# New $L_p$ Affine Isoperimetric Inequalities *

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**Abstract**

We prove new $L_p$ affine isoperimetric inequalities for all $p \in [-\infty, 1)$. We establish, for all $p \neq -n$, a duality formula which shows that $L_p$ affine surface area of a convex body $K$ equals $L_{\frac{n^2}{p}}$ affine surface area of the polar body $K^\circ$.

## 1 Introduction

An affine isoperimetric inequality relates two functionals associated with convex bodies (or more general sets) where the ratio of the functionals is invariant under non-degenerate linear transformations. These affine isoperimetric inequalities are more powerful than their better known Euclidean relatives.

This article deals with affine isoperimetric inequalities for the $L_p$ affine surface area. $L_p$ affine surface area was introduced by Lutwak in the ground breaking paper [26]. It is now at the core of the rapidly developing $L_p$ Brunn Minkowski theory. Contributions here include new interpretations of $L_p$ affine surface areas [32, 37, 38], the discovery of new ellipsoids [21, 28], the study of solutions of nontrivial ordinary and, respectively, partial differential equations (see e.g. Chen [9], Chou and Wang [10], Stancu [39, 40]), the study of the $L_p$ Christoffel-Minkowski problem by Hu, Ma and Shen [16], a new proof by Fleury, Guédon and Paouris [11] of a result by Klartag [18] on concentration of volume, and characterization theorems by Ludwig and Reitzner [23].

The case $p = 1$ is the classical affine surface area which goes back to Blaschke [6]. Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention too (e.g. [5, 20, 25, 31, 36]). It is fundamental in the theory of valuations (see e.g. [1, 2, 22, 17]), in approximation of convex bodies by polytopes [14, 38, 24] and it is the subject of the affine Plateau problem solved in $\mathbb{R}^3$ by Trudinger and Wang [41, 43].

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The classical affine isoperimetric inequality which gives an upper bound for the affine surface area in terms of volume proved to be the key ingredient in many problems (e.g. [12, 13, 27, 34]). In particular, it was used to show the uniqueness of self-similar solutions of the affine curvature flow and to study its asymptotic behavior by Andrews [3, 4], Sapiro and Tannenbaum [35].

$L_p$ affine isoperimetric inequalities were first established by Lutwak for $p > 1$ in [26]. There has been a growing body of work in this area since from which we quote only Lutwak, Yang and Zhang [29, 30] and Campi and Gronchi [8].

Here we derive new $L_p$ affine isoperimetric inequalities for all $p \in [-\infty, 1)$. We give new interpretations of $L_p$ affine surface areas. We establish, for all $p \neq -n$, a duality formula which shows that $L_p$ affine surface area of a convex body $K$ equals $L_{\frac{n^2}{p}}$ affine surface area of the polar body $K^\circ$. This formula was proved in [15] for $p > 0$.

From now on we will always assume that the centroid of a convex body $K$ in $\mathbb{R}^n$ is at the origin. We write $K \in C^2_+$ if $K$ has $C^2$ boundary with everywhere strictly positive Gaussian curvature. For real $p \neq -n$, we define the $L_p$ affine surface area $as_p(K)$ of $K$ as in [26] ($p > 1$) and [38] ($p < 1$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x) \tag{1.1}$$

and

$$as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n}} d\mu_K(x) \tag{1.2}$$

provided the above integrals exist. $N_K(x)$ is the outer unit normal vector at $x$ to $\partial K$, the boundary of $K$. $\kappa_K(x)$ is the Gaussian curvature at $x \in \partial K$ and $\mu_K$ denotes the usual surface area measure on $\partial K$. $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ which induces the Euclidian norm $\| \cdot \|$. In particular, for $p = 0$

$$as_0(K) = \int_{\partial K} \langle x, N_K(x)\rangle \, d\mu_K(x) = n|K|,$$

where $|K|$ stands for the $n$-dimensional volume of $K$. More generally, for a set $M$, $|M|$ denotes the Hausdorff content of its appropriate dimension. For $p = 1$

$$as_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} d\mu_K(x)$$

is the classical affine surface area which is independent of the position of $K$ in space.

If the boundary of $K$ is sufficiently smooth then (1.1) and (1.2) can be written as integrals over the boundary $\partial B^n_2 = S^{n-1}$ of the Euclidean unit ball $B^n_2$ in $\mathbb{R}^n$

$$as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u).$$

$\sigma$ is the usual surface area measure on $S^{n-1}$. $h_K(u)$ is the support function of direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at this point $x \in \partial K$ that has $u$ as outer normal. In particular, for $p = \pm\infty$,

$$as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n|K^\circ| \tag{1.3}$$

where $K^\circ = \{y \in \mathbb{R}^n, \langle x, y\rangle \leq 1, \forall x \in K\}$ is the polar body of $K$.

In Sections 2 and 3 we give new geometric interpretations of the $L_p$ affine surface areas and obtain as a consequence

**Corollary 3.1** *Let $K$ be a convex body in $C^2_+$ and let $p \neq -n$ be a real number. Then*

$$as_p(K) = as_{\frac{n^2}{p}}(K^\circ).$$

In Section 4 we prove the following new $L_p$ affine isoperimetric inequalities. For $p \geq 1$ they were proved by Lutwak [26].

**Theorem 4.2** *Let $K$ be a convex body with centroid at the origin.*
*(i) If $p \geq 0$, then*

$$\frac{as_p(K)}{as_p(B^n_2)} \leq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}},$$

*with equality if and only if $K$ is an ellipsoid. For $p = 0$, equality holds trivially for all $K$.*
*(ii) If $-n < p < 0$, then*

$$\frac{as_p(K)}{as_p(B^n_2)} \geq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}},$$

*with equality if and only if $K$ is an ellipsoid.*
*(iii) If $K$ is in addition in $C^2_+$ and if $p < -n$, then*

$$c^{\frac{np}{n+p}} \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B^n_2)}.$$

The constant $c$ in (iii) is the constant from the Inverse Santaló inequality due to Bourgain and Milman [7]. This constant has recently been improved by Kuperberg [19]. We give examples that the above isoperimetric inequalities cannot be improved.

In Theorem 4.1 we show a monotonicity behavior of the quotient $\left(\frac{as_r(K)}{n|K|}\right)^{\frac{n+r}{r}}$, namely

$$\left(\frac{as_r(K)}{n|K|}\right) \leq \left(\frac{as_t(K)}{n|K|}\right)^{\frac{r(n+t)}{t(n+r)}}.$$

and as a consequence obtain

**Corollary 4.1** *Let $K$ be convex body in $\mathbb{R}^n$ with centroid at the origin.*
*(i) For all $p \geq 0$*

$$as_p(K)\ as_p(K^\circ) \leq n^2|K|\ |K^\circ|.$$

*(ii) For $-n < p < 0$,*

$$as_p(K)as_p(K^\circ) \geq n^2|K|\ |K^\circ|.$$

*If $K$ is in addition in $C^2_+$, inequality (ii) holds for all $p < -n$.*

## 2 $L_{-\frac{n}{n+2}}$ affine surface area of the polar body

It was proved in [32] that for a convex body $K \in C^2_+$

$$\begin{aligned}\lim_{\delta\to 0} c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} &= \int_{S^{n-1}} \frac{d\sigma(u)}{f_K(u)^{\frac{1}{n+1}}\ h_K(u)^{n+1}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{n+2}{n+1}}}{\langle x, N_K(x)\rangle^{n+1}} d\mu_K(x) \\ &= as_{-n(n+2)}(K), \end{aligned} \tag{2.4}$$

where $c_n = 2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$ and $K_\delta$ is the convex floating body [36]: The intersection of all halfspaces $H^+$ whose defining hyperplanes $H$ cut off a set of volume $\delta$ from $K$.

Assumptions on the boundary of $K$ are needed in order that (2.4) holds.
To see that, consider $B^n_\infty = \{x \in \mathbb{R}^n : \max_{1\leq i\leq n}|x_i| \leq 1\}$. As $\kappa_{B^n_\infty}(x) = 0$ a.e. on $\partial B^n_\infty$,

$$\int_{\partial B^n_\infty} \frac{\kappa_{B^n_\infty}(x)^{\frac{n+2}{n+1}}}{\langle x, N_{B^n_\infty}(x)\rangle^{n+1}} d\mu_{B^n_\infty}(x) = 0.$$

However

$$\lim_{\delta\to 0} c_n \frac{|((B^n_\infty)_\delta)^\circ| - |(B^n_\infty)^\circ|}{\delta^{\frac{2}{n+1}}} = \infty. \tag{2.5}$$

Indeed, writing $K$ for $B^n_\infty$, we will construct a 0-symmetric convex body $K_1$ such that $K_\delta \subseteq K_1 \subseteq K$. Then $K^\circ \subseteq K_1^\circ \subseteq K_\delta^\circ$. Therefore, to show (2.5), it is enough to show that

$$\lim_{\delta\to 0} c_n \frac{|K_1^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \infty.$$

Let $R^+ = \{(x_j)_{j=1}^n : x_j \geq 0, 1 \leq j \leq n\}$. It is enough to consider $K^+ = R^+ \cap K$ and to construct $(K_1)^+ = K_1 \cap R^+$.
We define $(K_1)^+$ to be the intersection of $R^+$ with the half-spaces $H_i^+$, $1 \leq i \leq n+1$, where $H_i = \{(x_j)_{j=1}^n : x_i = 1\}$, $1 \leq i \leq n$, and $H_{n+1} = \left\{(x_j)_{j=1}^n : \sum_{j=1}^n x_j = n - (n!\delta)^{\frac{1}{n}}\right\}$, $\delta > 0$ sufficiently small. Notice that the hyperplane $H_{n+1}$ (orthogonal to the vector $(1, \ldots, 1)$ ) cuts off a set of volume exactly $\delta$ from $K$ and therefore $K_\delta \subset K_1$.

Moreover, $K_1^\circ$ can be written as a convex hull:

$$K_1^\circ = \text{co}\bigg( \{\pm e_i, 1 \leq i \leq n\} \cup \left\{ \frac{1}{s} \ (\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_j = \pm 1, 1 \leq j \leq n \right\} \bigg),$$

where $s = n - (n!\delta)^{\frac{1}{n}}$. Hence

$$|K_1^\circ| = \frac{2^n}{n!} \cdot \frac{n}{n - (n!\delta)^{\frac{1}{n}}}$$

and therefore

$$\lim_{\delta \to 0} \frac{|K_1^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \frac{2^n}{n!} \ \lim_{\delta \to 0} \delta^{\frac{-2}{n+1}} \ \frac{(n!\delta)^{\frac{1}{n}}}{(n - (n!\delta)^{\frac{1}{n}})} = \infty.$$

Now we show

**Theorem 2.1** *Let $K$ be a convex body in $C^2_+$ such that $0 \in int(K)$. Then*

$$\lim_{\delta \to 0} \ c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = as_{-\frac{n}{n+2}}(K^\circ).$$

As a corollary of (2.4) and Theorem 2.1 we get that for a convex body $K \in C^2_+$

$$as_{-n(n+2)}(K) = as_{-\frac{n}{n+2}}(K^\circ). \tag{2.6}$$

This is a special case for $p = -n(n+2)$ of the formula $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$ proved in [15] for $p > 0$. We will show in the next section that this formula holds for all $p < 0, p \neq -n$ for convex bodies with sufficiently smooth boundary. For $p = 0$ (and $K \in C^2_+$) the formula holds trivially as $as_0(K) = n|K|$ and $as_\infty(K^\circ) = n|K|$ (see [38]).

For the proof of Theorem 2.1 we need the following lemmas.

**Lemma 2.1** *Let $K \in C^2_+$. Then for any $x \in \partial K^\circ$, we have*

$$\lim_{\delta \to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{n \ \delta^{\frac{2}{n+1}}} \left[ \left( \frac{\|x_\delta\|}{\|x\|} \right)^n - 1 \right] = \frac{\langle x, N_{K^\circ}(x)\rangle^2}{c_n \ (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}}$$

*where $x_\delta \in \partial (K_\delta)^\circ$ is in the ray passing through $0$ and $x$.*

**Proof**
Since $K$, and hence also $K_\delta$, are in $C^2_+$ one has that $K^\circ$ and $(K_\delta)^\circ$ are in $C^2_+$. Therefore, for $x \in \partial K^\circ$ there exists a unique $y \in \partial K$, such that, $\langle x, y\rangle = 1$, namely $y = \dfrac{N_{K^o}(x)}{\langle N_{K^o}(x), x\rangle}$. $y$ has outer normal vector $N_K(y) = \dfrac{x}{\|x\|}$ and $\frac{1}{\|x\|} = \langle y, N_K(y)\rangle$.

Similarly, for $x_\delta \in \partial(K_\delta)^\circ$ there exists a unique $y_\delta$ in $\partial K_\delta$ such that $\langle x_\delta, y_\delta\rangle = 1$, namely $y_\delta = \dfrac{N_{(K_\delta)^o}(x_\delta)}{\langle N_{(K_\delta)^o}(x_\delta), x_\delta\rangle}$, $y_\delta$ has outer normal vector $N_{K_\delta}(y_\delta) = \dfrac{x_\delta}{\|x_\delta\|} = \dfrac{x}{\|x\|}$ and $\frac{1}{\|x_\delta\|} = \langle y_\delta, N_{K_\delta}(y_\delta)\rangle$.

Let $y' = [0, y] \cap \partial K_\delta$ ($[z_1, z_2]$ denotes the line segment from $z_1$ to $z_2$) and let $y'_\delta \in \partial K$ be such that $y_\delta = [0, y'_\delta] \cap K_\delta$.

We have

$$\begin{aligned} \frac{1}{\|x\|} &= \langle y, N_K(y)\rangle \geq \langle y'_\delta, N_K(y)\rangle = \langle y'_\delta, \frac{x}{\|x\|}\rangle, \\ \frac{1}{\|x_\delta\|} &= \langle y_\delta, N_{K_\delta}(y_\delta)\rangle \geq \langle y', N_{K_\delta}(y_\delta)\rangle = \langle y', \frac{x}{\|x\|}\rangle. \end{aligned}$$

Hence

$$\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] = \left[\left(\frac{\langle y, N_K(y)\rangle}{\langle y_\delta, N_{K_\delta}(y_\delta)\rangle}\right)^n - 1\right] \geq \left[\left(\frac{\langle y'_\delta, \frac{x}{\|x\|}\rangle}{\langle y_\delta, \frac{x}{\|x\|}\rangle}\right)^n - 1\right] = \left[\left(\frac{\|y'_\delta\|}{\|y_\delta\|}\right)^n - 1\right],$$

$$\begin{aligned} \left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] = \left[\left(\frac{\langle y, N_K(y)\rangle}{\langle y_\delta, N_{K_\delta}(y_\delta)\rangle}\right)^n - 1\right] &\leq \left[\left(\frac{\langle y, \frac{x}{\|x\|}\rangle}{\langle y', \frac{x}{\|x\|}\rangle}\right)^n - 1\right] \\ &= \left[\left(\frac{\|y\|}{\|y'\|}\right)^n - 1\right] \end{aligned} \tag{2.7}$$

and therefore

$$\frac{\langle x, N_{K^\circ}(x)\rangle}{n}\left[\left(\frac{\|y'_\delta\|}{\|y_\delta\|}\right)^n - 1\right] \leq \frac{\langle x, N_{K^\circ}(x)\rangle}{n}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \leq \frac{\langle x, N_{K^\circ}(x)\rangle}{n}\left[\left(\frac{\|y\|}{\|y'\|}\right)^n - 1\right].$$

We first consider the lower bound.

$$\lim_{\delta\to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{n\, \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \geq \lim_{\delta\to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{\langle y'_\delta, N_{K_\delta}(y'_\delta)\rangle}\frac{\langle y'_\delta, N_{K_\delta}(y'_\delta)\rangle}{n\, \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|y'_\delta\|}{\|y_\delta\|}\right)^n - 1\right].$$

As $\delta \to 0$, $y'_\delta \to y$. As $K$ is in $C^2_+$, $N_{K_\delta}(y'_\delta) \to N_K(y)$ as $\delta \to 0$.

Therefore $\lim_{\delta\to 0}\langle y'_\delta, N_{K_\delta}(y'_\delta)\rangle = \langle y, N_K(y)\rangle$. By Lemma 7 and Lemma 10 of [36],

$$\lim_{\delta\to 0} \frac{\langle y'_\delta, N_{K_\delta}(y'_\delta)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|y'_\delta\|}{\|y_\delta\|}\right)^n - 1\right] = \frac{(\kappa_K(y))^{\frac{1}{n+1}}}{c_n}.$$

Hence

$$\lim_{\delta\to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \geq \frac{\langle x, N_{K^\circ}(x)\rangle}{\langle y, N_K(y)\rangle}\frac{(\kappa_K(y))^{\frac{1}{n+1}}}{c_n} = \frac{\langle x, N_{K^\circ}(x)\rangle^2}{c_n\ (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}}.$$

The last equation follows from the fact that if $K \in C^2_+$, then, for any $y \in \partial K$, there is a unique point $x \in \partial K^\circ$ such that $\langle x, y\rangle = 1$ and [15]

$$\langle y, N_K(y)\rangle\langle x, N_{K^\circ}(x)\rangle = (\kappa_K(y)\kappa_{K^\circ}(x))^{\frac{1}{n+1}}\ . \tag{2.8}$$

Similarly, one gets for the upper bound

$$\lim_{\delta\to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \leq \frac{\langle x, N_{K^\circ}(x)\rangle^2}{c_n\ (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}},$$

hence altogether

$$\lim_{\delta\to 0} \frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] = \frac{\langle x, N_{K^\circ}(x)\rangle^2}{c_n\ (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}}.$$

**Lemma 2.2** *Let $K \in C^2_+$. Then we have*

$$\frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \leq c(K,n),$$

*where $c(K,n)$ is a constant (depending on $K$ and $n$ only) and $x$ and $x_\delta$ are as in Lemma 2.1.*

**Proof** By (2.7)

$$\begin{aligned}\frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] &\leq \frac{\langle x, N_{K^\circ}(x)\rangle}{\langle y, N_K(y)\rangle}\frac{\langle y, N_K(y)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[\left(\frac{\|y\|}{\|y'\|}\right)^n - 1\right] \\ &\leq \frac{\langle x, N_{K^\circ}(x)\rangle}{\langle y, N_K(y)\rangle}\left(\frac{\|y\|}{\|y'\|}\right)^n\frac{\langle y, N_K(y)\rangle}{n\ \delta^{\frac{2}{n+1}}}\left[1 - \left(\frac{\|y'\|}{\|y\|}\right)^n\right].\end{aligned}$$

Since $K_\delta$ is increasing to $K$ as $\delta \to 0$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$, $0 \in \text{int}(K_\delta)$. Therefore there exits $\alpha > 0$ such that $B_2^n(0,\alpha) \subset K_\delta \subset K \subset B_2^n(0,\frac{1}{\alpha})$ for all $\delta < \delta_0$. $B_2^n(0,r)$ is the $n$-dimensional Euclidean ball centered at 0 with radius $r$.

Hence for $\delta < \delta_0$

$$\frac{\langle x, N_{K^\circ}(x)\rangle}{n\ \delta^{\frac{2}{n+1}}} \left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] \quad \leq \quad \alpha^{-2(n+1)}\ \frac{\langle y, N_K(y)\rangle}{n\ \delta^{\frac{2}{n+1}}} \left[1 - \left(\frac{\|y'\|}{\|y\|}\right)^n\right] \leq C' r(y)^{-\frac{n-1}{n+1}}$$

due to Lemma 6 in [36]. Here $r(y)$ is the radius of the biggest Euclidean ball contained in $K$ and touching $\partial K$ at $y$.

Since $K$ is $C^2_+$, by the Blaschke rolling theorem (see [34]) there is $r_0 > 0$ such that $r_0 \leq \min_{y\in\partial K} r(y)$. We put $c(K,n) = C' r_0^{-\frac{n-1}{n+1}}$.

**Proof of Theorem 2.1.**

$$\frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} = \frac{1}{n\ \delta^{\frac{2}{n+1}}} \int_{\partial K^o} \langle x, N_{K^\circ}(x)\rangle \left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] d\mu_{K^\circ}(x).$$

Combining Lemma 2.1, Lemma 2.2 and Lebesgue's convergence theorem, gives Theorem 2.1:

$$\begin{aligned}
\lim_{\delta\to 0} \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{\frac{2}{n+1}}} &= \lim_{\delta\to 0} \frac{1}{n\ \delta^{\frac{2}{n+1}}} \int_{\partial K^o} \langle x, N_{K^\circ}(x)\rangle \left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] d\mu_{K^\circ}(x) \\
&= \int_{\partial K^o} \lim_{\delta\to 0} \frac{1}{n\ \delta^{\frac{2}{n+1}}} \langle x, N_{K^\circ}(x)\rangle \left[\left(\frac{\|x_\delta\|}{\|x\|}\right)^n - 1\right] d\mu_{K^\circ}(x) \\
&= \int_{\partial K^o} \frac{\langle x, N_{K^\circ}(x)\rangle^2}{c_n\ (\kappa_{K^\circ}(x))^{\frac{1}{n+1}}}\, d\mu_{K^\circ}(x) \\
&= \frac{1}{c_n}\ as_{\frac{-n}{n+2}}(K^\circ).
\end{aligned}$$

**Remark**

The proof of Theorem 2.1 provides a uniform method to evaluate

$$\lim_{t\to 0} \frac{|(K_t)^\circ| - |K^\circ|}{t^{\frac{2}{n+1}}}$$

where $K_t$ is a family convex bodies constructed from the convex body $K$ such that $K_t \subset K$ or- similarly- such that $K \subset K_t$. In particular, we can apply this method to prove the analog statements as in (2.4) and Theorem 2.1 if we take as $K_t$ the illumination body of $K$ [42], or the Santaló body of $K$ [31], or the convolution body of $K$ [33] - and there are many more.

## 3 $L_p$ affine surface areas

We now prove that for all $p \neq -n$ and all $K \in C^2_+$, $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$. To do so, we use the surface body of a convex body which was introduced in [37, 38]. We also give a new geometric interpretation of $L_p$ affine surface area for all $p \neq -n$.

**Definition 3.1** *Let $s \geq 0$ and $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function. The surface body $K_{f,s}$ is the intersection of all the closed half-spaces $H^+$ whose defining hyperplanes $H$ cut off a set of $f\mu_K$-measure less than or equal to $s$ from $\partial K$. More precisely,*

$$K_{f,s} = \bigcap_{\int_{\partial K \cap H^-} f d\mu_K \leq s} H^+.$$

**Theorem 3.1** *Let $K$ be a convex body in $C^2_+$ and such that $0$ is the center of gravity of $K$. Let $f : \partial K \to \mathbb{R}$ be an integrable function such that $f(x) > c$ for all $x \in \partial K$ and some constant $c > 0$. Let $\beta_n = 2\left(|B_2^{n-1}|\right)^{\frac{2}{n-1}}$. Then*

$$\lim_{s\to 0} \beta_n \frac{|(K_{f,s})^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} = \int_{S^{n-1}} \frac{d\sigma(u)}{h_K(u)^{n+1} f_K(u)^{\frac{1}{n-1}} \left(f(N_K^{-1}(u))\right)^{\frac{2}{n-1}}}$$

*where $N_K : \partial K \to S^{n-1}$, $x \to N_K(x) = u$ is the Gauss map.*

**Proof**
Let $u \in S^{n-1}$. Let $x \in \partial K$ be such that $N_K(x) = u$ and let $x_s \in \partial K_{f,s}$ be such that $N_{K_{f,s}}(x_s) = u$. Let $H_\Delta = H(x - \Delta u, u)$ be the hyperplane through $x - \Delta u$ with outer normal vector $u$. Since $K$ has everywhere strictly positive Gaussian curvature, by Lemma 21 in [38] almost everywhere on $\partial K$,

$$\lim_{\Delta\to 0} \frac{1}{|\partial K \cap H_\Delta^-|} \int_{\partial K \cap H_\Delta^-} |f(x) - f(y)| \, d\mu_K(y) = 0.$$

This implies that

$$\lim_{\Delta\to 0} \frac{1}{|\partial K \cap H_\Delta^-|} \int_{\partial K \cap H_\Delta^-} f(y) \, d\mu_{\partial K}(y) = f(x). \tag{3.9}$$

Let $b_s = h_K(u) - h_{K_{f,s}}(u)$. As $H(x - b_s u, u) = H(x_s, u)$ (the hyperplane through $x_s$ with outer normal $u$) and as $b_s \to 0$ as $s \to 0$, (3.9) implies

$$\lim_{s\to 0} \frac{1}{|\partial K \cap H^-(x_s, u)|} \int_{\partial K \cap H^-(x_s,u)} f(y) \, d\mu_K(y) = f(x). \tag{3.10}$$

Hence there exists $s_1$ small enough, such that for all $s < s_1$,

$$s \le \int_{\partial K \cap H^-(x_s,u)} f(y)\, d\mu_K(y) \le (1+\varepsilon) f(x) |\partial K \cap H^-(x_s,u)|. \tag{3.11}$$

As $\partial K$ has everywhere strictly positive Gaussian curvature, the indicatrix of Dupin exists everywhere on $\partial K$ and is an ellipsoid. It then follows from (3.11) with Lemmas 1.2, 1.3 and 1.4 in [37] that there exists $0 < s_2 < s_1$ such that for all $0 < s < s_2$

$$s \le (1+\varepsilon)\ f(x)\ |B_2^{n-1}|\ \sqrt{f_K(u)}\ \ (2b_s)^{\frac{n-1}{2}}\ ,$$

or, equivalently

$$\frac{b_s}{s^{\frac{2}{n-1}}} \ge \frac{1 - c_1\ \varepsilon}{\beta_n\ f(N_K^{-1}(u))^{\frac{2}{n-1}}\ f_K(u)^{\frac{1}{n-1}}}, \tag{3.12}$$

where $c_1$ is an absolute constant.

Let now $x_s' \in [0,x] \cap \partial K_{f,s}$. Then $\langle x_s', u\rangle \le h_{K_{f,s}}(u)$. Therefore $b_s = h_K(u) - h_{K_{f,s}}(u) \le \langle x - x_s', u\rangle$.

Hence for $s$ sufficiently small

$$\begin{aligned} \frac{b_s}{s^{\frac{2}{n-1}}} \le \frac{\langle x - x_s', u\rangle}{s^{\frac{2}{n-1}}} &\le \frac{\langle x, u\rangle}{s^{\frac{2}{n-1}}} \left(1 - \frac{\|x_s'\|}{\|x\|}\right) \le \frac{\langle x, u\rangle}{s^{\frac{2}{n-1}}}\ \frac{\|x_s' - x\|}{\|x\|} \\ &\le (1+\varepsilon)\ \frac{\langle x, N_K(x)\rangle}{n\ s^{\frac{2}{n-1}}} \left[1 - \left(\frac{\|x_s'\|}{\|x\|}\right)^n\right]. \end{aligned} \tag{3.13}$$

The last inequality follows as $1 - \left(\frac{\|x_s'\|}{\|x\|}\right)^n \ge (1-\varepsilon)\ \frac{n\ \|x_s' - x\|}{\|x\|}$ for sufficiently small $s$. By Lemma 23 in [38]

$$\lim_{s\to 0} \frac{1}{ns^{\frac{2}{n-1}}} \langle x, N_K(x)\rangle \left[1 - \left(\frac{\|x_s'\|}{\|x\|}\right)^n\right] = \frac{1}{\beta_n f(N_K^{-1}(u))^{\frac{2}{n-1}} f_K(u)^{\frac{1}{n-1}}}. \tag{3.14}$$

Thus we get from (3.12), (3.13) and (3.14) that

$$\lim_{s\to 0} \frac{b_s}{s^{\frac{2}{n-1}}} = \frac{1}{\beta_n f(N_K^{-1}(u))^{\frac{2}{n-1}} f_K(u)^{\frac{1}{n-1}}}. \tag{3.15}$$

As $(1-t)^{-n} \ge 1 + nt$ for $0 \le t < 1$ and by (3.15),

$$\begin{aligned} \lim_{s\to 0} \frac{\beta_n}{ns^{\frac{2}{n-1}}} \left([h_{K_{f,s}}(u)]^{-n} - [h_K(u)]^{-n}\right) &= \lim_{s\to 0} \frac{\beta_n}{ns^{\frac{2}{n-1}}} [h_K(u)]^{-n} \left[\left(1 + \frac{b_s}{h_K(u)}\right)^{-n} - 1\right] \\ &\ge \lim_{s\to 0} \frac{\beta_n}{[h_K(u)]^{n+1}} \frac{b_s}{s^{\frac{2}{n-1}}} \\ &= \frac{1}{[h_K(u)]^{n+1}\ f(N_K^{-1}(u))^{\frac{2}{n-1}}\ f_K(u)^{\frac{1}{n-1}}}. \end{aligned} \tag{3.16}$$

As $h_{K_{f,s}}(u) \geq \langle x'_s, u\rangle$,

$$\frac{h_{K_{f,s}}(u)}{h_K(u)} \geq \frac{\langle x'_s, u\rangle}{\langle x, u\rangle} = \frac{\|x'_s\|}{\|x\|}. \tag{3.17}$$

Since $K \in C^2_+$, $h_{K_{f,s}}(u) \to h_K(u)$ as $s \to 0$. Therefore,

$$\lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} \left([h_{K_{f,s}}(u)]^{-n} - [h_K(u)]^{-n}\right) = \lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} [h_{K_{f,s}}(u)]^{-n} \left[1 - \left(\frac{h_{K_{f,s}}(u)}{h_K(u)}\right)^n\right]$$

$$\leq \lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} [h_{K_{f,s}}(u)]^{-n} \left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right] = \lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} [h_{K_{f,s}}(u)]^{-n} \frac{\langle x, u\rangle}{\langle x, u\rangle} \left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right]$$

$$= \lim_{s\to 0} \frac{1}{h_K(u)\ [h_{K_{f,s}}(u)]^n}\ \lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} \langle x, N_K(x)\rangle \left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right]$$

$$= \frac{1}{[h_K(u)]^{n+1} f(N_K^{-1}(u))^{\frac{2}{n-1}} f_K(u)^{\frac{1}{n-1}}} \tag{3.18}$$

where the last equality follows from (3.14).

Altogether, (3.16) and (3.18) give

$$\lim_{s\to 0} \frac{\beta_n}{n s^{\frac{2}{n-1}}} \left([h_{K_{f,s}}(u)]^{-n} - [h_K(u)]^{-n}\right) = \frac{1}{[h_K(u)]^{n+1}\ f(N_K^{-1}(u))^{\frac{2}{n-1}}\ f_K(u)^{\frac{1}{n-1}}}.$$

Therefore

$$\begin{aligned}
\lim_{s\to 0} \beta_n \frac{|(K_{f,s})^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} &= \lim_{s\to 0} \frac{\beta_n}{n\ s^{\frac{2}{n-1}}} \int_{S^{n-1}} \left[\left(\frac{1}{h_{K_{f,s}}(u)}\right)^n - \left(\frac{1}{h_K(u)}\right)^n\right] d\sigma(u) \\
&= \int_{S^{n-1}} \lim_{s\to 0} \frac{\beta_n}{n\ s^{\frac{2}{n-1}}} \left[\left(\frac{1}{h_{K_{f,s}}(u)}\right)^n - \left(\frac{1}{h_K(u)}\right)^n\right] d\sigma(u) \\
&= \int_{S^{n-1}} \frac{d\sigma(u)}{h_K(u)^{n+1} f_K(u)^{\frac{1}{n-1}} \left(f(N_K^{-1}(u))\right)^{\frac{2}{n-1}}},
\end{aligned}$$

provided we can interchange integration and limit.

We show this next. To do so, we show that for all $u \in S^{n-1}$ and all sufficiently small $s > 0$,

$$\frac{1}{n\ s^{\frac{2}{n-1}}} \left[\left(\frac{1}{h_{K_{f,s}}(u)}\right)^n - \left(\frac{1}{h_K(u)}\right)^n\right] \leq g(u)$$

with $\int_{S^{n-1}} g(u)\ d\sigma(u) < \infty$. As $0 \in int(K)$, the interior of $K$, there exists $\alpha > 0$ such that for all $s$ sufficiently small $B_2^n(0,\alpha) \subset K_{f,s} \subset K \subset B_2^n(0,\frac{1}{\alpha})$. Therefore, $\alpha \le h_{K_{f,s}}(u) \le h_K(u) \le \frac{1}{\alpha}$ and $\alpha \le \frac{1}{h_K(u)} \le \frac{1}{h_{K_{f,s}}(u)} \le \frac{1}{\alpha}$.

With (3.17), we thus get for all $s > 0$,

$$\begin{aligned}
&\frac{1}{n\ s^{\frac{2}{n-1}}}\ \left(\left(h_{K_{f,s}}(u)\right)^{-n} - \left(h_K(u)\right)^{-n}\right)\\
&= \frac{1}{n\ s^{\frac{2}{n-1}}}\left(h_{K_{f,s}}(u)\right)^{-n}\left(1 - \frac{\left(h_{K_{f,s}}(u)\right)^n}{\left(h_K(u)\right)^n}\right)\\
&\le \frac{\alpha^{-n}}{n\ s^{\frac{2}{n-1}}}\left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right]\\
&\le \alpha^{-(n+1)}\ \ \frac{\langle x, u\rangle}{n\ s^{\frac{2}{n-1}}}\left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right].
\end{aligned}$$

By Lemma 17 in [38] there exists $s_3$ such that for all $s \le s_3$

$$\frac{\langle x, u\rangle}{s^{\frac{2}{n-1}}}\left[1 - \left(\frac{\|x'_s\|}{\|x\|}\right)^n\right] \le \frac{C}{(M_f(x))^{\frac{2}{n-1}} r(x)},$$

where $C$ is an absolute constant and, as in the proof of Lemma 2.2, $r(x)$ is the biggest Euclidean ball contained in $K$ that touches $\partial K$ at $x$. Thus, as $\partial K$ is $C^2_+$, by Blaschke's rolling theorem (see [34]) there is $r_0$ such that $r(x) \ge r_0$.

$$M_f(x) = \inf_{0<s} \frac{\int_{\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s))} f d\mu_K}{|\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s)|}$$

is the *minimal function*. It was introduced in [38]. By the assumption on $f$, $M_f(x) \ge c$. Thus altogether

$$\frac{1}{n\ s^{\frac{2}{n-1}}}\ \left(\left(h_{K_{f,s}}(u)\right)^{-n} - \left(h_K(u)\right)^{-n}\right) \le \frac{\alpha^{-(n+1)}\ C}{n\ c^{\frac{2}{n-1}}\ r_0} = g(u),$$

which, as a constant, is integrable.

**Theorem 3.2** *Let $K$ be a convex body in $C^2_+$ and such that $0$ is the center of gravity of $K$. Let $f: \partial K \to \mathbb{R}$ be an integrable function such that $f(y) > c$ for all $y \in \partial K$ and some constant $c > 0$. Let $\beta_n = 2\left(|B_2^{n-1}|\right)^{\frac{2}{n-1}}$. Then*

$$\lim_{s\to 0} \beta_n \frac{|(K_{f,s})^\circ - |K^\circ|}{s^{\frac{2}{n-1}}} = \int_{\partial K^\circ} \left(\frac{\langle x, N_{K^\circ}(x)\rangle}{\langle y(x), N_K(y(x))\rangle}\right)\left(\frac{\kappa_K(y(x))^{\frac{1}{n-1}}}{f(y(x))^{\frac{2}{n-1}}}\right)\ d\mu_{K^\circ}(x)$$

*Here $y(x) \in \partial K$ is such that $\langle y(x), x\rangle = 1$.*

**Proof**

We follow the pattern of the proof of Theorem 3.1 integrating now over $\partial K^\circ$ instead of $S^{n-1}$.

As a corollary we get the following geometric interpretation of $L_p$ affine surface area.

**Corollary 3.1** *Let $K \in C^2_+$ be a convex body. For $p \in \mathbb{R}$, $p \neq -n$, let $f_p : \partial K \to \mathbb{R}$ be defined by $f_p(y) = \kappa_K(y)^{\frac{n^2+p}{2(n+p)}} \langle y, N_K(y)\rangle^{\frac{-(n-1)(n^2+2n+p)}{2(n+p)}}$. Then*

*(i)*

$$\lim_{s\to 0} \beta_n \frac{|(K_{f_p,s})^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} = as_{\frac{n^2}{p}}(K^\circ).$$

*(ii)*

$$\lim_{s\to 0} \beta_n \frac{|(K_{f_p,s})^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} = as_p(K).$$

*(iii)*

$$as_p(K) = as_{\frac{n^2}{p}}(K^\circ).$$

**Proof**

Notice first that $f_p(y)$ verifies the conditions of Theorems 3.1 and 3.2.

(i) For $x \in \partial K^\circ$, let now $y(x)$ be the (unique) element in $\partial K$ such that $\langle x, y(x)\rangle = 1$. Then, by Theorem 3.2, with $f(y(x)) = f_p(y(x))$, and with (2.8)

$$\begin{aligned}
\lim_{s\to 0} \beta_n \frac{|\left(K_{f_p,s}\right)^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} &= \int_{\partial K^\circ} \langle x, N_{K^\circ}(x)\rangle \frac{\langle y(x), N_K(y(x))\rangle^{\frac{n(n+1)}{n+p}}}{\kappa_K(y(x))^{\frac{n}{n+p}}}\, d\mu_{K^\circ}(x) \\
&= \int_{\partial K^\circ} \frac{\kappa_{K^\circ}(x)^{\frac{n}{n+p}}}{\langle x, N_{K^\circ}(x)\rangle^{\frac{n^2-p}{n+p}}}\, d\mu_{K^\circ}(x) = as_{\frac{n^2}{p}}(K^\circ).
\end{aligned}$$

(ii) For $u \in S^{n-1}$, let now $y \in \partial K$ be such that $N_K(y) = u$. Then $f_p(N_K^{-1}(u)) = f_K(u)^{-\frac{n^2+p}{2(n+p)}} h_K(u)^{\frac{-(n-1)(n^2+2n+p)}{2(n+p)}}$. By Theorem 3.1 with $f(N_K^{-1}(u)) = f_p(N_K^{-1}(u))$

$$\lim_{s\to 0} \beta_n \frac{|\left(K_{f_p,s}\right)^\circ| - |K^\circ|}{s^{\frac{2}{n-1}}} = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}}\, d\sigma(u) = as_p(K).$$

(iii) follows from (i) and (ii).

# 4 Inequalities

**Theorem 4.1** *Let $s \neq -n, r \neq -n, t \neq -n$ be real numbers. Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at the origin and such that $\mu_K\{x \in \partial K : \kappa_K(x) = 0\} = 0$.*

*(i) If $\frac{(n+r)(t-s)}{(n+t)(r-s)} > 1$, then*

$$as_r(K) \leq \big(as_t(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \big(as_s(K)\big)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}.$$

*(ii) If $\frac{(n+r)t}{(n+t)r} > 1$, then*

$$\left(\frac{as_r(K)}{n|K|}\right) \leq \left(\frac{as_t(K)}{n|K|}\right)^{\frac{r(n+t)}{t(n+r)}}.$$

**Proof**

(i) By Hölder's inequality -which enforces the condition $\frac{(n+r)(s-t)}{(n+t)(s-r)} > 1$

$$\begin{aligned} as_r(K) &= \int_{\partial K} \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x)\rangle^{\frac{n(r-1)}{n+r}}} \, d\mu_K(x) \\ &= \int_{\partial K} \left(\frac{\kappa_K(x)^{\frac{t}{n+t}}}{\langle x, N_K(x)\rangle^{\frac{n(t-1)}{n+t}}}\right)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \left(\frac{\kappa_K(x)^{\frac{s}{n+s}}}{\langle x, N_K(x)\rangle^{\frac{n(s-1)}{n+s}}}\right)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}} d\mu_K(x) \\ &\leq \big(as_t(K)\big)^{\frac{(r-s)(n+t)}{(t-s)(n+r)}} \big(as_s(K)\big)^{\frac{(t-r)(n+s)}{(t-s)(n+r)}}. \end{aligned}$$

(ii) Similarly, again using Hölder's inequality -which now enforces the condition $\frac{(n+r)t}{(n+t)r} > 1$,

$$\begin{aligned} as_r(K) &= \int_{\partial K} \frac{\kappa_K(x)^{\frac{r}{n+r}}}{\langle x, N_K(x)\rangle^{\frac{n(r-1)}{n+r}}} \, d\mu_K(x) = \int_{\partial K} \left(\frac{\kappa_K(x)^{\frac{t}{n+t}}}{\langle x, N_K(x)\rangle^{\frac{n(t-1)}{n+t}}}\right)^{\frac{r(n+t)}{t(n+r)}} \frac{d\mu_K(x)}{\langle x, N_K(x)\rangle^{\frac{(r-t)n}{(n+r)t}}} \\ &\leq \big(as_t(K)\big)^{\frac{r(n+t)}{t(n+r)}} \big(n\,|K|\big)^{\frac{(t-r)n}{(n+r)t}}. \end{aligned}$$

Condition $\frac{(n+r)(t-s)}{(n+t)(r-s)} > 1$ implies 8 cases: $-n < s < r < t$, $s < -n < t < r$, $r < t < -n < s$, $t < r < s < -n$, $s < r < t < -n$, $r < s < -n < t$, $t < -n < s < r$ and $-n < t < r < s$.

Note also that (ii) describes a monotonicity condition for $\left(\frac{as_r(K)}{n|K|}\right)^{\frac{n+r}{r}}$: if $0 < r < t$, or $r < t < -n$, or $-n < r < t < 0$ then

$$\left(\frac{as_r(K)}{n|K|}\right)^{\frac{n+r}{r}} \le \left(\frac{as_t(K)}{n|K|}\right)^{\frac{n+t}{t}}.$$

We now analyze various subcases of Theorem 4.1 (i) and (ii). For $r = 0$, if $\frac{n(s-t)}{s(n+t)} > 1$

$$n|K| \le \big(as_t(K)\big)^{\frac{s(n+t)}{n(s-t)}} \big(as_s(K)\big)^{\frac{t(n+s)}{n(t-s)}}.$$

For $s = 0$, if $\frac{t(n+r)}{r(n+t)} > 1$,

$$as_r(K) \le (n|K|)^{\frac{n(t-r)}{t(n+r)}} \big(as_t(K)\big)^{\frac{r(n+t)}{t(n+r)}}. \tag{4.19}$$

For $s \to \infty$, if $\frac{n+r}{n+t} > 1$,

$$as_r(K) \le \big(as_\infty(K)\big)^{\frac{r-t}{n+r}} \big(as_t(K)\big)^{\frac{n+t}{n+r}}. \tag{4.20}$$

For $r \to \infty$, if $\frac{t-s}{n+t} > 1$ and if $K$ is in $C^2_+$,

$$as_\infty(K) = n|K^\circ| \le \big(as_t(K)\big)^{\frac{n+t}{t-s}} \big(as_s(K)\big)^{\frac{n+s}{s-t}}. \tag{4.21}$$

As for all convex bodies $K$, $as_\infty(K) \le n|K^\circ|$ (see [38]), it follows from (4.19) that, for all convex body $K$ with centroid at origin,

$$as_r(K) \le (n|K|)^{\frac{n}{n+r}} \ (n|K^\circ|)^{\frac{r}{n+r}}, \quad r > 0 \tag{4.22}$$

and from (4.20),

$$n|K|\big(n|K^\circ|\big)^{\frac{t}{n}} \le \big(as_t(K)\big)^{\frac{n+t}{n}}, \quad -n < t < 0. \tag{4.23}$$

Similarly, (4.21) implies that, if in addition $K$ is in $C^2_+$,

$$n|K^\circ|\big(n|K|\big)^{\frac{n}{t}} \le \big(as_t(K)\big)^{\frac{n+t}{t}}, \quad t < -n \tag{4.24}$$

(4.22) can also be obtained from Proposition 4.6 of [26] and Theorem 3.2 of [15].

Inequalities (4.22), (4.23) and (4.24) yield the following Corollary which was proved by Lutwak [26] in the case $p \ge 1$.

**Corollary 4.1** *Let $K$ be convex body in $\mathbb{R}^n$ with centroid at the origin.*

*(i) For all $p \geq 0$*

$$as_p(K)\ as_p(K^\circ) \leq n^2|K|\ |K^\circ|.$$

*(ii) For $-n < p < 0$,*

$$as_p(K)as_p(K^\circ) \geq n^2|K|\ |K^\circ|.$$

*If $K$ is in addition in $C^2_+$, inequality (ii) holds for all $p < -n$.*

Thus, using Santaló inequality in (i), for $p \geq 0$, $as_p(K)as_p(K^\circ) \leq as_p(B^n_2)^2$, and inverse Santaló inequality in (ii), for $-n < p < 0$, $as_p(K)as_p(K^\circ) \geq c^n as_p(B^n_2)^2$. $c$ is the constant in the inverse Santaló inequality [7, 19].

**Proof**

(i) follows immediately form (4.22). (ii) follows from (4.23) if $-n < p < 0$ and from (4.24) if $p < -n$.

Lutwak [26] proved for $p \geq 1$

$$\frac{as_p(K)}{as_p(B^n_2)} \leq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}}$$

with equality if and only if $K$ is an ellipsoid. We now generalize these $L_p$-affine isoperimetric inequalities to $p < 1$.

**Theorem 4.2** *Let $K$ be a convex body with centroid at the origin.*

*(i) If $p \geq 0$, then*

$$\frac{as_p(K)}{as_p(B^n_2)} \leq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}},$$

*with equality if and only if $K$ is an ellipsoid. For $p = 0$, equality holds trivially for all $K$.*

*(ii) If $-n < p < 0$, then*

$$\frac{as_p(K)}{as_p(B^n_2)} \geq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}},$$

*with equality if and only if $K$ is an ellipsoid.*

*(iii) If $K$ is in addition in $C^2_+$ and if $p < -n$, then*

$$\frac{as_p(K)}{as_p(B^n_2)} \geq c^{\frac{np}{n+p}} \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}}.$$

*where $c$ is the constant in the inverse Santaló inequality [7, 19].*

We cannot expect to get a strictly positive lower bound in Theorem 4.2 (i), even if $K$ is in $C^2_+$: Consider, in $\mathbb{R}^2$, the convex body $K(R,\varepsilon)$ obtained as the intersection of four Euclidean balls with radius $R$ centered at $(\pm(R-1),0)$, $(0,\pm(R-1))$, $R$ arbitrarily large. We then "round" the corners by putting there arcs of Euclidean balls of radius $\varepsilon$, $\varepsilon$ arbitrarily small. To obtain a body in $C^2_+$, we "bridge" between the $R$-arcs and $\varepsilon$-arcs by $C^2_+$-arcs on a set of arbitrarily small measure. Then $as_p(K(R,\varepsilon)) \leq \frac{16}{R^{\frac{p}{2+p}}} + 4\pi\ \varepsilon^{\frac{2}{2+p}}$. A similar construction can be done in higher dimensions.

This example also shows that, likewise, we cannot expect finite upper bounds in Theorem 4.2 (ii) and (iii). If $-2 < p < 0$, then $as_p(K(R,\varepsilon)) \geq 2^{\frac{3(p+1)}{2+p}} R^{\frac{-p}{2+p}}$. If $p < -2$, then $-2 < \frac{4}{p} < 0$ and thus

$$as_p(K(R,\varepsilon)^\circ) = as_{\frac{4}{p}}(K(R,\varepsilon)) \geq R^{\frac{-2}{p+2}}\ 2^{\frac{12+3p}{4+2p}}.$$

Note also that in part (iii) we cannot remove the constant $c^{\frac{np}{n+p}}$. Indeed, if $p \to -\infty$, the inequality becomes $c^n |B^n_2|^2 \leq |K||K^\circ|$.

**Proof of Theorem 4.2**

(i) The case $p = 0$ is trivial. We prove the case $p > 0$. Combining inequality (4.22), the Blaschke Santaló inequality, and $as_q(B^n_2) = n|B^n_2|^{\frac{n}{n+q}}|B^n_2|^{\frac{q}{n+q}}$, one obtains

$$\frac{as_p(K)}{as_p(B^n_2)} \leq \left(\frac{|K^\circ|}{|B^n_2|}\right)^{\frac{p}{n+p}} \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n}{n+p}} \leq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}}.$$

This proves the inequality. The equality case follows from the equality case for the Blaschke Santaló inequality.

(ii) Combining inequality (4.23) and $\big(as_p(B^n_2)\big)^{\frac{n+p}{n}} = n|B^n_2|\big(n|B^n_2|\big)^{\frac{p}{n}}$, one gets, for $-n < p < 0$,

$$\left(\frac{as_p(K)}{as_p(B^n_2)}\right)^{\frac{n+p}{n}} \geq \left(\frac{|K|}{|B^n_2|}\right) \left(\frac{|K^\circ|}{|B^n_2|}\right)^{\frac{p}{n}} \geq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n}}$$

where the last inequality follows from the Blaschke Santaló inequality. As $\frac{p}{n} < 0$,

$$(|K|\ |K^\circ|)^{\frac{p}{n}} \geq (|B^n_2|\ |B^n_2|)^{\frac{p}{n}}.$$

As $n + p > 0$,

$$\frac{as_p(K)}{as_p(B^n_2)} \geq \left(\frac{|K|}{|B^n_2|}\right)^{\frac{n-p}{n+p}}.$$

The equality case follows from the equality case for the Blaschke Santaló inequality.

(iii) Similarly, combining (4.24), $n|B_2^n| = (as_p(B_2^n))^{\frac{n+p}{p-1}} (as(B_2^n))^{\frac{n+1}{1-p}}$, and the Inverse Santaló inequality, we get, for $p < -n$,

$$\left(\frac{as_p(K)}{as_p(B_2^n)}\right)^{\frac{n+p}{p}} \geq \left(\frac{|K^\circ|}{|B_2^n|}\right) \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n}{p}} \geq c^n \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{p}}.$$

As $\frac{n+p}{p} > 0$,

$$\frac{as_p(K)}{as_p(B_2^n)} \geq c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

The $L_{-n}$ *affine surface area* was defined in [32] for convex bodies $K$ in $C_+^2$ and with centroid at the origin by

$$as_{-n}(K) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}.$$

More generally, one could define the $L_{-n}$ affine surface area for any convex body $K$ with centroid at the origin by $as_{-n}(K) = \sup_{x \in \partial K} \frac{\langle x, N_K(x)\rangle^{\frac{n+1}{2}}}{\kappa_K(x)^{\frac{1}{2}}}$. But as in most cases then $as_{-n}(K) = \infty$, it suffices to consider $K$ in $C_+^2$.

A statement similar to Theorem 4.1 holds.

**Proposition 4.1** *Let $K$ be a convex body in $C_+^2$ with centroid at the origin. Let $p \neq -n$ and $s \neq -n$ be real numbers.*

*(i) If $\frac{n(s-p)}{(n+p)(n+s)} \geq 0$, then*

$$as_p(K) \leq \big(as_{-n}(K)\big)^{\frac{2n(s-p)}{(n+p)(n+s)}} \; as_s(K).$$

*(ii) If $\frac{n(s-p)}{(n+p)(n+s)} \leq 0$, then*

$$as_p(K) \geq \big(as_{-n}(K)\big)^{\frac{2n(s-p)}{(n+p)(n+s)}} \; as_s(K).$$

*(iii) The $L_{-n}$ affine isoperimetric inequality holds*

$$\frac{as_{-n}(K)}{as_{-n}(B_2^n)} \geq \frac{|K|}{|B_2^n|}.$$

**Proof** (i) and (ii)

$$\begin{aligned} as_p(K) &= \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} \, d\mu_K(x) \\ &= \int_{\partial K} \left(\frac{\kappa_K(x)^{\frac{s}{n+s}}}{\langle x, N_K(x)\rangle^{\frac{n(s-1)}{n+s}}}\right) \left(\frac{\langle x, N_K(x)\rangle^{\frac{n+1}{2}}}{\kappa_K(x)^{\frac{1}{2}}}\right)^{\frac{2n(s-p)}{(n+p)(n+s)}} d\mu_K(x) \end{aligned}$$

which is

$$\leq \left(as_{-n}(K)\right)^{\frac{2n(s-p)}{(n+p)(n+s)}} \; as_s(K), \quad \text{if } \tfrac{n(s-p)}{(n+p)(n+s)} \geq 0,$$

and

$$\geq \left(as_{-n}(K)\right)^{\frac{2n(s-p)}{(n+p)(n+s)}} \; as_s(K), \quad \text{if } \tfrac{n(s-p)}{(n+p)(n+s)} \leq 0.$$

(iii) Note that $\frac{n(s-p)}{(n+p)(n+s)} > 0$ implies that $s > p > -n$ or $p < s < -n$ or $s < -n < p$. If $p = 0$ and $s \to \infty$, then

$$as_{-n}(K) \geq \sqrt{\frac{|K|}{|K^\circ|}}. \tag{4.25}$$

This gives the $L_{-n}$ affine isoperimetric inequality

$$\frac{as_{-n}(K)}{as_{-n}(B_2^n)} = as_{-n}(K) \geq \sqrt{\frac{|K|}{|K^\circ|}} \geq \sqrt{\frac{|K|^2}{|K|\;|K^\circ|}} \geq \sqrt{\frac{|K|^2}{|B_2^n|^2}} = \frac{|K|}{|B_2^n|}.$$

Analogous to corollary 4.1, an immediate consequence of (4.25) is the following corollary. It can also be proved directly using (2.8).

**Corollary 4.2** *Let $K$ be a convex body in $C^2_+$ with centroid at the origin. Then*

$$as_{-n}(K^\circ)as_{-n}(K) \geq as_{-n}(B_2^n)^2.$$

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