x_n$ . The set of points where  $H$  is not well-defined has measure 0. Let  $H^+$  be the halfspace containing 0. Then

$$\begin{aligned} & \mathbb{P}^{N-n}(\{(x_{n+1}, \dots, x_N) \mid [x_1, \dots, x_n] \text{ is facet of } [x_1, \dots, x_N] \text{ and } 0 \in [x_1, \dots, x_N]\}) \\ &= \left( \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right)^{N-n}. \end{aligned}$$

Therefore, the above expression equals

$$\begin{aligned} & \binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ & \times \text{vol}_{n-1} [(1-\gamma)S^{n-1} \cap H^- \cap \text{cone}(x_1, \dots, x_n)] d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_n) + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n. \\ &= \binom{N}{n} \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B_2^n))^n} \int_{\xi \in S^{n-1}} \int_{p=0}^1 \int_{\partial(B_2^n \cap H)} \cdots \int_{\partial(B_2^n \cap H)} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ & \times \frac{\text{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{n/2}} \text{vol}_{n-1} [(1-\gamma)S^{n-1} \cap H^- \cap \text{cone}(x_1, \dots, x_n)] \\ & \times d\mu_{\partial(B_2^n \cap H)}(x_1) \cdots d\mu_{\partial(B_2^n \cap H)}(x_n) dp d\mu_{\partial B_2^n}(\xi) + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n. \end{aligned}$$

For the last equality we have used Lemma 5. It was shown in [21] that for  $p \leq 1 - \frac{1}{n}$ ,

$$\left( \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right)^{N-n} \leq \exp \left( -\frac{N-n}{n \frac{n+1}{2}} \right)$$

and the rest of the expression is bounded. Thus, there is a positive constant  $c_n$  such that for all  $n \in \mathbb{N}$

$$\begin{aligned} I_1 &\leq \binom{N}{n} \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B_2^n))^n} \int_{\xi \in S^{n-1}} \int_{p=1-\frac{1}{n}}^1 \int_{\partial(B_2^n \cap H)} \cdots \int_{\partial(B_2^n \cap H)} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ &\quad \times \frac{\text{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{n/2}} \text{vol}_{n-1}[(1-\gamma)S^{n-1} \cap H^- \cap \text{cone}(x_1, \dots, x_n)] \\ &\quad \times d\mu_{\partial(B_2^n \cap H)}(x_1) \cdots d\mu_{\partial(B_2^n \cap H)}(x_n) dp d\mu_{\partial B_2^n}(\xi) \\ &\quad + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp \left( -\frac{N-n}{n \frac{n+1}{2}} \right). \end{aligned} \quad (10)$$

Now we consider

$$\begin{aligned} I_2 &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1}[(1-\gamma)B_2^n \cap \partial P_N] d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &\geq \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \text{vol}_{n-1}[(1-\gamma)B_2^n \cap \partial P_N] \mathbb{1}_{\{0 \in \text{int}(P_N)\}} d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N) \\ &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \sum_{\{j_1, \dots, j_n\} \subset \{1, \dots, N\}} \psi_{j_1 \dots j_n}(x_1, \dots, x_N) d\mathbb{P}(x_1) \cdots d\mathbb{P}(x_N), \end{aligned}$$

where the map  $\psi_{j_1 \dots j_n} : \prod_{i=1}^N \partial B_2^n \rightarrow \mathbb{R}$  is defined by

$$\psi_{j_1 \dots j_n}(x_1, \dots, x_N) = \begin{cases} 0, & \text{if } [x_{j_1}, \dots, x_{j_n}] \text{ is not an } (n-1)\text{-dimensional face of } [x_1, \dots, x_N] \\ 0, & \text{if } 0 \notin \text{int}([x_1, \dots, x_N]) \\ \text{vol}_{n-1}[(1-\gamma)B_2^n \cap [x_{j_1}, \dots, x_{j_n}]], & \text{otherwise.} \end{cases}$$

We proceed now for  $I_2$  as above for  $I_1$ , also using Lemma 5, and get that the previous integral is greater than or equal

$$\begin{aligned} &\binom{N}{n} \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B_2^n))^n} \int_{\xi \in S^{n-1}} \int_{p=0}^1 \int_{\partial(B_2^n \cap H)} \cdots \int_{\partial(B_2^n \cap H)} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ &\quad \times \frac{\text{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{n/2}} \text{vol}_{n-1}[(1-\gamma)B_2^n \cap H \cap \text{cone}(x_1, \dots, x_n)] \\ &\quad \times d\mu_{\partial(B_2^n \cap H)}(x_1) \cdots d\mu_{\partial(B_2^n \cap H)}(x_n) dp d\mu_{\partial B_2^n}(\xi). \end{aligned} \quad (11)$$

Therefore, with (10) and (11),

$$\begin{aligned}
\mathbb{E}[\Delta_s((1-\gamma)B_2^n, P_N)] &\leq 2 \binom{N}{n} \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B_2^n))^n} \\
&\times \int_{\xi \in S^{n-1}} \int_{1-\frac{1}{n}}^1 \int_{\partial(B_2^n \cap H)} \cdots \int_{\partial(B_2^n \cap H)} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \frac{\text{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{n/2}} \\
&\times \left[ \text{vol}_{n-1}[(1-\gamma)S^{n-1} \cap H^- \cap \text{cone}(x_1, \dots, x_n)] - \text{vol}_{n-1}[(1-\gamma)B_2^n \cap [x_1, \dots, x_n]] \right] \\
&\times d\mu_{\partial(B_2^n \cap H)}(x_1) \cdots d\mu_{\partial(B_2^n \cap H)}(x_n) dp d\mu_{\partial B_2^n}(\xi) \\
&+ \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp\left(-\frac{N-n}{n \frac{n+1}{2}}\right).
\end{aligned}$$

We notice that

$$\begin{aligned}
\text{vol}_{n-1}[(1-\gamma)S^{n-1} \cap H^- \cap \text{cone}(x_1, \dots, x_n)] &\leq \\
&\left(\frac{1-\gamma}{p}\right)^{n-1} \text{vol}_{n-1}((1-\gamma)B_2^n \cap [x_1, \dots, x_n]).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[\Delta_s((1-\gamma)B_2^n, P_N)] &\leq 2 \binom{N}{n} \frac{(n-1)!}{(\text{vol}_{n-1}(\partial B_2^n))^n} \\
&\times \int_{\xi \in S^{n-1}} \int_{1-\frac{1}{n}}^1 \int_{\partial(B_2^n \cap H)} \cdots \int_{\partial(B_2^n \cap H)} \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \frac{(\text{vol}_{n-1}[x_1, \dots, x_n])^2}{(1-p^2)^{n/2}} \\
&\times \max\left\{0, \left(\frac{1-\gamma}{p}\right)^{n-1} - 1\right\} d\mu_{\partial(B_2^n \cap H)}(x_1) \cdots d\mu_{\partial(B_2^n \cap H)}(x_n) dp d\mu_{\partial B_2^n}(\xi) \\
&+ \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp\left(-\frac{N-n}{n \frac{n+1}{2}}\right).
\end{aligned}$$

By Lemma 6 this equals

$$\begin{aligned}
&2 \binom{N}{n} \frac{n}{(n-1)^{n-1}} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\text{vol}_{n-1}(\partial B_2^n))^n} \int_{\partial B_2^n} \int_{1-\frac{1}{n}}^1 \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\
&\times \max\left\{0, \left(\frac{1-\gamma}{p}\right)^{n-1} - 1\right\} \frac{r^{n^2-2}}{(1-p^2)^{n/2}} dp d\mu_{\partial B_2^n}(\xi) \\
&+ \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp\left(-\frac{N-n}{n \frac{n+1}{2}}\right),
\end{aligned}$$

where  $r$  denotes the radius of  $B_2^n \cap H$ . The expression  $B_2^n \cap H$  is a function of the distance  $p$  of the hyperplane  $H$  from the origin. Since the integral does not depend on the direction  $\xi$  and



$r^2 + p^2 = 1$ , this last expression is equal to

$$\begin{aligned} & 2 \binom{N}{n} \frac{n}{(n-1)^{n-1}} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\text{vol}_{n-1}(\partial B_2^{n-1}))^{n-1}} \int_{1-\frac{1}{n}}^1 \left[ \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^+)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ & \times \max \left\{ 0, \left( \frac{1-\gamma}{p} \right)^{n-1} - 1 \right\} r^{n^2-n-2} dp \\ & + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp \left( -\frac{N-n}{n^{\frac{n+1}{2}}} \right), \end{aligned}$$

which equals

$$\begin{aligned} & 2 \binom{N}{n} \frac{n}{(n-1)^{n-1}} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\text{vol}_{n-1}(\partial B_2^n))^{n-1}} \int_{1-\frac{1}{n}}^{1-\gamma} \left[ 1 - \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^-)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \\ & \times \left[ \left( \frac{1-\gamma}{p} \right)^{n-1} - 1 \right] r^{n^2-n-2} dp + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp \left( -\frac{N-n}{n^{\frac{n+1}{2}}} \right). \end{aligned}$$

Since  $p \geq 1 - \frac{1}{n}$  and, by (6),  $\gamma$  is of the order  $N^{-\frac{2}{n-1}}$ , we have for sufficiently large  $N$

$$\left( \frac{1-\gamma}{p} \right)^{n-1} - 1 \leq n(1-\gamma-p).$$

Therefore, the previous expression can be estimated by

$$\begin{aligned} & 2 \binom{N}{n} \frac{n^2}{(n-1)^{n-1}} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\text{vol}_{n-1}(\partial B_2^n))^{n-1}} \int_{1-\frac{1}{n}}^{1-\gamma} \left[ 1 - \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^-)}{\text{vol}_{n-1}(\partial B_2^n)} \right]^{N-n} \frac{1-\gamma-p}{r^{n+2-n^2}} dp \\ & + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp \left( -\frac{N-n}{n^{\frac{n+1}{2}}} \right). \end{aligned}$$

Let  $\phi : [0, 1] \rightarrow [0, \infty)$  be the function defined by

$$\phi(p) = \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^-)}{\text{vol}_{n-1}(\partial B_2^n)}$$

where  $H$  is a hyperplane with distance  $p$  from the origin. As in [21], we now choose

$$s = \phi(p) = \frac{\text{vol}_{n-1}(\partial B_2^n \cap H^-)}{\text{vol}_{n-1}(\partial B_2^n)}$$

as our new variable under the integral. We apply Lemma 7 in order to change the variable under the integral and get that the above expression is smaller or equal to

$$\begin{aligned} & \binom{N}{n} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\text{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n^2}{(n-1)^{n-1}} \int_{\phi(1-\frac{1}{n})}^{\phi(1-\frac{1}{n})} (1-s)^{N-n} (1-\gamma-p) r^{(n-1)^2} ds \\ & + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp \left( -\frac{N-n}{n^{\frac{n+1}{2}}} \right), \end{aligned} \tag{12}$$

where  $\phi(p)$  is the normalized surface area of the cap with distance  $p$  of the hyperplane to 0. Before we proceed, we want to estimate  $\phi(1 - \gamma)$ . The radius  $r$  and the distance  $p$  satisfy  $1 = p^2 + r^2$ . It was shown in [21] that

$$r^{n-1} \frac{\text{vol}_{n-1}(B_2^{n-1})}{\text{vol}_{n-1}(\partial B_2^n)} \leq \phi(\sqrt{1-r^2}) \leq \frac{1}{\sqrt{1-r^2}} r^{n-1} \frac{\text{vol}_{n-1}(B_2^{n-1})}{\text{vol}_{n-1}(\partial B_2^n)}.$$

We include the argument from [21] for completeness. We compare  $\phi$  with the surface area of the intersection  $B_2^n \cap H$  of the Euclidean ball and the hyperplane  $H$ . We have

$$\frac{\text{vol}_{n-1}(B_2^n \cap H)}{\text{vol}_{n-1}(\partial B_2^n)} = r^{n-1} \frac{\text{vol}_{n-1}(B_2^{n-1})}{\text{vol}_{n-1}(\partial B_2^n)}.$$

Since the orthogonal projection onto  $H$  maps  $\partial B_2^n \cap H^-$  onto  $B_2^n \cap H$ , the left hand inequality follows.

The right hand inequality follows again by considering the orthogonal projection onto  $H$ . The surface area of a surface element of  $\partial B_2^n \cap H^-$  equals the surface area of the one it is mapped to in  $B_2^n \cap H$  divided by the cosine of the angle between the normal to  $H$  and the normal to  $\partial B_2^n$  at the given point. The cosine is always greater than  $\sqrt{1-r^2}$ .

For  $p = 1 - \gamma$  we have  $r = \sqrt{2\gamma - \gamma^2} \leq \sqrt{2\gamma}$ . Therefore we get by (5),

$$\begin{aligned} \phi(1 - \gamma) &\leq \frac{2^{\frac{n-1}{2}}}{1 - \gamma} \frac{\text{vol}_{n-1}(B_2^{n-1})}{\text{vol}_{n-1}(\partial B_2^n)} \left\{ \frac{n}{n-1} \frac{N^{-\frac{2}{n-1}}}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-2}(\partial B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n + \frac{2}{n-1})}{2(n-2)!} \right\}^{\frac{n-1}{2}} \\ &= \frac{N^{-1}}{1 - \gamma} \left\{ \frac{n}{n+1} \frac{\Gamma(n + \frac{2}{n-1})}{(n-1)!} \right\}^{\frac{n-1}{2}}. \end{aligned} \quad (13)$$

The quantity  $\gamma$  is of the order  $N^{-\frac{2}{n-1}}$ , so  $1/(1 - \gamma)$  is as close to 1 as we desire for  $N$  large enough. Moreover, for all  $n \in \mathbb{N}$

$$\left( \frac{n}{n+1} \right)^{\frac{n-1}{2}} \leq 1.$$

Therefore, for all  $n \in \mathbb{N}$  and  $N$  large enough

$$\phi(1 - \gamma) \leq \frac{1}{N} \left\{ \frac{\Gamma(n + \frac{2}{n-1})}{(n-1)!} \right\}^{\frac{n-1}{2}}.$$

For all  $n \in \mathbb{N}$  with  $n \geq 2$ ,

$$\left\{ \frac{\Gamma(n + \frac{2}{n-1})}{(n-1)!} \right\}^{\frac{n-1}{2}} \leq 2n. \quad (14)$$

We verify the estimate. Stirling's formula tells us that for all  $x > 0$

$$\sqrt{2\pi x} x^x e^{-x} < \Gamma(x+1) < \sqrt{2\pi x} x^x e^{-x} e^{\frac{1}{12x}}.$$

Therefore,

$$\frac{\Gamma(n + \frac{2}{n-1})}{(n-1)!} \leq \left(1 + \frac{2}{(n-1)^2}\right)^{n - \frac{1}{2} + \frac{2}{n-1}} (n-1)^{\frac{2}{n-1}} e^{-\frac{2}{n-1}} e^{\frac{1}{12(n-1 + \frac{2}{n-1})}}$$

and

$$\left(\frac{\Gamma(n + \frac{2}{n-1})}{(n-1)!}\right)^{\frac{n-1}{2}} \leq \frac{n-1}{e} \left(1 + \frac{2}{(n-1)^2}\right)^{\frac{(n-1)(2n-1)}{4}} \left(1 + \frac{2}{(n-1)^2}\right) e^{\frac{n-1}{24(n-1 + \frac{2}{n-1})}}.$$

The right hand expression is asymptotically equal to  $(n-1)e^{1/24}$  and (14) follows. Altogether,

$$\phi(1-\gamma) \leq \frac{2n}{N}. \quad (15)$$

Since  $p = \sqrt{1-r^2}$ , we get for all  $r$  with  $0 \leq r \leq 1$

$$1 - \gamma - p = 1 - \gamma - \sqrt{1-r^2} \leq \frac{1}{2}r^2 + r^4 - \gamma.$$

This estimate is equivalent to  $1 - \frac{1}{2}r^2 - r^4 \leq \sqrt{1-r^2}$ . The left hand side is negative for  $r \geq 0.9$  and thus the inequality holds for  $r$  with  $0.9 \leq r \leq 1$ . For  $r$  with  $0 \leq r \leq 0.9$  we square both sides. Thus the integral (12) is smaller or equal to

$$\begin{aligned} & \binom{N}{n} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\text{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n^2}{(n-1)^{n-1}} \int_{\phi(1-\gamma)}^{\phi(1-\frac{1}{n})} (1-s)^{N-n} \left(\frac{1}{2}r^2 + r^4 - \gamma\right) r^{(n-1)^2} ds \\ & + \text{vol}_{n-1}(\partial B_2^n) 2^{-N+1} n N^n + c_n \exp\left(-\frac{N-n}{n^{\frac{n+1}{2}}}\right). \end{aligned}$$

Now we evaluate the integral of this expression. Again, we proceed exactly as in [21] with the obvious modifications. We include the arguments for completeness. We use Lemma 8. By differentiation we verify that  $(\frac{1}{2}r^2 + r^4 - \gamma)r^{(n-1)^2}$  is a monotone function of  $r$ . Here we use that  $\frac{1}{2}r^2 + r^4 - \gamma$  is nonnegative.

$$\begin{aligned} & \int_{\phi(1-\gamma)}^{\phi(1-\frac{1}{n})} (1-s)^{N-n} \left[\frac{1}{2}r^2 + r^4 - \gamma\right] r^{(n-1)^2} ds \leq \frac{1}{2} \int_0^1 (1-s)^{N-n} \left[s \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})}\right]^{n-1 + \frac{2}{n-1}} ds \\ & + \int_0^1 (1-s)^{N-n} \left(s \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{n-1 + \frac{4}{n-1}} ds - \int_0^1 (1-s)^{N-n} \gamma \left(s \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} ds \\ & + \int_0^{\phi(1-\gamma)} (1-s)^{N-n} \gamma \left(s \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})}\right)^{n-1} ds. \end{aligned}$$

By (4),

$$\begin{aligned}
& \int_{\phi(1-\gamma)}^1 (1-s)^{N-n} \left( \frac{1}{2}r^2 + r^4 - \gamma \right) r^{(n-1)^2} ds \\
& \leq \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \\
& \quad + \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})} \\
& \quad - \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} \frac{\Gamma(N-n+1)\Gamma(n)}{\Gamma(N+1)} \\
& \quad \quad \times \frac{N^{-\frac{2}{n-1}}}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}} \frac{\Gamma(n+\frac{2}{n-1})}{2(n-2)!} \\
& \quad + \gamma \cdot \phi(1-\gamma) \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} \max_{s \in [0, \phi(1-\gamma)]} (1-s)^{N-n} s^{n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{\phi(1-\gamma)}^1 (1-s)^{N-n} \left( \frac{1}{2}r^2 + r^4 - \gamma \right) r^{(n-1)^2} ds \tag{16} \\
& \leq \frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \\
& \quad + \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{4}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})} \\
& \quad - \frac{1}{2} \frac{n-1}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma\left(n+\frac{2}{n-1}\right)}{\Gamma(N+1)} N^{-\frac{2}{n-1}} \\
& \quad + \gamma \cdot \phi(1-\gamma) \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} \max_{s \in [0, \phi(1-\gamma)]} (1-s)^{N-n} s^{n-1}.
\end{aligned}$$

The second summand is asymptotically equal to

$$\begin{aligned}
& \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{4}{n-1}} \frac{(N-n)!(n-1)!n^{\frac{4}{n-1}}}{N!(N+1)^{\frac{4}{n-1}}} \\
& = \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{4}{n-1}} \frac{n^{-1+\frac{4}{n-1}}}{\binom{N}{n}(N+1)^{\frac{4}{n-1}}}. \tag{17}
\end{aligned}$$

This summand is of the order  $N^{-\frac{4}{n-1}}$ , while the others are of the order  $N^{-\frac{2}{n-1}}$ .

We consider the sum of the first and third summands, which is equal to

$$\frac{1}{2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} \left( 1 - \frac{n-1}{n+1} \frac{\Gamma(N+1+\frac{2}{n-1})}{\Gamma(N+1)N^{\frac{2}{n-1}}} \right).$$

Since  $\Gamma(N+1+\frac{2}{n-1})$  is asymptotically equal to  $(N+1)^{\frac{2}{n-1}}\Gamma(N+1)$ , the sum of the first and third summand is for large  $N$  of the order

$$\frac{2}{n+1} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})}, \quad (18)$$

which in turn is of the order

$$\frac{1}{n^2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \binom{N}{n}^{-1} N^{-\frac{2}{n-1}}. \quad (19)$$

We consider now the fourth summand. By (6) and (15) the fourth summand is less than

$$bN^{-\frac{2}{n-1}} \frac{n-1}{e^{23/24}N} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} \max_{s \in [0, \phi(1-\gamma)]} (1-s)^{N-n} s^{n-1}. \quad (20)$$

The maximum of the function  $(1-s)^{N-n} s^{n-1}$  is attained at  $(n-1)/(N-1)$  and the function is increasing on the interval  $[0, (n-1)/(N-1)]$ . Therefore, since  $\phi(1-\gamma) < (n-1)/(N-1)$  the maximum of this function over the interval  $[0, \phi(1-\gamma)]$  is attained at  $\phi(1-\gamma)$ . By (15) we have  $\phi(1-\gamma) \leq e^{\frac{1}{24}} \frac{n-1}{eN}$  and thus for  $N$  sufficiently big

$$\begin{aligned} \max_{s \in [0, \phi(1-\gamma)]} (1-s)^{N-n} s^{n-1} &\leq \left( 1 - \frac{n-1}{e^{23/24}N} \right)^{N-n} \left( e^{\frac{1}{24}} \frac{n-1}{eN} \right)^{n-1} \\ &\leq \exp \left( \frac{n-1}{24} - \frac{(n-1)(N-n)}{e^{23/24}N} \right) \left( \frac{n}{eN} \right)^{n-1} \\ &\leq \exp \left( -\frac{n-1}{4} \right) \left( \frac{n}{eN} \right)^{n-1}. \end{aligned}$$

Thus we get with a new constant  $b$  that (20) is smaller than or equal to

$$bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-\frac{n}{4}} \frac{n^n e^{-n}}{N^n}.$$

This is asymptotically equal to

$$bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-\frac{n}{4}} \frac{1}{\binom{N}{n} \sqrt{2\pi n}}. \quad (21)$$

Altogether, (12) for  $N$  sufficiently big can be estimated by

$$\begin{aligned} \binom{N}{n} \frac{(\text{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\text{vol}_{n-1}(\partial B_2^n))^{n-2}} \frac{n^2}{(n-1)^{n-1}} \left\{ \frac{1}{n^2} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1+\frac{2}{n-1}} \binom{N}{n}^{-1} N^{-\frac{2}{n-1}} \right. \\ \left. + bN^{-\frac{2}{n-1}} \left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{n-1} e^{-\frac{n}{4}} \frac{1}{\binom{N}{n} \sqrt{2\pi n}} \right\}. \end{aligned}$$

This can be estimated by a constant times

$$(\text{vol}_{n-1}(\partial B_2^n))n^2 \left\{ \frac{1}{n^2} N^{-\frac{2}{n-1}} + bN^{-\frac{2}{n-1}} e^{-\frac{n}{4}} \frac{1}{\sqrt{2\pi n}} \right\}. \quad (22)$$

Finally, it should be noted that we have been estimating the approximation of  $(1 - \gamma)B_2^n$  and not that of  $B_2^n$ . Therefore we need to multiply (22) by  $(1 - \gamma)^{-(n-1)}$ . By (6),

$$(1 - \gamma)^{n-1} \geq 1 - b \frac{n-1}{N^{\frac{2}{n-1}}},$$

so that we have for all  $N \geq (2b(n-1))^{\frac{n-1}{2}}$  that  $(1 - \gamma)^{-(n-1)} \leq 2$ .  $\square$

## 4 Proof of Theorem 2

For the proof of Theorem 2 we need several more ingredients. Throughout this section, we denote by  $\|\cdot\|_2$  the Euclidean norm on  $\mathbb{R}^n$  and by  $B_2^n(\xi, r)$  the  $n$ -dimensional Euclidean ball with radius  $r$  centered at  $\xi$ .

For a polytope  $P$ , the map  $T : \partial P \cap B_2^n \rightarrow \partial B_2^n$  is such that it maps an element  $x$  with a unique outer normal  $N(x)$  onto the following element of  $\partial B_2^n$

$$x \mapsto T(x) = \partial B_2^n \cap \{x + sN(x) : s \geq 0, N(x) \text{ normal at } x\}. \quad (23)$$

Points not having a unique normal have measure 0 and their image is prescribed in an arbitrary way.

**Lemma 9.** *For all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $M \in \mathbb{N}$  with  $M \geq 3$ , all polytopes  $P_M$  in  $\mathbb{R}^n$  with facets  $F_i$ ,  $i = 1, \dots, M$  and for all  $i = 1, \dots, M$  we have*

$$\text{vol}_{n-1}(T(F_i \cap B_2^n)) - \text{vol}_{n-1}(F_i \cap B_2^n) \geq \frac{1}{32} \frac{(\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}.$$

**Proof.** In the case that  $F_i \cap B_2^n$  is the empty set, the inequality holds since both sides of the inequality equal 0.

Let  $\xi_i$ ,  $i = 1, \dots, M$ , be the outer normals of  $P_M$  to  $F_i$  and let  $t_i \in \mathbb{R}$  be such that  $H(t_i \xi_i, \xi_i)$  is the hyperplane containing  $F_i$ . By definition, the volume radius of  $F_i \cap B_2^n$  is

$$r_i = \left( \frac{\text{vol}_{n-1}(F_i \cap B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{1}{n-1}}. \quad (24)$$

We decompose the set  $F_i$  into the two sets

$$F_i^1 = F_i \cap B_2^n(t_i \xi_i, \frac{r_i}{2}) \quad \text{and} \quad F_i^2 = F_i \cap (B_2^n(t_i \xi_i, \frac{r_i}{2}))^c.$$

$F_i^1$  may be the empty set but, as we shall see during the proof,  $F_i^2$  is never empty provided  $F_i \cap B_2^n$  is nonempty. The map  $T$  stretches an infinitesimal surface element at  $x$  by the factor  $\frac{1}{|\langle \xi_i, T(x) \rangle|}$ . Therefore,

$$\text{vol}_{n-1}(T(F_i \cap B_2^n)) = \int_{F_i \cap B_2^n} \frac{dx}{|\langle \xi_i, T(x) \rangle|} = \int_{F_i^1 \cap B_2^n} \frac{dx}{|\langle \xi_i, T(x) \rangle|} + \int_{F_i^2 \cap B_2^n} \frac{dx}{|\langle \xi_i, T(x) \rangle|}. \quad (25)$$

For all  $x \in F_i^2 \cap B_2^n$  we have

$$|\langle \xi_i, T(x) \rangle| \leq \sqrt{1 - \frac{1}{4}r_i^2}. \quad (26)$$

We verify this. There is  $s \geq 0$  with  $T(x) = x + s\xi_i$ . This implies  $\|x + s\xi_i\|_2 = 1$ , and consequently

$$s + \langle x, \xi_i \rangle = \sqrt{1 - \|x\|_2^2 + \langle x, \xi_i \rangle^2}.$$

Moreover,  $x \in (B_2^n(t_i\xi_i, \frac{r_i}{2}))^c$  means

$$\frac{r_i^2}{4} < \|x - t_i\xi_i\|_2^2 = \|x\|_2^2 - 2t_i\langle x, \xi_i \rangle + t_i^2 = \|x\|_2^2 - \langle x, \xi_i \rangle^2.$$

Thus,

$$\langle \xi_i, T(x) \rangle = \langle \xi_i, x + s\xi_i \rangle = \langle \xi_i, x \rangle + s = \sqrt{1 - \|x\|_2^2 + \langle x, \xi_i \rangle^2} < \sqrt{1 - \frac{r_i^2}{4}}$$

and we have shown (26). By (25) and (26),

$$\begin{aligned} \text{vol}_{n-1}(T(F_i \cap B_2^n)) &\geq \text{vol}_{n-1}(F_i^1 \cap B_2^n) + \frac{\text{vol}_{n-1}(F_i^2 \cap B_2^n)}{\sqrt{1 - \frac{r_i^2}{4}}} \\ &\geq \text{vol}_{n-1}(F_i^1 \cap B_2^n) + \text{vol}_{n-1}(F_i^2 \cap B_2^n) \sqrt{1 + \frac{r_i^2}{4}}. \end{aligned}$$

Since  $r_i \leq 1$ ,

$$\begin{aligned} \text{vol}_{n-1}(T(F_i \cap B_2^n)) &\geq \text{vol}_{n-1}(F_i^1 \cap B_2^n) + \left(1 + \frac{r_i^2}{16}\right) \text{vol}_{n-1}(F_i^2 \cap B_2^n) \\ &= \text{vol}_{n-1}(F_i \cap B_2^n) + \frac{r_i^2}{16} \text{vol}_{n-1}(F_i^2 \cap B_2^n). \end{aligned}$$

Since  $F_i^1 \subseteq B_2^n(t_i\xi_i, \frac{r_i}{2})$ , we have  $\text{vol}_{n-1}(F_i^1) \leq \frac{r_i^{n-1}}{2^{n-1}} \text{vol}_{n-1}(B_2^{n-1})$ . With (24)

$$\begin{aligned} \text{vol}_{n-1}(T(F_i \cap B_2^n)) &\geq \text{vol}_{n-1}(F_i \cap B_2^n) + \frac{r_i^2}{16} \left( \text{vol}_{n-1}(F_i \cap B_2^n) - \frac{r_i^{n-1}}{2^{n-1}} \text{vol}_{n-1}(B_2^{n-1}) \right) \\ &\geq \text{vol}_{n-1}(F_i \cap B_2^n) + \frac{(\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}}}{16 (\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} - \frac{(\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}}}{2^{n+3} (\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}. \end{aligned}$$

Therefore,

$$\text{vol}_{n-1}(T(F_i \cap B_2^n)) - \text{vol}_{n-1}(F_i \cap B_2^n) \geq \frac{1}{32} \frac{(\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}.$$

□

**Proposition 10.** For all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $M \in \mathbb{N}$  with  $M \geq 3$ , all polytopes  $P_M$  in  $\mathbb{R}^n$  with at most  $M$  facets we have

$$\text{vol}_{n-1}(\partial B_2^n \cap P_M^c) - \text{vol}_{n-1}(\partial P_M \cap B_2^n) \geq \frac{1}{32} \frac{(\text{vol}_{n-1}(B_2^n \cap \partial P_M))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} \frac{1}{M^{\frac{2}{n-1}}}.$$

**Proof.** Let  $T$  be as in (23). Then

$$\text{vol}_{n-1}(\partial B_2^n \cap P_M^c) - \text{vol}_{n-1}(\partial P_M \cap B_2^n) \geq \text{vol}_{n-1} \left( \bigcup_{i=1}^M T(F_i \cap B_2^n) \right) - \text{vol}_{n-1} \left( \bigcup_{i=1}^M (F_i \cap B_2^n) \right).$$

Since the intersection of two sets  $F_i$  and  $F_{i'}$  is a nullset and by Lemma 9,

$$\begin{aligned} & \text{vol}_{n-1}(\partial B_2^n \cap P_M^c) - \text{vol}_{n-1}(\partial P_M \cap B_2^n) \\ & \geq \sum_{i=1}^M \text{vol}_{n-1}(T(F_i \cap B_2^n)) - \sum_{i=1}^M \text{vol}_{n-1}(F_i \cap B_2^n) \geq \frac{1}{32} \sum_{i=1}^M \frac{(\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}. \end{aligned}$$

As

$$\sum_{i=1}^M \text{vol}_{n-1}(F_i \cap B_2^n) = \text{vol}_{n-1}(B_2^n \cap \partial P_M),$$

by Hölder's inequality

$$\sum_{i=1}^M (\text{vol}_{n-1}(F_i \cap B_2^n))^{\frac{n+1}{n-1}} \geq \frac{(\text{vol}_{n-1}(B_2^n \cap \partial P_M))^{\frac{n+1}{n-1}}}{M^{\frac{2}{n-1}}}.$$

Therefore,

$$\text{vol}_{n-1}(\partial B_2^n \cap P_M^c) - \text{vol}_{n-1}(\partial P_M \cap B_2^n) \geq \frac{1}{32} \frac{(\text{vol}_{n-1}(B_2^n \cap \partial P_M))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} \frac{1}{M^{\frac{2}{n-1}}}.$$

□

Let  $R : \mathbb{R}^n \rightarrow S^{n-1}$ ,  $x \mapsto R(x) = \frac{x}{\|x\|_2}$  be the radial projection.

**Lemma 11.** For all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $M \in \mathbb{N}$  with  $M \geq 3$ , all polytopes  $P_M$  in  $\mathbb{R}^n$  with  $0 \in \text{int}(P_M) \subseteq 2B_2^n$  and with facets  $F_i$ ,  $i = 1, \dots, M$  and for all  $i = 1, \dots, M$

$$\text{vol}_{n-1}(F_i \cap (B_2^n)^c) - \text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) \geq \frac{1}{128} \frac{(\text{vol}_{n-1}(F_i \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}.$$

**Proof.** Let  $\xi_i$ ,  $i = 1, \dots, M$ , be the normals to  $F_i$  and let  $t_i \in \mathbb{R}$  be such that  $H(t_i \xi_i, \xi_i)$  is the hyperplane containing  $F_i$ .

Since  $0$  is an interior point of  $P_M$ ,  $R$  maps  $\partial P_M$  bijectively onto  $\partial B_2^n$ . In particular,  $R$  maps  $\partial P_M \cap (B_2^n)^c$  up to a nullset bijectively onto  $\partial B_2^n \cap P_M$ . The map  $R$  stretches an infinitesimal surface element at  $x$  by the factor  $\frac{\langle \xi_i, \frac{x}{\|x\|_2} \rangle}{\|x\|_2^{\frac{n-1}{2}}}$ .



The volume radius of  $F_i \cap (B_2^n)^c$  is

$$\rho_i = \left( \frac{\text{vol}_{n-1}(F_i \cap (B_2^n)^c)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{1}{n-1}}. \quad (27)$$

For all  $x \in F_i \cap (B_2^n)^c$  we have  $\|x\|_2 > 1$ . Thus,

$$\text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) = \int_{F_i \cap (B_2^n)^c} \frac{\left\langle \xi_i, \frac{x}{\|x\|_2} \right\rangle}{\|x\|_2^{n-1}} dx \leq \int_{F_i \cap (B_2^n)^c} \left\langle \xi_i, \frac{x}{\|x\|_2} \right\rangle dx. \quad (28)$$

We decompose the set  $F_i \cap (B_2^n)^c$  into two sets

$$A_i = F_i \cap (B_2^n)^c \cap B_2^n(t_i \xi_i, \frac{\rho_i}{2}) \quad \text{and} \quad B_i = F_i \cap (B_2^n)^c \cap (B_2^n(t_i \xi_i, \frac{\rho_i}{2}))^c.$$

For all  $x \in F_i \cap (B_2^n(t_i \xi_i, \frac{\rho_i}{2}))^c$  we have

$$\left\langle \xi_i, \frac{x}{\|x\|_2} \right\rangle \leq \sqrt{1 - \frac{\rho_i^2}{4\|x\|_2^2}}. \quad (29)$$

We verify this. The inequality  $\|x - t_i \xi_i\|_2 > \frac{\rho_i}{2}$  implies

$$\frac{\rho_i^2}{4} < \|x\|_2^2 - 2t_i \langle x, \xi_i \rangle + t_i^2 = \|x\|_2^2 - \langle x, \xi_i \rangle^2.$$

Thus (29) follows. By (28) and (29),

$$\text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) \leq \int_{A_i} \left\langle \xi_i, \frac{x}{\|x\|_2} \right\rangle dx + \int_{B_i} \left\langle \xi_i, \frac{x}{\|x\|_2} \right\rangle dx \leq \int_{A_i} dx + \int_{B_i} \sqrt{1 - \frac{\rho_i^2}{4\|x\|_2^2}} dx.$$

Since  $P_M \subseteq 2B_2^n$ ,

$$\begin{aligned} \text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) &\leq \text{vol}_{n-1}(A_i) + \text{vol}_{n-1}(B_i) \sqrt{1 - \frac{\rho_i^2}{16}} \\ &\leq \text{vol}_{n-1}(F_i \cap (B_2^n)^c) - \frac{\rho_i^2}{64} \text{vol}_{n-1}(B_i). \end{aligned}$$

Since  $\text{vol}_{n-1}(A_i) \leq \frac{\rho_i^{n-1}}{2^{n-1}} \text{vol}_{n-1}(B_2^{n-1})$ , we have  $\text{vol}_{n-1}(B_i) \geq \text{vol}_{n-1}(F_i \cap (B_2^n)^c) - \frac{\rho_i^{n-1}}{2^{n-1}} \text{vol}_{n-1}(B_2^{n-1})$ . Therefore, with (27),

$$\begin{aligned} \text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) &\leq \left(1 - \frac{\rho_i^2}{64}\right) \text{vol}_{n-1}(F_i \cap (B_2^n)^c) + \frac{\rho_i^{n+1}}{2^{n+5}} \text{vol}_{n-1}(B_2^{n-1}) \\ &= \text{vol}_{n-1}(F_i \cap (B_2^n)^c) - \frac{(\text{vol}_{n-1}(F_i \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}} \left(\frac{1}{64} - \frac{1}{2^{n+5}}\right). \end{aligned}$$

□

**Proposition 12.** For all  $n \in \mathbb{N}$  with  $n \geq 2$ , all  $M \in \mathbb{N}$  with  $M \geq 3$ , all polytopes  $P_M$  in  $\mathbb{R}^n$  with at most  $M$  facets and with  $0 \in \text{int}(P_M) \subseteq 2B_2^n$

$$\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c) - \text{vol}_{n-1}(\partial B_2^n \cap P_M) \geq \frac{1}{128} \frac{(\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}}.$$

**Proof.** By Lemma 11,

$$\begin{aligned} & \text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c) - \text{vol}_{n-1}(\partial B_2^n \cap P_M) \\ & \geq \sum_{i=1}^M \left[ \text{vol}_{n-1}(F_i \cap (B_2^n)^c) - \text{vol}_{n-1}(R(F_i \cap (B_2^n)^c)) \right] \geq \frac{1}{128} \sum_{i=1}^M \frac{(\text{vol}_{n-1}(F_i \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}. \end{aligned}$$

As

$$\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c) = \sum_{i=1}^M \text{vol}_{n-1}(F_i \cap (B_2^n)^c),$$

Hölder's inequality implies

$$\left( \sum_{i=1}^M (\text{vol}_{n-1}(F_i \cap (B_2^n)^c))^{\frac{n+1}{n-1}} \right)^{\frac{n-1}{n+1}} M^{\frac{2}{n+1}} \geq \sum_{i=1}^M \text{vol}_{n-1}(F_i \cap (B_2^n)^c) = \text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c).$$

Consequently,

$$\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c) - \text{vol}_{n-1}(\partial B_2^n \cap P_M) \geq \frac{1}{128} \frac{(\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}}.$$

□

**Proof of Theorem 2.** We may assume that the origin is an interior point of  $P_M$ . If not, then  $P_M$  is contained in a Euclidean half ball and, by convexity, the surface area of  $P_M$  is smaller than that of the half ball,  $\text{vol}_{n-1}(\partial P_M) \leq \frac{1}{2} \text{vol}_{n-1}(\partial B_2^n) + \text{vol}_{n-1}(B_2^{n-1})$ . So, for sufficiently large  $M$ ,

$$\Delta_s(B_2^n, P_M) \geq \text{vol}_{n-1}(\partial B_2^n) - \text{vol}_{n-1}(\partial P_M) \geq \frac{1}{2} \text{vol}_{n-1}(\partial B_2^n) - \text{vol}_{n-1}(B_2^{n-1}) \geq \frac{\text{vol}_{n-1}(\partial B_2^n)}{M^{\frac{2}{n-1}}}.$$

In the same way, we see that for sufficiently large  $M$  we may assume that  $\text{vol}_{n-1}(\partial P_M) \geq \frac{1}{2} \text{vol}_{n-1}(\partial B_2^n)$ .

Moreover, we may assume that  $P_M \subseteq 2B_2^n$ . If not, there is  $x_0 \in P_M$  with  $\|x_0\|_2 \geq 2$ . For  $M$  sufficiently big we may assume that  $\frac{1}{2}B_2^n \subseteq P_M$ . Therefore,

$$\Delta_s(B_2^n, P_M) \geq \text{vol}_{n-1}(\partial[x_0, \frac{1}{2}B_2^n] \cap (B_2^n)^c),$$

where  $[x_0, \frac{1}{2}B_2^n]$  denotes the convex hull of the point  $x_0$  with the Euclidean ball of radius  $\frac{1}{2}$ .

By Propositions 10 and 12,

$$\begin{aligned} \Delta_s(B_2^n, P_M) &= \text{vol}_{n-1}(\partial B_2^n \cap P_M^c) - \text{vol}_{n-1}(\partial P_M \cap B_2^n) \\ &\quad + \text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c) - \text{vol}_{n-1}(\partial B_2^n \cap P_M) \\ &\geq \frac{1}{32} \frac{(\text{vol}_{n-1}(B_2^n \cap \partial P_M))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}} + \frac{1}{128} \frac{(\text{vol}_{n-1}(\partial P_M \cap (B_2^n)^c))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}}. \end{aligned}$$

By Hölder's inequality,

$$\Delta_s(B_2^n, P_M) \geq \frac{1}{128 \cdot 2^{\frac{2}{n-1}}} \frac{(\text{vol}_{n-1}(\partial P_M))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}}.$$

For sufficiently large  $M$  we have  $\text{vol}_{n-1}(\partial P_M) \geq \frac{1}{2} \text{vol}_{n-1}(\partial B_2^n)$ . Therefore,

$$\Delta_s(B_2^n, P_M) \geq \frac{1}{2^{12}} \frac{(\text{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\text{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}} M^{\frac{2}{n-1}}}.$$

There is a constant  $c > 0$  such that for all  $n \in \mathbb{N}$  with  $n \geq 2$

$$\left( \frac{\text{vol}_{n-1}(\partial B_2^n)}{\text{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{2}{n-1}} \geq c.$$

Therefore, with a new constant  $c$ ,

$$\Delta_s(B_2^n, P_M) \geq c \frac{\text{vol}_{n-1}(\partial B_2^n)}{M^{\frac{2}{n-1}}}.$$

□

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