Relative entropies for convex bodies *

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Abstract

We introduce a new class of (not necessarily convex) bodies and show, among other things, that these bodies provide yet another link between convex geometric analysis and information theory. Namely, they give geometric interpretations of the relative entropy of the cone measures of a convex body and its polar and related quantities.

Such interpretations were first given by Paouris and Werner for symmetric convex bodies in the context of the L_p-centroid bodies. There, the relative entropies appear after performing second order expansions of certain expressions. Now, no symmetry assumptions are needed. Moreover, using the new bodies, already first order expansions make the relative entropies appear. Thus, these bodies detect "faster" details of the boundary of a convex body than the L_p-centroid bodies.

1 Introduction.

It has been observed in recent years that there is a close connection between convex geometric analysis and information theory. An example is the parallel between geometric inequalities for convex bodies and inequalities for probability densities. For instance, the Brunn-Minkowski inequality and the entropy power inequality follow both in a very similar way from the sharp Young inequality (see. e.g., [3]).

Further connections between convexity and information theory were established by Lutwak, Yang, and Zhang ([21, 24, 26]). They showed in [24]

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that the Cramer-Rao inequality corresponds to an inclusion of the Legendre ellipsoid and the polar L_2-projection body. The latter is a basic notion from the L_p-Brunn-Minkowski theory. This theory evolved rapidly over the last years and due to a number of highly influential works (see, e.g., [5], [7], [8], [10] - [29], [31], [33] - [42], [45]), it is now a central part of modern convex geometry. In fact, this affine geometry of bodies pertains to some questions that had been considered Euclidean in nature. For example, the famous Busemann-Petty Problem (finally laid to rest in [4, 6, 31, 43, 44]), was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [19].

Two fundamental notions within the L_p-Brunn-Minkowski theory are L_p-affine surface areas, introduced by Lutwak in [20], and L_p-centroid bodies introduced by Lutwak and Zhang in [22]. See Section 3 for the definition of those quantities. Based on these quantities, Paouris and Werner [30] established yet another relation between affine convex geometry and information theory. They proved that the exponential of the relative entropy of the cone measures of a *symmetric* convex body and its polar equals a limit of normalized L_p-affine surface areas. Moreover, they introduce a new affine invariant quantity Ω_K (see also Section 3 for the definition).

Here we introduce a new class of (not necessarily convex) bodies which we call *mean width bodies.* We describe some of their properties. For instance, we show that they are always star shaped and that they provide geometric interpretations of L_p-affine surface areas. Many such geometric interpretations have been given (see e.g. [28, 35, 36, 40, 41, 42]). The twist here is that these new geometric interpretations of affine invariants for *convex* bodies are expressed in terms of *not necessarily convex bodies* (see also [42]).

More importantly, these bodies provide yet another link between convex geometric analysis and information theory: The main result of the paper shows that these new bodies give geometric interpretations of both, the relative entropy of the cone measures of a *not necessarily symmetric* convex body and its polar and the quantity Ω_K. Such interpretations were first given by Paouris and Werner [30] only for *symmetric* convex bodies in the context of the L_p-centroid bodies. There the relative entropies appear after performing a *second order expansion* of certain expressions. The remarkable fact now is that, using the mean width bodies, already a *first order expansion* makes them appear. Thus, these new bodies detect "faster" details of the boundary of a convex body than the L_p-centroid bodies.

1.1 Notation

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm. $B_2^n(x,r)$ is the Euclidean ball centered at x with radius r. We write $B_2^n = B_2^n(0,1)$ for the Euclidean unit ball centered at 0 and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. Some of our definitions, the below L_p-affine surface area among them, require a fixed reference point. Thus, throughout the paper, we will assume without loss of generality that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. $K^* = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } \ x \in K\}$ is the polar body of K. The normalized cone measure V_K on ∂K, the boundary of K, is defined as follows: For every measurable set $A \subseteq \partial K$

$$V_K(A) = \frac{1}{|K|} |\{ta : \ a \in A, t \in [0,1]\}|.$$

Let L be a subset of $\mathbb{R}^n$ that contains 0. Then L is called star shaped, if the line segment $[0,x] \subset L$ for all $x \in L$.

We write $K \in C_+^2$, if K has C^2 boundary ∂K with everywhere strictly positive Gaussian curvature κ_K. For a point $x \in \partial K$, $N_K(x)$ is the outer unit normal at x to K. S_K is the usual surface area measure on ∂K. The usual surface area measure on S^{n-1} is denoted by ω. σ is its normalization: $\sigma(A) = \frac{\omega(A)}{\omega(S^{n-1})}$ for all Borel measurable sets $A \subset S^{n-1}$.

For u and x in $\mathbb{R}^n$, $H = H(x,\xi)$ is the hyperplane through x orthogonal to ξ. $H^+ = H^+(x,\xi) = \{y \in \mathbb{R}^n : \langle y, \xi \rangle \geq \langle x, \xi \rangle\}$ and $H^- = H^-(x,\xi) = \{y \in \mathbb{R}^n : \langle y, \xi \rangle \leq \langle x, \xi \rangle\}$ are the two closed half spaces generated by H.

Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. Then $h_K(u)$ is the support function of direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at this point $x \in \partial K$ that has u as outer normal. The mean width $W(K)$ of a convex body K in $\mathbb{R}^n$ is defined as

$$W(K) = 2 \int_{S^{n-1}} h_K(u) d\sigma(u).$$

For a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \leq p \leq \infty$, the L_p-centroid body $Z_p(K)$, introduced in [22], is this convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}.$$

L_p-affine surface area $\Omega_p(K)$ of K was introduced by Lutwak in the ground breaking paper [20] for $p > 1$ and for general p by Schütt and

Werner [36]. For real $p \neq -n$, we define $\Omega_p(K)$ as in [20] ($p > 1$) and [36] ($p < 1, p \neq -n$) by

$$\Omega_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} dS_K(x) \tag{1}$$

and

$$\Omega_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} dS_K(x), \tag{2}$$

provided the above integrals exist. In particular, for $p = 0$

$$\Omega_0(K) = \int_{\partial K} \langle x, N_K(x)\rangle \, dS_K(x) = n|K|.$$

The case $p = 1$ is the classical affine surface area which is independent of the position of K in space and which goes back to Blaschke.

$$\Omega_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} \, dS_K(x).$$

Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention too (e.g. [17, 20, 27, 34, 39]).

Let (X, μ) be a measure space and let $dP = pd\mu$ and $dQ = qd\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. The Kullback-Leibler divergence or relative entropy from P to Q is defined as (see [2])

$$D_{KL}(P\|Q) = \int_X p \log \frac{p}{q} d\mu. \tag{3}$$

2 Mean width bodies.

Let K be a convex body in $\mathbb{R}^n$. It is easy to see ([9]) that the mean width of $W(K)$ can be written as

$$W(K) = \frac{2}{\omega(S^{n-1})} \int_{\mathbb{R}^n \setminus K^*} \|\xi\|^{-(n+1)} d\xi \tag{4}$$

and therefore, for convex bodies M and K with $K \subset M$,

$$W(M) - W(K) = \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus M^*} \|\xi\|^{-(n+1)} d\xi. \tag{5}$$

Let $f : K^* \to \mathbb{R}$ be a positive, integrable function. We generalize (4) to

$$W_f(K) = \frac{2}{\omega(S^{n-1})} \int_{\mathbb{R}^n \setminus K^*} f(\xi) d\xi. \tag{6}$$

Therefore, for convex bodies M and K with $K \subset M$, (5) generalizes to

$$W_f(M) - W_f(K) = \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus M^*} f(\xi) d\xi. \tag{7}$$

In the following easy lemma we will need another notation.
Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive function. Recall that f is said to be homogeneous of degree α, if for all $r \geq 0$,

$$f(ru) = r^\alpha f(u).$$

Lemma 2.1. *Let K and M be convex bodies in $\mathbb{R}^n$ such that $K \subset M$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive, integrable function that is homogeneous of degree α.*

(i) Let $\alpha \neq -n$. Then

$$W_f(M) - W_f(K) = \frac{2}{(\alpha + n)} \int_{S^{n-1}} f(u) \left[\frac{1}{h_K^{\alpha+n}(u)} - \frac{1}{h_M^{\alpha+n}(u)} \right] d\sigma(u).$$

(ii) Let $\alpha = -n$. Then

$$W_f(M) - W_f(K) = 2 \int_{S^{n-1}} f(u) \log \left[\frac{h_M(u)}{h_K(u)} \right] d\sigma(u).$$

Proof. We use α-homogeneity and get

$$\begin{aligned} W_f(M) - W_f(K) &= \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus M^*} f(\xi) d\xi \\ &= \frac{2}{\omega(S^{n-1})} \int_{S^{n-1}} \int_{\frac{1}{h_M(u)}}^{\frac{1}{h_K(u)}} f(ru) r^{n-1} dr d\omega(u) \\ &= \frac{2}{\omega(S^{n-1})} \int_{S^{n-1}} \int_{\frac{1}{h_M(u)}}^{\frac{1}{h_K(u)}} f(u) r^{n+\alpha-1} dr d\omega(u). \end{aligned}$$

Integration then yields (i) and (ii).

If we let $f(u) = \frac{1}{h_K^n(u)}$ (or $f(u) = \frac{1}{h_M^n(u)}$) in Lemma 2.1 (ii), then $f(ru) = \frac{r^{-n}}{h_K^n(u)} = r^{-n} f(u)$. Thus this f is homogeneous of degree $-n$.

Consider now the measure space $(X, \mu) = (S^{n-1}, \omega)$ and for convex bodies K and M in $\mathbb{R}^n$ put

$$p_K = \frac{1}{n|K^*|h_K^n}, \qquad p_M = \frac{1}{n|M^*|h_M^n}. \tag{8}$$

Then $dP_K = p_K d\omega$ and $dP_M = p_M d\omega$ are probability measures on S^{n-1} and Lemma 2.1 (ii) becomes

$$\begin{aligned} W_{\frac{1}{h_K^n}}(M) - W_{\frac{1}{h_K^n}}(K) &= \frac{2}{n}|K^*| \int_{S^{n-1}} \frac{1}{|K^*|h_K^n} \log\left(\frac{h_M^n}{h_K^n}\right) d\sigma \\ &= \frac{2|K^*|}{\omega(S^{n-1})} \int_{S^{n-1}} p_K \left(\log\frac{p_K}{p_M} + \log\left(\frac{|K^*|}{|M^*|}\right)\right) d\omega \\ &= \frac{2|K^*|}{\omega(S^{n-1})} \left(D_{KL}(P_K \| P_M) + \log\left(\frac{|K^*|}{|M^*|}\right)\right). \end{aligned}$$

Hence we get

Corollary 2.2. *Let K and M be convex bodies in $\mathbb{R}^n$ such that $K \subset M$ and let p_K and p_M be the probability densities given in (8). Then*

$$\int_{K^* \setminus M^*} \frac{1}{h_K^n(\xi)} \frac{d\xi}{|K^*|} = D_{KL}(P_K \| P_M) + \log\left(\frac{|K^*|}{|M^*|}\right).$$

We now want to apply the above considerations for a specific M. Namely, for $x \in \mathbb{R}^n$, let $K_x = [x, K]$ be the convex hull of x and K. For $x \in K$, $K_x = K$. Therefore, we will consider only $x \notin K$. Let $t \geq 0$ and let

$$K[t] = \{x \in \mathbb{R}^n : w(x) \leq t\} \tag{9}$$

where

$$w(x) = W(K_x) - W(K) = \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus K_x^*} \|\xi\|^{-(n+1)} d\xi. \tag{10}$$

The bodies $K[t]$ have been used by several authors (e.g. by Böröczky and Schneider [1] and Glasauer and Gruber [9]) in connection with approximation of convex bodies by polytopes. We generalize them to the *mean width bodies* as follows.

Definition 2.3. *Let $f : K^* \to \mathbb{R}$ be a positive, integrable function. Let*

$$w_f(x) = W_f(K_x) - W_f(K) = \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus K_x^*} f(\xi) d\xi \tag{11}$$

Then we call

$$K_f[t] = \{x \in \mathbb{R}^n : w_f(x) \le t\}. \tag{12}$$

the mean width bodies of K with respect to f.

Thus, for instance, for $\beta \in \mathbb{R}$ and $f_\beta(\xi) = \|\xi\|^{-\beta}$ we get

$$K_{f_\beta}[t] = \left\{ x \in \mathbb{R}^n : \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus K_x^*} \|\xi\|^{-\beta} dx \le t \right\}, \tag{13}$$

which, in the particular case $\beta = n+1$, gives the bodies (9) above.

As $K_x = [x, K]$, $K_x^* = K^* \cap \{y \in \mathbb{R}^n : \langle y, x \rangle \le 1\}$. Thus, putting $H^+ \left(\frac{x}{\|x\|^2}, x \right) = \{y \in \mathbb{R}^n : \langle y, x \rangle \le 1\}$, K_x^* is obtained from K^* by cutting off a cap $K^* \cap H^- \left(\frac{x}{\|x\|^2}, x \right)$ of K^*:

$$K_x^* = K^* \cap H^+ \left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|} \right).$$

and

$$K^* \setminus K_x^* = K^* \cap H^- \left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|} \right).$$

Therefore

$$K_f[t] \quad = \quad \left\{ x \in \mathbb{R}^n : \frac{2}{\omega(S^{n-1})} \int_{K^* \cap H^- \left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|} \right)} f(\xi) d\xi \le t \right\}. \tag{14}$$

Remarks 1: Properties of $K_f[t]$

(i) It is clear that for all f and for all $t \ge 0$, $K \subset K_f[t]$ and that $K_{f_\beta}[0] = K$ for all β. However, it can happen that K is a proper subset of $K_f[0]$.

To see that, let $K = B_\infty^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \max_{1 \le i \le n} |x_i| \le 1\}$.
Then $K^* = B_1^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \le 1\}$.
Define $f : B_1^n \to \mathbb{R}$, $(x_1, \dots, x_n) \to f((x_1, \dots, x_n))$ by

$$f(x) = \begin{cases} 0, & x_n \ge 0 \\ 1, & \text{otherwise.} \end{cases}$$

Then $(0,\ldots,0,\frac{3}{2}) \in K_f[0]$ but $(0,\ldots,0,\frac{3}{2}) \notin K$.

(ii) $K_f[t]$ need neither be bounded nor convex. Indeed, let $K = B^2_\infty$. Define $f : B_1^2 \to \mathbb{R}$, $(x_1,x_2) \to f((x_1,x_2))$ by

$$f(x) = \begin{cases} \frac{1}{2}, & x_2 \geq 0 \\ 1, & \text{otherwise.} \end{cases}$$

If $t \geq \frac{1}{\pi}$, $K_f[t] = \mathbb{R}^2$. If $\frac{3}{4\pi} \leq t < \frac{1}{\pi}$, $\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \subset K_f[t]$. If $\frac{1}{2\pi} \leq t < \frac{3}{4\pi}$, $\{(0,x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \subset K_f[t]$. Thus $K_f[t]$ is unbounded in those cases. If $t < \frac{1}{2\pi}$, then $K_f[t]$ is bounded.

Moreover, with the same K and f: $\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \subset K_f[\frac{3}{4\pi}]$ and $\left(0, -\frac{1}{1-\sqrt{3}/2}\right) \in K_f[\frac{3}{4\pi}]$. Let $x_0 = \left(\frac{1}{1-\sqrt{3}/2}, \frac{-1}{1-\sqrt{3}/2}\right)$. Then $w_f(x_0) = \sqrt{3}\left(1-\sqrt{3}/16\right) > \frac{3}{4\pi}$. Therefore, $K_f[\frac{3}{4\pi}]$ is not convex.

(iii) Formulas (11) and (14) show that to define $K_f[t]$, we cut off a set of "weighted volume" t of K^*. Thus $K_f[t]$ resembles the convex floating body of K^*.

Recall that for $0 \leq \delta \leq \frac{|K|}{2}$, the *convex floating body* K_δ of K is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume at most δ from K [34]:

$$K_\delta = \bigcap_{|H^- \cap K| \leq \delta} H^+.$$

For $\beta = 0$, we get in formula (14),

$$\begin{aligned} K_{f_0}[t] &= \{x \in \mathbb{R}^n : \frac{2}{\omega(S^{n-1})} \int_{K^* \cap H^-\left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|}\right)} d\xi \leq t\} \\ &= \left\{ x \in \mathbb{R}^n : \left| K^* \cap H^-\left(\frac{x}{\|x\|^2}, \frac{x}{\|x\|}\right)\right| \leq \frac{t\omega(S^{n-1})}{2} \right\} \end{aligned}$$

However, $K_{f_0}[t]$ is not a convex floating body of K^*.

Indeed, it is easy to see that for the Euclidean ball $B = rB_2^n$ in $\mathbb{R}^n$ with radius r, $B_{f_0}[t]$, for small t, is a Euclidean ball with radius of order

$$r\left(1 + k_n r^{\frac{2n}{n+1}} t^{\frac{2}{n+1}}\right),$$

where $k_n = \frac{1}{2}\left(\frac{n(n+1)|B_2^n|}{2|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$. $(B^*)_\delta$, for small δ, is a ball with radius of order

$$\frac{1}{r}\left(1 - c_n r^{\frac{2n}{n+1}} \delta^{\frac{2}{n+1}}\right),$$

where $c_n = \frac{1}{2}\left(\frac{n+1}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$ (see e.g. [34]) and B_δ, for small δ, is a ball with radius of order

$$r\left(1 - \frac{c_n}{r^{\frac{2n}{n+1}}}\delta^{\frac{2}{n+1}}\right),$$

(see also e.g. [34]).

Also, $K_{f_0}[t]$ is different from the *illumination body* K^δ which, for $\delta \geq 0$, is defined as follows [39]:

$$K^\delta = \{x \in \mathbf{R}^n : |\mathrm{co}[x, K] \backslash K| \leq \delta\}.$$

Again, this can be seen by considering the Euclidean ball rB_2^n. $(rB_2^n)^\delta$, for small δ, is a Euclidean ball with radius of order

$$r\left(1 + \frac{d_n}{r^{\frac{2n}{n+1}}}\delta^{\frac{2}{n+1}}\right),$$

where $d_n = \frac{1}{2}\left(\frac{n(n+1)}{|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$ [39].

We have seen that $K_f[t]$ need not be convex. But it is always star shaped.

Lemma 2.4. *Let K be a convex body in $\mathbb{R}^n$ and let $f : K^* \to \mathbb{R}$ be a positive, integrable function.*

(i) $K_f[t]$ is star shaped.

(ii) $K_f[t] = \bigcap_{s>0} K_f[t+s]$.

Proof. (i) Let $x \in K_f[t]$ and let $y \in [0, x]$. Then $K_y = [y, K] \subset [x, K] = K_x$ and consequently $K^* \setminus K_y^* \subset K^* \setminus K_x^*$. As $f \geq 0$ on K^*, we therefore get

$$\frac{2}{\omega(S^{n-1})}\int_{K^* \setminus K_y^*} f(\xi)d\xi \leq \frac{2}{\omega(S^{n-1})}\int_{K^* \setminus K_x^*} f(\xi)d\xi \leq t$$

and thus $y \in K_f[t]$.

(ii) For all $s > 0$, $K_f[t] \subset K_f[t+s]$. Therefore, we only need to show that $\bigcap_{s>0} K_f[t+s] \subset K_f[t]$. Let thus $x \in \bigcap_{s>0} K_f[t+s]$. Then for all $s > 0$, $w_f(x) \leq t + s$. Letting $s \to 0$, we get $w_f(x) \leq t$.

Additional conditions on f ensure convexity of $K_f[t]$. This is shown in the next lemma whose proof is the same as the corresponding one in [1].

Lemma 2.5. *Let K be a convex body in $\mathbb{R}^n$ and let $f : S^{n-1} \to \mathbb{R}$ be a positive, integrable function that is homogeneous of degree α. Then $K_f[t]$ is convex for all $\alpha \leq -(n+1)$.*

Proof. Let x and y be in $K_f[t]$ and let $0 < \lambda < 1$. For $t \in \mathbb{R}$, $t \geq 0$, the function $g(t) = t^\gamma$ is convex if $\gamma \geq 1$. Therefore, and as $K_{(1-\lambda)x+\lambda y} \subseteq (1-\lambda)K_x + \lambda K_y$, we get for $\alpha \leq -(n+1)$

$$\frac{h_{K_{(1-\lambda)x+\lambda y}}^{-(\alpha+n)}}{-(\alpha+n)} \leq \frac{\left((1-\lambda)\ h_{K_x} + \lambda\ h_{K_y}\right)^{-(\alpha+n)}}{-(\alpha+n)} \leq \frac{(1-\lambda)\ h_{K_x}^{-(\alpha+n)} + \lambda\ h_{K_y}^{-(\alpha+n)}}{-(\alpha+n)}.$$

Hence for $\alpha \leq -(n+1)$,

$$\begin{aligned}
&\frac{2}{-(\alpha+n)} \int_{S^{n-1}} f(u) h_{K_{(1-\lambda)x+\lambda y}}^{-(\alpha+n)}(u) d\sigma(u) \\
&\leq \frac{2}{-(\alpha+n)} \left[(1-\lambda) \int_{S^{n-1}} f(u) h_{K_x}^{-(\alpha+n)}(u) d\sigma(u) \right. \\
&\left. +\lambda \int_{S^{n-1}} f(u) h_{K_y}^{-(\alpha+n)}(u) d\sigma(u)\right] \\
&\leq (1-\lambda)\left[\frac{2}{-(\alpha+n)} \int_{S^{n-1}} f(u) h_K^{-(\alpha+n)}(u) d\sigma(u) + t\right] \\
&\qquad +\lambda\left[\frac{2}{-(\alpha+n)} \int_{S^{n-1}} f(u) h_K^{-(\alpha+n)}(u) d\sigma(u) + t\right] \\
&= \frac{2}{-(\alpha+n)} \int_{S^{n-1}} f(u) h_K^{-(\alpha+n)}(u) d\sigma + t.
\end{aligned}$$

Remark. If $\alpha > -(n+1)$, then $K_f[t]$ need not be convex. An example is the cube in $\mathbb{R}^2$ and the f given in Remark 1 (ii).

Now we give conditions that guarantee that $K_f[t]$ is bounded.

Lemma 2.6. *Let K be a convex body in $\mathbb{R}^n$ and let $f : K^* \to \mathbb{R}$ be a strictly positive, integrable function. Then*

(i) $K_f[0] = K$.

(ii) There exists t_0 such that for all $t \leq t_0$, $K_f[t]$ is bounded.

(iii) Let $t \leq t_0$, where t_0 is as in (ii). Then we have for all $x \in \partial K_f[t]$ that $w_f(x) = t$.

Proof.
(i) We only have to show that $K_f[0] \subset K$. Let $x \in K_f[0]$. Then $w_f(x) = \frac{2}{\omega(S^{n-1})} \int_{K^* \setminus K_x^*} f(\xi) d\xi = 0$. As $f > 0$ on K^*, this can only happen if $m(K^* \setminus K_x^*) = 0$. As $K_x^* \subset K^*$ is closed and convex, this can only happen if $K_x^* = K^*$, or, equivalently, $K_x = K$, or $x \in K$.

(ii) This follows immediately from (i), Lemma 2.4 (ii) and the fact that, as K is a convex body, there exists $\alpha > 0$ such that

$$B_2^n(0, \alpha) \subset K \subset B_2^n\left(0, \frac{1}{\alpha}\right). \tag{15}$$

As $K = K_f[0] = \bigcap_{t>0} K_f[t]$, there exists t_0 such that for all $t \leq t_0$, $K_f[t] \subset 2K \subset B_2^n\left(0, \frac{2}{\alpha}\right)$.

(iii) Let $t \leq t_0$ and let $x \in \partial K_f[t]$. Suppose $w_f(x) < t$. Let $y \in \{ax : a \geq 1\}$. Then $K_x = [x, K] \subset K_y = [y, K]$, hence $K_y^* \subset K_x^*$ and therefore $\int_{K^* \setminus K_y^*} f(\xi) d\xi \geq \int_{K^* \setminus K_x^*} f(\xi) d\xi$. As $f > 0$ on K^*, we can choose $y = ax$ with $a > 1$ such that $\frac{2}{\omega(S^{n-1})} \int_{K^* \setminus K_y^*} f(\xi) d\xi = t$. This implies that $x \notin \partial K_f[t]$, a contradiction.

3 Relative entropies of cone measures and affine surface areas

In this section we present new geometric interpretations of important affine invariants, namely the L_p-affine surface areas. Many such geometric interpretations have been given (see e.g. [28, 35, 36, 40, 41, 42]). The remarkable fact here is that these geometric interpretations of affine invariants for *convex* bodies are expressed in terms of *not necessarily convex* bodies, a phenomenon which already occurred in [42].

We also give new geometric interpretations for the relative entropies of cone measures of convex bodies. Geometric interpretations for those quantities were given first in [30] in terms of L_p-centroid bodies. However, in the context of the L_p-centroid bodies, the relative entropies appeared only after performing a second order expansion of certain expressions. Now, using the mean width bodies, already a first order expansion makes them appear. Thus, these bodies detect “faster” more detail of the boundary of a convex body than the L_p-centroid bodies.

Theorem 3.1. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let $f : K^* \to \mathbb{R}$ be a continuous function such that $f(y) \geq c$ for all $y \in K^*$ and some constant $c > 0$. Then*

$$\lim_{t \to 0} \frac{|K_f[t]| - |K|}{k_n \ t^{\frac{2}{n+1}}} = \int_{\partial K} \frac{\langle x, N_K(x)\rangle^2 dS_K(x)}{f(y(x))\kappa_K(x)^{\frac{1}{n+1}}}.$$

$k_n = \frac{1}{2}\left(\frac{n(n+1)|B_2^n|}{2|B_2^{n-1}|}\right)^{\frac{2}{n+1}}$ *and $y(x) \in \partial K^*$ is such that $\langle y(x), x\rangle = 1$.*

Remark.
We put $N_K(x) = u$. Then $\langle x, N_K(x)\rangle = h_K(u)$ and $y(x) = \frac{u}{h_K(u)}$. As $dS_K = f_K d\omega$, we therefore also have

$$\lim_{t \to 0} \frac{|K_f[t]| - |K|}{k_n \ t^{\frac{2}{n+1}}} = \int_{S^{n-1}} \frac{h_K(u)^2 d\omega(u)}{f_K(u)^{\frac{n+2}{n+1}} f\left(\frac{u}{h_K(u)}\right)}. \tag{16}$$

Theorem 3.1 leads to the announced new geometric interpretations of the above mentioned quantities. For that, we need the following functions. For $p \in \mathbb{R}$, $p \neq -n$, let $g_{\Omega_p} : \partial K^* \to \mathbb{R}$ be defined by

$$g_{\Omega_p}(y) = \left(\frac{\langle x, N_K(x)\rangle}{\kappa_K(x)^{\frac{1}{n+1}}}\right)^{\frac{n+p(n+2)}{n+p}}, \tag{17}$$

where, for $y \in \partial K^*$, $x = x(y) \in \partial K$ is such that $\langle x, y\rangle = 1$.

For $\beta \in \mathbb{R}$, let $f_\beta : K^* \to \mathbb{R}$ be defined by

$$f_\beta(y) = \frac{1}{\|y\|^\beta} = \langle x, N_K(x)\rangle^\beta, \tag{18}$$

where, again, for $y \in \partial K^*$, $x = x(y) \in \partial K$ is such that $\langle x, y\rangle = 1$.

Then we have

Corollary 3.2. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+.*

(i) For $p \in \mathbb{R}$, $p \neq -n$, let g_{Ω_p} as in (17). Then

$$\lim_{t \to 0} \frac{|K_{g_{\Omega_p}}[t]| - |K|}{k_n \ t^{\frac{2}{n+1}}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}} dS_K(x)}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} \ = \ \Omega_p(K).$$

(ii) For $\beta \in \mathbb{R}$, let f_β be as in (18). Then

$$\lim_{t\to 0} \frac{|K_{f_\beta}[t]| - |K|}{k_n\ t^{\frac{2}{n+1}}} = \int_{\partial K} \frac{dS_K(x)}{\kappa_K(x)^{\frac{1}{n+1}} \langle x, N_K(x)\rangle^{\beta-2}}$$

Proof. As ∂K is in C^2_+, the functions g_{Ω_p} and f_β satisfy the conditions of Theorem 3.1. The proof of the corollary then follows immediately from Theorem 3.1.

Remarks

(i) For $\beta = 0$, we get in Corollary 3.2 (ii) the $\Omega_{-\frac{n}{n+2}}$-affine surface area of K.

(ii) As $\kappa_K(rx) = r^{-(n-1)}\kappa_K(x)$, it makes most sense to put $f_K(ru) = f_{rK}(u) = r^{n-1}f_K(u)$ and define $n-1$ to be the degree of homogeneity of the function f_K. Then g_{Ω_p} is homogeneous of degree $\frac{2n(n+p(n+2))}{(n+1)(n+p)}$ and f_β is homogeneous of degree β. Thus, by Lemma 2.5, $K_{g_{\Omega_p}}[t]$ is convex if $-n < p \leq -n\frac{(n+1)^2+1}{(n+1)^2+n+2}$ and $K_{f_\beta}[t]$ is convex if $\beta \leq -(n+1)$.

Let K a convex body in $\mathbb{R}^n$ that is C^2_+. Let

$$p_K(x) = \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n\ n|K^*|}, \quad q_K(x) = \frac{\langle x, N_K(x)\rangle}{n\ |K|}. \tag{19}$$

Then

$$P_K = p_K\ S_K \quad \text{and} \quad Q_K = q_K\ S_K \tag{20}$$

are probability measures on ∂K that are absolutely continuous with respect to S_K.

The next proposition is well known. See e.g. [30] for a proof. There, $N_K : \partial K \to S^{n-1}$, $x \to N_K(x)$ is the Gauss map and V_K is the cone measure of K.

Proposition 3.3. *Let K a convex body in $\mathbb{R}^n$ that is C^2_+. Let P_K and Q_K be the probability measures on ∂K defined by (20). Then*

$$P_K = N_K^{-1} N_{K^*} V_{K^*} \quad \textit{and} \quad Q_K = V_K,$$

or, equivalently, for every measurable subset A in ∂K

$$P_K(A) = V_{K^*}\left(N_{K^*}^{-1}\big(N_K(A)\big)\right) \quad \textit{and} \quad Q_K(A) = V_K(A).$$

Thus this proposition shows that the measure Q_K defined in (20) is the cone measure V_K of K and that the measure P_K defined in (20), though a measure on ∂K, can be viewed, in the sense of Proposition 3.3, as the "cone measure" of K^* modulo the respective Gauss maps.

For a convex body K in $\mathbb{R}^n$ that is in C^2_+, we use, in the next corollaries, also the following notations.

Let $x \in \partial K$ and let $r_i(x)$, $1 \leq i \leq n-1$ be the principal radii of curvature. We put

$$r = \inf_{x\in\partial K} \min_{1\leq i\leq n-1} r_i(x) \quad \text{and} \quad R = \sup_{x\in\partial K} \max_{1\leq i\leq n-1} r_i(x). \tag{21}$$

Note that if K be a convex body in $\mathbb{R}^n$ that is in C^2_+, then $0 < r \leq R < \infty$. Note also that $r = R$ iff K is a Euclidean ball with radius r.

For $y \in \partial K^*$, let $x = x(y) \in \partial K$ be such that $\langle x, y\rangle = 1$. Define $ent_1 : \partial K^* \to \mathbb{R}$ by

$$ent_1(y) = \frac{\kappa_K(x)^{-\frac{n+2}{n+1}} \langle x, N_K(x)\rangle^{n+1}}{\log\left(\frac{R^{2n}|K|\ \kappa_K(x)}{r^{2n}|K^*|\ \langle x, N_K(x)\rangle^{n+1}}\right)}, \tag{22}$$

and $ent_2 : \partial K^* \to \mathbb{R}$ by

$$ent_2(y) = \frac{\kappa_K(x)^{-\frac{1}{n+1}}}{\log\left(\frac{R^{2n}|K|\kappa_K(x)}{r^{2n}|K^*|\langle x, N_K(x)\rangle^{n+1}}\right)}. \tag{23}$$

Corollary 3.4. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let r, R be as in (21) and ent_1 as in (22).*

Then

$$\begin{aligned}\lim_{t\to 0} \frac{|K_{ent_1}[t]| - |K|}{k_n\ t^{\frac{2}{n+1}}} &= \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} \log \frac{R^{2n}|K|\kappa_K(x)}{r^{2n}|K^*|\langle x, N_K(x)\rangle^{n+1}} dS_K(x) \\ &= n|K^*| \left[[D_{KL}(P_K\|Q_K) + 2n \log\left(\frac{R}{r}\right)\right]\end{aligned}$$

Corollary 3.5. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let r, R be as in (21) and ent_2 as in (23). Then*

$$\begin{aligned}\lim_{t\to 0}\frac{|K_{ent_2}[t]|-|K|}{k_n\ t^{\frac{2}{n+1}}} &= -\int_{\partial K}\langle x, N_K(x)\rangle \log\frac{r^{2n}|K^*|\langle x, N_K(x)\rangle^{n+1}}{R^{2n}|K|\kappa_K(x)}dS_K(x)\\ &= -n|K|\left[D_{KL}(Q_K||P_K)-2n\log\left(\frac{R}{r}\right)\right]\end{aligned}$$

Proof of Corollaries 3.4 and 3.5. As ∂K is in C^2_+, $0<r\le R<\infty$ and we have for all $x\in\partial K$ that

$$B^n_2(x-rN_K(x),r)\subset K\subset B^n_2(x-RN_K(x),R).$$

Suppose first that $r=R$. Then K is a Euclidean ball with radius r and the right hand sides of the identities in the corollary are equal to 0. Moreover, in this case, ent_1 and ent_2 are identically equal to ∞. Therefore, for all $t\ge 0$, $K_{ent_1}[t]=K$ and $K_{ent_2}[t]=K$ and hence for all $t\ge 0$, $|K_{ent_1}[t]|-|K|=0$ and $|K_{ent_2}[t]|-|K|=0$. Therefore, the corollary holds trivially in this case.

Suppose now that $r<R$. Then, as

$$1\le\frac{R^{2n}|K|\ \kappa_K(x)}{r^{2n}|K^*|\ \langle x, N_K(x)\rangle^{n+1}}\le\left(\frac{R}{r}\right)^{4n},$$

we get that the functions ent_1 and ent_2 satisfy the conditions of Theorem 3.1. The proof of the corollaries then follows immediately from Theorem 3.1.

In [30], the following new affine invariant Ω_K was introduced and its relation to the relative entropies was established.

Let K a convex body in $\mathbb{R}^n$ with centroid at the origin.

$$\Omega_K=\lim_{p\to\infty}\left(\frac{\Omega_p(K)}{n|K^*|}\right)^{n+p}.$$

Let p_K and q_K be the densities defined in (19). It was proved in [30] that for a convex body K in $\mathbb{R}^n$ that is C^2_+.

$$D_{KL}(P_K\|Q_K)=\log\left(\frac{|K|}{|K^*|}\Omega_K^{-\frac{1}{n}}\right)\tag{24}$$

and

$$D_{KL}(Q_K \| P_K) = \log\left(\frac{|K^*|}{|K|}\Omega_{K^*}^{-\frac{1}{n}}\right). \tag{25}$$

In [30], geometric interpretations in terms of L_p-centroid bodies were given in the case of symmetric convex bodies for the new affine invariants Ω_K. These interpretations are in the spirit of Corollary 3.2: As $p \to \infty$, the quantities Ω_K and the related relative entropies appear in appropriately chosen volume differences of K and its L_p-centroid bodies. However, in the context of the L_p-centroid bodies, a second order expansion was needed for the volume differences in order to make these terms appear. Now, it follows from Corollaries 3.4, 3.5 and 3.6 that no symmetry assumptions are needed and that already a first order expansion gives such geometric interpretations, if one uses the mean width bodies instead of the L_p-centroid body.

Corollary 3.6. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let the functions ent_1 and ent_2 be as in (22) and (23). Then*

$$\lim_{t\to 0} \frac{|K_{ent_1}[t]| - |K|}{k_n \ t^{\frac{2}{n+1}}} - 2n^2|K^*|\log\left(\frac{R}{r}\right) = n|K^*|\log\left(\frac{|K|}{|K^*|}\Omega_K^{-\frac{1}{n}}\right).$$

and

$$\lim_{t\to 0} \frac{|K_{ent_2}[t]| - |K|}{k_n \ t^{\frac{2}{n+1}}} - 2n^2|K|\log\left(\frac{R}{r}\right) = n|K|\log\left(\frac{|K|}{|K^*|}\Omega_{K^*}^{\frac{1}{n}}\right).$$

4 Proof of Theorem 3.1

To prove Theorem 3.1, we need the following lemmas. The first one, Lemma 4.1, is well known.

Lemma 4.1. *Let $\mathcal{E}_n(x_0, a)$ be an ellipsoid in $\mathbb{R}^n$ centered at x_0 and with axes parallel to the coordinate axes and of lengths $a_1, \dots, a_n$. Let $0 < \Delta < a_n$. Let*

$$C(\mathcal{E}_n, \Delta) = \mathcal{E}_n \cap H(x_0 + (a_n - \Delta)e_n, e_n)$$

be a cap of $\mathcal{E}_n(x_0, a)$ of height Δ. Then

$$\frac{2^{\frac{n+1}{2}}\left(1-\frac{\Delta}{2a_n}\right)^{\frac{n-1}{2}}|B_2^{n-1}|}{n+1}\prod_{i=1}^{n-1}\frac{a_i}{\sqrt{a_n}}\,\Delta^{\frac{n+1}{2}} \leq |C(\mathcal{E}_n,\Delta)|$$
$$\leq \frac{2^{\frac{n+1}{2}}|B_2^{n-1}|}{n+1}\prod_{i=1}^{n-1}\frac{a_i}{\sqrt{a_n}}\,\Delta^{\frac{n+1}{2}}$$

In the next few lemmas and throughout the remainder of the paper we will use the following notation.

Let K be a convex body in $\mathbb{R}^n$. Let $f : K^* \to \mathbb{R}$ be an integrable function and for $t \geq 0$, let $K_f[t]$ be a mean width body of K. For $x \in \partial K$, let

$$x_t = \{\gamma x : \gamma \geq 0\} \cap \partial K_f[t]. \tag{26}$$

Let $y(x) \in \partial K^*$ be such that $\langle y(x), x\rangle = 1$. Let m be the Lebesgue measure on $\mathbb{R}^n$ and let m_f be the measure (on K^*) defined by $m_f = \frac{2f}{\omega(S^{n-1})}\, m$, i.e. for all $A \subset K^*$

$$m_f(A) = \frac{2}{\omega(S^{n-1})}\int_A f(\xi)d\xi.$$

Lemma 4.2. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let $f : K^* \to \mathbb{R}$ be an integrable function such that $f(y) \geq c$ for all $y \in K^*$ and some constant $c > 0$. Let x_t be as in (26). Then the functions*

$$\frac{1}{t^{\frac{2}{n+1}}}\left(\frac{\|x_t\|}{\|x\|} - 1\right)$$

are uniformly (in t) bounded by an integrable function.

Proof. We can assume that $t \leq t_0$ where t_0 is given by Lemma 2.6. Then $K_f[t]$ is bounded and hence

$$K_f[t] \subset B_2^n(0, a) \tag{27}$$

for some $a > 0$. As $f \geq c$ on K^*, we get with (14)

$$\begin{aligned} t &\geq \frac{2}{\omega(S^{n-1})}\int_{K^*\cap H\left(\frac{x_t}{\|x_t\|^2},\frac{x}{\|x\|}\right)^-} f(\xi)d\xi \\ &\geq \frac{2c}{\omega(S^{n-1})}\left|K^*\cap H^-\left(\frac{x_t}{\|x_t\|^2},\frac{x}{\|x\|}\right)\right|. \end{aligned}$$

As K is in C^2_+, K^* is in C^2_+. Thus, by the Blaschke rolling theorem (see [32]), there exists $r_0 > 0$ such that for all $y \in \partial K^*$, $B^n_2(y - r_0 N_{K^*}(y), r_0) \subset K^*$. Let now $y(x) \in \partial K^*$ be such that $\langle x, y(x)\rangle = 1$. Then $N_{K^*}(y(x)) = \frac{x}{\|x\|}$ and thus

$$
\begin{aligned}
t &\geq \frac{2c}{\omega(S^{n-1})} \left| B^n_2\left(y(x) - r_0\frac{x}{\|x\|}, r_0\right) \cap H^-\left(\frac{x_t}{\|x_t\|^2}, \frac{x}{\|x\|}\right)\right| \\
&\geq \frac{2^{\frac{n+3}{2}}\ c\ r_0^{\frac{n-1}{2}}\ \left|B^{n-1}_2\right|}{(n+1)\ \omega(S^{n-1})} \left(\frac{1}{\|x\|} - \frac{1}{\|x_t\|}\right)^{\frac{n+1}{2}},
\end{aligned}
$$

where we have used that $\left|B^n_2\left(y(x) - r_0\frac{x}{\|x\|}, r_0\right) \cap H^-\left(\frac{x_t}{\|x_t\|^2}, \frac{x}{\|x\|}\right)\right|$ is the volume of a cap of height $\frac{1}{\|x\|} - \frac{1}{\|x_t\|} = \frac{\|x_t - x\|}{\|x_t\|\|x\|}$ of the ball $B^n_2\left(y(x) - r_0\frac{x}{\|x\|}, r_0\right)$ which we have estimated from below using Lemma 4.1. We assume also that t is so small that $\frac{1}{\|x\|} - \frac{1}{\|x_t\|} < r_0$.

As x and x_t are colinear, $\frac{\|x_t\|}{\|x\|} - 1 = \frac{\|x_t - x\|}{\|x\|}$ and hence

$$
\begin{aligned}
\frac{1}{t^{\frac{2}{n+1}}}\ \left(\frac{\|x_t\|}{\|x\|} - 1\right) &= \frac{1}{t^{\frac{2}{n+1}}}\ \frac{\|x_t - x\|}{\|x\|} \leq \left(\frac{(n+1)\ \omega(S^{n-1})}{c\ \left|B^{n-1}_2\right|}\right)^{\frac{2}{n+1}}\ \frac{r_0^{-\frac{n-1}{n+1}}}{2^{\frac{n+3}{n+1}}} \|x_t\| \\
&\leq \left(\frac{(n+1)\ \omega(S^{n-1})}{c\ \left|B^{n-1}_2\right|}\right)^{\frac{2}{n+1}}\ \frac{r_0^{-\frac{n-1}{n+1}}}{2^{\frac{n+3}{n+1}}}\ a. \qquad (28)
\end{aligned}
$$

In the last inequality we have used (27). The expression (28) is a constant and thus integrable.

Lemma 4.3. *Let K be a convex body in $\mathbb{R}^n$ that is in C^2_+. Let $f : K^* \to \mathbb{R}$ be a continuous, positive function. Then for all $x \in \partial K$ one has*

$$
\lim_{t\to 0} \frac{\langle x, N_K(x)\rangle}{n\ k_n\ t^{\frac{2}{n+1}}}\ \left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right] = \frac{\langle x, N_K(x)\rangle^2}{\kappa_K(x)^{\frac{1}{n+1}} f(y(x))^{\frac{2}{n+1}}},
$$

where $k_n = \frac{1}{2}\left(\frac{n(n+1)|B^n_2|}{2|B^{n-1}_2|}\right)^{\frac{2}{n+1}}$ and $y(x) \in \partial K^*$ is such that $\langle x, y(x)\rangle = 1$.

Proof. Let $x \in \partial K$. Let x_t be as in (26). As x and x_t are collinear and as $(1+s)^n \geq 1 + ns$ for $s \in [0,1)$, one has for small enough t,

$$
\frac{\langle x, N_K(x)\rangle}{n} \left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right] = \frac{\langle x, N_K(x)\rangle}{n} \left[\left(1 + \frac{\|x_t - x\|}{\|x\|}\right)^n - 1\right] \geq \Delta(x,t),
$$

where $\Delta(x,t) = \left\langle \frac{x}{\|x\|}, N_K(x) \right\rangle \|x_t - x\| = \langle x_t - x, N_K(x)\rangle$.
Similarly, as $(1+s)^n \leq 1 + ns + 2^n s^2$ for $s \in [0,1)$, one has for t small enough,

$$\frac{\langle x, N_K(x)\rangle}{n}\left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right] \leq \Delta(x,t)\left[1 + \frac{2^n}{n}\ \left(\frac{\|x_t - x\|}{\|x\|}\right)\right]. \tag{29}$$

Hence for $\varepsilon > 0$ there exists $t_\varepsilon \leq t_0$, t_0 from Lemma 2.6, such that for all $0 < t \leq t_\varepsilon$

$$1 \leq \frac{\langle x, N_K(x)\rangle \left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right]}{n\ \Delta(x,t)} \leq 1 + \varepsilon.$$

By Lemma 2.6 (iii), $m_f(K^* \setminus K^*_{x_t}) = t$ and thus

$$1 \leq \frac{\langle x, N_K(x)\rangle \left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right] \left(m_f(K^* \setminus K^*_{x_t})\right)^{\frac{2}{n+1}}}{n\ \Delta(x,t)\ t^{\frac{2}{n+1}}} \leq 1 + \varepsilon.$$

Let now $y = y(x) \in \partial K^*$ be such that $\langle x, y\rangle = 1$. Thus $y = \frac{N_K(x)}{\langle x, N_K(x)\rangle}$ and $N_{K^*}(y) = \frac{x}{\|x\|}$. As f is continuous on K^*, there exists $\delta > 0$ such that for all $z \in B_2^n(y,\delta)$,

$$f(y) - \varepsilon < f(z) < f(y) + \varepsilon.$$

We choose t so small that $K^* \setminus K^*_{x_t} \subset B_2^n(y,\delta)$. Then

$$\begin{aligned}
&\frac{2\,(f(y(x)) - \varepsilon)}{\omega(S^{n-1})}\left|K^* \setminus K^*_{x_t}\right| \leq \\
&\qquad m_f\left(K^* \setminus K^*_{x_t}\right) = \frac{2}{\omega(S^{n-1})}\int_{K^* \setminus K^*_{x_t}} f d\xi \\
&\qquad\qquad \leq \frac{2\,(f(y(x)) + \varepsilon)}{\omega(S^{n-1})}\left|K^* \setminus K^*_{x_t}\right|
\end{aligned}$$

and we get with (new) absolute constants c_1 and c_2 that

$$\begin{aligned}
1 - c_1\varepsilon &\leq \frac{\langle x, N_K(x)\rangle \left[\left(\frac{\|x_t\|}{\|x\|}\right)^n - 1\right] \left(\frac{2f(y(x))}{\omega(S^{n-1})}\left|K^* \setminus K^*_{x_t}\right|\right)^{\frac{2}{n+1}}}{n\ \Delta(x,t)\ t^{\frac{2}{n+1}}} \\
&\leq 1 + c_2\varepsilon.
\end{aligned} \tag{30}$$

As K and hence K^* is in C^2_+, $\kappa_{K^*}(y) > 0$. It is well known (see [35]) that then there exists an ellipsoid $\mathcal{E} = \mathcal{E}(y - a_n N_{K^*}(y), a)$ centered at $y -$

$a_n N_{K^*}(y)$ and with half axes of lengths $a_1, \dots, a_n$ which approximates ∂K^* in a neighborhood of y. For the computations that follow, we can assume without loss of generality that $N_{K^*}(y) = e_n$ and that the other axes of $\mathcal{E}$ coincide with $e_1, \dots, e_{n-1}$. Thus (see [35]), for $\varepsilon > 0$ given, there exists Δ_ε such that for all $\Delta \le \Delta_\varepsilon$

$$\begin{aligned} \mathcal{E}\big(y - (1-\varepsilon)a_n N_{K^*}(y), (1-\varepsilon)a\big) \ \cap H_\Delta^- \\ \subseteq K^* \cap H_\Delta^- \subseteq \\ \mathcal{E}\big(y - (1+\varepsilon)a_n N_{K^*}(y), (1+\varepsilon)a\big) \cap H_\Delta^-, \end{aligned} \tag{31}$$

where $H_\Delta = H(y - \Delta e_n, e_n)$. Also (see [35]),

$$\kappa_{K^*}(y) = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}. \tag{32}$$

As $x_t \to x$ as $t \to 0$, we can choose t so small that $K^* \setminus K^*_{x_t} = K^* \cap H^-\left(\frac{x_t}{\|x_t\|^2}, \frac{x}{\|x\|}\right)$ is contained in $H^-(y - \Delta e_n, e_n)$. Hence, by (31),

$$\begin{aligned} \left| \mathcal{E}\big(y - (1-\varepsilon)a_n N_{K^*}(y), (1-\varepsilon)a\big) \cap H^-\left(\frac{x_t}{\|x_t\|^2}, \frac{x}{\|x\|}\right) \right| \le \left| K^* \setminus K^*_{x_t} \right| \le \\ \left| \mathcal{E}\big(y - (1+\varepsilon)a_n N_{K^*}(y), (1+\varepsilon)a\big) \cap H^-\left(\frac{x_t}{\|x_t\|^2}, \frac{x}{\|x\|}\right) \right|. \end{aligned}$$

By Lemma 4.1, with (32), and as $\frac{1}{\|x\|} - \frac{1}{\|x_t\|} = \frac{\Delta(x,t)}{\|x_t\| \langle x, N_K(x) \rangle}$, we get with new absolute constants c_1 and c_2

$$\begin{aligned} (1 - c_1\varepsilon) \frac{2^{\frac{n+1}{2}} \left|B_2^{n-1}\right|}{(n+1)\left(\kappa_{K^*}(y)\right)^{\frac12}} \left(\frac{\Delta(x,t)}{\|x_t\| \langle x, N_K(x) \rangle} \right)^{\frac{n+1}{2}} \le \left| K^* \setminus K^*_{x_t} \right| \le \\ (1 + c_2\varepsilon) \frac{2^{\frac{n+1}{2}} \left|B_2^{n-1}\right|}{(n+1)\left(\kappa_{K^*}(y)\right)^{\frac12}} \left(\frac{1}{\|x\|} - \frac{1}{\|x_t\|} \right)^{\frac{n+1}{2}} \\ = (1 + c_2\varepsilon) \frac{2^{\frac{n+1}{2}} \left|B_2^{n-1}\right|}{(n+1)\left(\kappa_{K^*}(y)\right)^{\frac12}} \left(\frac{\Delta(x,t)}{\|x_t\| \langle x, N_K(x) \rangle} \right)^{\frac{n+1}{2}}. \end{aligned}$$

Hence, again with new absolute constants c_1 and c_2, (30) becomes

$$1 - c_1\varepsilon \le \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right] 2 \ \left(\frac{2 f(y) |B_2^{n-1}}{(n+1)\omega(S^{n-1})} \right)^{\frac{2}{n+1}}}{n \ t^{\frac{2}{n+1}} \left(\kappa_{K^*}(y)\right)^{\frac{1}{n+1}} \|x_t\| \langle x, N_K(x) \rangle} \le 1 + c_2\varepsilon.$$

Therefore, as $\|x_t\| \to \|x\|$ as $t \to 0$,

$$\lim_{t\to 0} \frac{\langle x, N_K(x)\rangle}{n\ t^{\frac{2}{n+1}}} \left[\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right] =$$

$$\frac{1}{2} \left(\frac{n(n+1)|B_2^n|}{2|B_2^{n-1}|} \right)^{\frac{2}{n+1}} \frac{\kappa_{K^*}(y)^{\frac{1}{n+1}} \|x\| \langle x, N_K(x)\rangle}{f(y)^{\frac{2}{n+1}}}.$$

Now we use that $\|x\| = \frac{1}{\langle y, N_{K^*}(y)\rangle}$ and that (see e.g. [42])

$$\frac{\kappa_{K^*}(y)^{\frac{1}{n+1}}}{\langle y, N_{K^*}(y)\rangle} = \frac{\langle x, N_K(x)\rangle}{\kappa_K(x)^{\frac{1}{n+1}}}.$$

We put $k_n = \frac{1}{2} \left(\frac{n(n+1)|B_2^n|}{2|B_2^{n-1}|} \right)^{\frac{2}{n+1}}$ and get that

$$\lim_{t\to 0} \frac{\langle x, N_K(x)\rangle}{n\ t^{\frac{2}{n+1}}} \left[\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right] = k_n \frac{\langle x, N_K(x)\rangle^2}{\kappa_K(x)^{\frac{1}{n+1}}\ f(y)^{\frac{2}{n+1}}}.$$

Proof of Theorem 3.1

It is well known (see e.g. [42]), that for a convex body K and a star shaped body L with $0 \in int(K)$ and $K \subset L$

$$|L| - |K| = \frac{1}{n} \int_{\partial K} \langle x, N_K(x)\rangle \left[\left(\frac{\|x'\|}{\|x\|} \right)^n - 1 \right] dS_K(x)$$

where $x \in \partial K$, $x' \in \partial L$ and $x = \partial K \cap [0, x']$.
Therefore,

$$|K_f[t]| - |K| = \frac{1}{n} \int_{\partial K} \langle x, N_K(x)\rangle \left(\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right) dS_K(x).$$

We now use Lemma 4.2 and Lebegue's theorem to interchange integration and limit and then Lemma 4.3 and get

$$\begin{aligned}
\lim_{t\to 0} \frac{|K_f[t]| - |K|}{t^{\frac{2}{n+1}}} &= \frac{1}{n} \lim_{t\to 0} \frac{1}{t^{\frac{2}{n+1}}} \int_{\partial K} \langle x, N_K(x)\rangle \left[\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right] dS_K(x) \\
&= \int_{\partial K} \lim_{t\to 0} \frac{\langle x, N_K(x)\rangle}{n\ t^{\frac{2}{n+1}}} \left[\left(\frac{\|x_t\|}{\|x\|} \right)^n - 1 \right] dS_K(x) \\
&= k_n \int_{\partial K} \frac{\langle x, N_K(x)\rangle^2}{\kappa_K(x)^{\frac{1}{n+1}}\ f(y)^{\frac{2}{n+1}}} dS_K(x).
\end{aligned}$$

This finishes the proof of Theorem 3.1.

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