

New affine measures of symmetry for convex bodies ^{*}

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Abstract

Grünbaum introduced measures of symmetry for convex bodies that measure how far a given convex body is from a centrally symmetric one. Here, we introduce new measures of symmetry that measure how far a given convex body is from one with “enough symmetries”.

To define these new measures of symmetry, we use affine covariant points. We give examples of convex bodies whose affine covariant points are “far apart”. In particular, we give an example of a convex body whose centroid and Santaló point are “far apart”.

1 Introduction.

The last quarter of a century has witnessed a virtual revolution in the study of convex bodies. While the study of the Euclidean aspects of these bodies dominated most of the 20th century, a number of highly influential works (see, e.g., [6], [8], [12], [15] - [17], [20] - [34], [38] - [43], [46]) redirected much of the research in this field to the study of the affine geometry of these bodies. In fact, some questions that had been considered Euclidean in nature turned out to be affine problems. For example, the famous Busemann-Petty Problem (finally laid to rest in [4, 7, 35, 44, 45]) was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [26].

In his seminal paper [14], Grünbaum initiated the systematic study of measures of symmetry of convex bodies. More formally, Grünbaum introduced measures of symmetry on the set of convex bodies (convex compact sets with non-empty interior), that take values between 0 and 1 and are 1 if and only if K is centrally symmetric. Thus, these measures tell how far a given convex body is from a centrally symmetric one.

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In this work we will propose a radically new approach: towards *affine measures* of symmetry based on affine invariants — or to be more precise affine covariants — of convex bodies. In Grünbaum’s view, a measure of symmetry might identify a highly symmetric object such as the regular simplex as “most asymmetric”. In our view, a measure of symmetry should identify convex bodies that have “sufficiently many” symmetries: the group of symmetries of that convex body has exactly one common fixed point. Since they may play a significant role in major open problems in convex geometry, convex bodies that lack sufficiently many symmetries have to be investigated. This has not been done before.

We call an affine map A a symmetry of a convex body K if $A(K) = K$ and say that a convex body has enough symmetries (compare with [9]) if there is only one point of the convex body that is left invariant under all these symmetries. Clearly, centrally symmetric bodies have enough symmetries, but also simplices. On simplices, Grünbaum’s measures of symmetry are small in general, whereas our measures are 1. Thus, the measures of symmetry that we introduce measure how far a convex body is from one with enough symmetries.

To define our measures of symmetry, we use affine covariant points. For a convex body K in \mathbb{R}^n , we call a point $a(K)$ *affine covariant* if for every nonsingular affine map T of \mathbb{R}^n we have

$$a(T(K)) = T(a(K)). \quad (1)$$

Examples of affine covariant points are the centroid, the Santaló point and the centers of the John ellipsoid and of the Löwner ellipsoid (see the definitions below).

Of particular importance is the Santaló point, which is the unique point $s(K) \in \text{int}(K)$, the interior of a convex body K , such that

$$\text{vol}_n(K^{s(K)}) = \min_{z \in \text{int}(K)} \text{vol}_n(K - z)^\circ = \min_{z \in \text{int}(K)} \text{vol}_n(K^z).$$

Here,

$$K^z = \{y \in \mathbb{R}^n; \langle y, x - z \rangle \leq 1 \text{ for every } x \in K\}$$

is the polar of K with respect to z . In the symmetric case, $s(K)$ is just the center of symmetry. The importance of the Santaló point is due to the Blaschke-Santaló inequality which provides an upper bound for (the affine invariant quantity) $\text{vol}_n(K^{s(K)})\text{vol}_n(K)$. To determine the exact lower bound for $\text{vol}_n(K^{s(K)})\text{vol}_n(K)$ is a major open problem. Mahler conjectured that in the symmetric case the lower bound is attained at the l_∞^n -unit ball and in the general case at the regular simplex. We refer to [5, 37] for an overview.

Thus, it is important to be able to decide how badly asymmetric a general convex set can be.

Let \mathcal{K}_n be the set of all convex bodies in \mathbb{R}^n . We consider here only affine covariant points π on \mathcal{K}_n such that for all $K \in \mathcal{K}_n$, $\pi(K) \in K$. For two affine covariant points π_1 and π_2 on \mathcal{K}_n , we define their affine invariant distance by

$$\begin{aligned} d(\pi_1(K), \pi_2(K)) &= 0, \text{ if } \pi_1(K) = \pi_2(K) \text{ and} \\ d(\pi_1(K), \pi_2(K)) &= \frac{\|\pi_1(K) - \pi_2(K)\|_2}{\text{vol}_1(\ell \cap K)}, \text{ if } \pi_1(K) \neq \pi_2(K), \end{aligned} \quad (2)$$

where ℓ is the line through $\pi_1(K)$ and $\pi_2(K)$ and $\|\cdot\|_2$ is the Euclidean norm on \mathbb{R}^n .

We then define as a measure of symmetry the map

$$K \rightarrow \phi_{\pi_1, \pi_2}(K) = 1 - d(\pi_1(K), \pi_2(K)) \quad (3)$$

on the set of convex bodies. Note that ϕ_{π_1, π_2} is continuous if π_1 and π_2 are continuous, takes values between 0 and 1 and is invariant under affine bijective maps. However, since all affine covariant points coincide when K is a simplex, the centrally symmetric bodies are not the only ones for which $\phi_{\pi_1, \pi_2}(K) = 1$. Consequently, there are bodies which, in this sense, are much more asymmetric than the simplex.

A natural problem is to determine

$$\max_{K \in \mathcal{K}_n} d(\pi_1(K), \pi_2(K)). \quad (4)$$

What are the bodies where this maximum is reached? What is the order of magnitude of this maximum in terms of n ? It appears that not much is known about this problem when π_1 and π_2 vary among all classical affine invariant points. We address these questions in this paper and give examples that show that affine covariant points can be far apart. Our main result shows that in the distance (4) the centroid g and the Santaló point s are far apart from one another. Namely we show

Theorem 1 *There is an absolute constant $0 \leq c < 1$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a convex body C_n in \mathbb{R}^{n+1} that satisfies*

$$\phi_{g,s}(C_n) \leq c.$$

In fact, the proof of Theorem 1 shows that

$$1 - c \leq d(g(C_n), s(C_n)) = \frac{\|g(C_n) - s(C_n)\|_2}{\text{vol}_1(\ell \cap C_n)} = \frac{\|g(C_n) - s(C_n)\|_2}{w_{C_n}(u)}. \quad (5)$$

Here, ℓ is the line through $g = g(C_n)$ and $s = s(C_n)$, $h_K(u) = \max_{x \in K} \langle u, x \rangle$ is the support function of K and $w_{C_n}(u) = h_{C_n}(u) + h_{C_n}(-u)$ is the width of C_n in direction of the unit vector u of ℓ .

The important point of the theorem is that the measure of symmetry of all C_n , $n \in \mathbb{N}$, is smaller than a constant that is strictly smaller than 1. We do not know whether there is a sequence of convex bodies K_n in \mathbb{R}^n , $n \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} \phi_{g,s}(K_n) = 0,$$

(we know that this is not the case for the bodies C_n of Theorem 1) or, more generally, whether there are two affine covariant points π_1 and π_2 such that

$$\lim_{n \rightarrow \infty} \phi_{\pi_1, \pi_2}(K_n) = 0.$$

The paper is organized as follows. Section 2 introduces more necessary notations. In Section 3, we give the construction of the convex bodies that satisfy Theorem 1 and state the main tool, Proposition 3, needed for its proof. Section 4 is devoted to the proof of Proposition 3. For this purpose, we collect and establish necessary lemmas, in particular various probabilistic lemmas involving volume estimates which are of interest in their own right. Section 5 gives another example of affine covariant points that are far apart.

2 Further Notation.

Throughout the paper we use the following notations.

For $a \in \mathbb{R}^n$ and $r > 0$, $B_2^n(a, r)$ is the Euclidean ball in \mathbb{R}^n centered at a with radius r . We write $B_2^n = B_2^n(0, 1)$ for the Euclidean unit ball and S^{n-1} for its boundary. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n . For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ if $1 \leq p < \infty$ and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ if $p = \infty$. $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ is the unit ball of the space $l_p^n = (\mathbb{R}^n, \|\cdot\|_p)$.

If A and B are convex subsets of \mathbb{R}^n , then

$$\text{co}[A, B] = \{\lambda a + (1 - \lambda)b : a \in A, b \in B, 0 \leq \lambda \leq 1\}$$

denotes the convex hull of A and B .

A convex body K in \mathbb{R}^n is called centrally symmetric if $K = 2x - K$ for some $x \in \mathbb{R}^n$. We denote the volume of K in \mathbb{R}^n by $\text{vol}_n(K)$ (if we want to emphasize the dimension) or by $|K|$. We write ∂K for the boundary of K .

Let K be a convex body in \mathbb{R}^n with 0 in its interior and $x \in \mathbb{R}^n$. Then $\|x\|_K = \inf\{\lambda : x \in \lambda K\}$ is the Minkowski functional of K .

For $\xi \in S^{n-1}$, let $\xi^\perp = \{x \in \mathbb{R}^n; \langle x, \xi \rangle = 0\}$. Let $s \in \mathbb{R}$. The $(n - 1)$ -dimensional section of K orthogonal to ξ through $s\xi$ is

$$K(s, \xi) = \{x \in K | \langle \xi, x \rangle = s\} = K \cap (s\xi + \xi^\perp).$$

We just write $K(s)$ if it is clear which direction ξ is meant.

The *centroid* $g(K)$ of a convex body K in \mathbb{R}^n is the point

$$g(K) = \frac{1}{\text{vol}_n(K)} \int_K x dx.$$

As remarked above, the *Santaló point* $s(K)$ of a convex body K is the unique point $x \in \text{int}(K)$ at which $\text{vol}_n(K^x)$ attains its minimum.

It is easy to see that both $g(K)$ and $s(K)$ are affine invariant. As $s(K)$ is defined implicitly, it is more difficult to locate the Santaló point than the centroid.

For convex bodies K and L in \mathcal{K}_n and natural numbers k , $0 \leq k \leq n$, the coefficients

$$V_{n-k,k}(K, L) = V(\underbrace{K, \dots, K}_{n-k}, \underbrace{L, \dots, L}_k) \quad (6)$$

in the expansion $|\lambda K + \mu L| = \sum_{k=0}^n \binom{n}{k} \lambda^{n-k} \mu^k V_{n-k,k}(K, L)$, $\lambda, \mu \geq 0$, are the *mixed volumes of K and L* (see [37]).

Finally, for quantities a and b we write $a \sim b$ if there are absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$.

3 The main result.

Our main theorem gives an example of a convex body whose centroid and Santaló point are far apart.

Theorem 1 *There is an absolute constant $0 \leq c < 1$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a convex body C_n in \mathbb{R}^{n+1} that satisfies*

$$\phi_{g,s}(C_n) \leq c.$$

The proof actually provides a sequence $C_n \in \mathcal{K}_{n+1}$ such that

$$\begin{aligned} \frac{1}{e} \frac{\sqrt{e\pi} - 2}{\sqrt{e\pi} + \frac{2}{e-1}} &\leq \liminf_{n \rightarrow \infty} d(g(C_n), s(C_n)) \\ &\leq \limsup_{n \rightarrow \infty} d(g(C_n), s(C_n)) \leq \frac{1}{e} \frac{\sqrt{e} - 1}{\sqrt{e} + \frac{1}{e-1}}. \end{aligned}$$

The left and right hand sides in these inequalities are of the order 0.083 and 0.107 respectively, and for the measure of symmetry we have asymptotically

$$0.893 \leq \phi_{g,s}(C) \leq 0.917.$$

We will frequently use the next lemma which is well known (see [37]).

Lemma 2 *For any convex body K , an interior point x of K is the Santaló point of K if and only if 0 is the centroid of $(K - x)^\circ$.*

This lemma can be rephrased as follows:

Let K be a convex body. Then 0 is the Santaló point of $K^{g(K)}$.

Indeed, $K^{g(K)} = (K - g(K))^\circ$ and $(K - g(K))^{\circ\circ} = K - g(K)$. Since 0 is the centroid of $(K - g(K))^{\circ\circ}$, it follows by Lemma 2 that 0 is the Santaló point of $(K - g(K))^\circ = K^{g(K)}$.

Remark. We do not know whether there exists a constant c , $0 < c < 1$, such that $d(s(K), g(K)) \leq c$ holds for all convex bodies K in \mathbb{R}^n . The best upper bound that we know is

$$d(s(K), g(K)) \leq 1 - \frac{2}{n+1},$$

or, in terms of the measure of symmetry,

$$\phi_{g,s}(K) \geq \frac{2}{n+1}.$$

To see this, let $g = g(K)$ be the centroid of K and $s = s(K)$ its Santaló point. It is well known (see [37], p. 308) that

$$K - g \subset n(g - K). \tag{7}$$

By Lemma 2, 0 is the centroid of K^s so with (7), $K^s \subset -nK^s$. Polarity with respect to 0 gives

$$K - s \subset n(s - K). \quad (8)$$

Let u be the unit vector parallel to $s - g$. Then

$$\max_{x \in K-g} \langle u, x \rangle = \max_{x \in K} \langle u, x \rangle - \langle u, g \rangle$$

and

$$\max_{x \in n(g-K)} \langle u, x \rangle = n \langle u, g \rangle + n \max_{x \in K} \langle u, -x \rangle = n \langle u, g \rangle - n \min_{x \in K} \langle u, x \rangle.$$

We use (7) to compare the two expressions above and get

$$\max_{x \in K} \langle u, x \rangle + n \min_{x \in K} \langle u, x \rangle \leq (n+1) \langle u, g \rangle. \quad (9)$$

On the other hand

$$\min_{x \in K-s} \langle u, x \rangle = \min_{x \in K} \langle u, x \rangle - \langle u, s \rangle$$

and

$$\min_{x \in n(s-K)} \langle u, x \rangle = n \langle u, s \rangle - n \max_{x \in K} \langle u, x \rangle.$$

Now we use (8) to compare the last two expressions,

$$\min_{x \in K} \langle u, x \rangle + n \max_{x \in K} \langle u, x \rangle \geq (n+1) \langle u, s \rangle. \quad (10)$$

Inequalities (9) and (10) give

$$\langle u, s - g \rangle \leq \frac{n-1}{n+1} \left(\max_{x \in K} \langle u, x \rangle - \min_{x \in K} \langle u, x \rangle \right)$$

or, as $u = \frac{s-g}{\|s-g\|_2}$,

$$\|s - g\|_2 \leq \frac{n-1}{n+1} \left(\max_{x \in K} \langle u, x \rangle - \min_{x \in K} \langle u, x \rangle \right).$$

□

Now we introduce the convex bodies which will serve as candidates for Theorem 1. Namely, for convex bodies K and L in \mathbb{R}^n that contain the origin in their interior and real numbers $a > 0$ and $b > 0$, we construct a convex body M_n in \mathbb{R}^{n+1}

$$M_n = \text{co}[(K, -a), (L, b)] = \{t(x, -a) + (1-t)(y, b) | x \in K, y \in L, 0 \leq t \leq 1\}. \quad (11)$$

The bodies used in Theorem 1 will be the polar bodies of M_n . The polar M_n° of M_n with respect to 0 can be described as follows: for $-\frac{1}{a} \leq s \leq \frac{1}{b}$, the sections of M_n° orthogonal to e_{n+1} through se_{n+1} are

$$M_n^\circ(s) = M_n^\circ(s, e_{n+1}) = (1+sa)K^\circ \cap (1-sb)L^\circ. \quad (12)$$

We show this:

$$\begin{aligned} M_n^\circ &= \{(z, s) \in \mathbb{R}^n \times \mathbb{R} \mid \forall x \in K, \langle z, x \rangle - sa \leq 1 \text{ and } \forall y \in L, \langle z, y \rangle + sb \leq 1\} \\ &= \left\{ (z, s) \mid z \in (1 + sa)K^\circ \cap (1 - sb)L^\circ \mid -\frac{1}{a} \leq s \leq \frac{1}{b} \right\} \end{aligned}$$

The body M_n in \mathbb{R}^{n+1} is the convex hull of two of its n -dimensional faces, K and L . In the following proposition we choose specific bodies for those faces. We choose them in such a way that their volume product differs greatly. This will have as effect that the centroid and the Santaló point of $M_n^{g(M_n)}$ are “far apart”. One face will be chosen to be an Euclidean ball and the other a cube, both centred on the x_{n+1} -axis and normalized so that their volume is 1.

Proposition 3 *Let $a = 1$ and $b = \frac{1}{e-1}$. Let $s_0 = -\frac{\sqrt{e}-1}{\sqrt{e}+\frac{1}{e-1}} = -0.290815\dots$ and $s_1 = \frac{2-\sqrt{e\pi}}{\sqrt{e\pi}+\frac{2}{e-1}} = -0.225705\dots$*

Let $K = \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}}$ and $L = \frac{1}{2} B_\infty^n$ and let M_n be the convex body in \mathbb{R}^{n+1} defined in (11).

Then

(i) $\lim_{n \rightarrow \infty} g(M_n) = 0.$

(ii) *The $(n+1)$ th coordinate $g(M_n^\circ)(n+1)$ of the centroid $g(M_n^\circ)$ of M_n° satisfies*

$$s_0 \leq \liminf_{n \rightarrow \infty} g(M_n^\circ)(n+1) \leq \limsup_{n \rightarrow \infty} g(M_n^\circ)(n+1) \leq s_1.$$

(iii) *The Santaló point of $M_n^{g(M_n)}$ satisfies $s(M_n^{g(M_n)}) = 0.$*

(iv)

$$\begin{aligned} -s_1 &\leq \liminf_{n \rightarrow \infty} \left| g(M_n^{g(M_n)})(n+1) - s(M_n^{g(M_n)})(n+1) \right| \\ &\leq \limsup_{n \rightarrow \infty} \left| g(M_n^{g(M_n)})(n+1) - s(M_n^{g(M_n)})(n+1) \right| \leq -s_0. \end{aligned}$$

As for the proof of Proposition 3, it follows from Lemma 2 that $s(M_n^{g(M_n)}) = 0$. The remaining part of the proof of the proposition is in the next section.

Proof of Theorem 1. We take $C_n = M_n^{g(M_n)}$, as defined in Proposition 3. \square

4 Proof of Proposition 3.

The proof of Proposition 3 is given at the end of this section.

Lemma 4 Let K and L be convex bodies in \mathbb{R}^n , $c > 0$ and $M_n = \text{co}[(K, 0), (L, c)] \subset \mathbb{R}^{n+1}$. Then the $(n+1)$ th coordinate $g(M_n)(n+1)$ of the centroid $g(M_n)$ of M_n satisfies

$$g(M_n)(n+1) = \frac{c}{n+2} \frac{\sum_{k=0}^n (k+1) V_{n-k,k}(K, L)}{\sum_{k=0}^n V_{n-k,k}(K, L)}.$$

Proof. By definition

$$g(M_n)(n+1) = \frac{\int_0^c w |M_n(w)| dw}{\int_0^c |M_n(w)| dw}.$$

Note that $\text{co}[(K, 0), (L, c)] = \left\{ \left((1 - \frac{w}{c})K + \frac{w}{c}L, w \right) \mid 0 \leq w \leq c \right\}$. Therefore

$$g(M_n)(n+1) = \frac{\int_0^c w |(1 - \frac{w}{c})K + \frac{w}{c}L| dw}{\int_0^c |(1 - \frac{w}{c})K + \frac{w}{c}L| dw} = c \frac{\int_0^1 t |(1-t)K + tL| dt}{\int_0^1 |(1-t)K + tL| dt}.$$

Now we use the mixed volume formula (6) to get

$$\begin{aligned} g(M_n)(n+1) &= \frac{c \int_0^1 \sum_{k=0}^n \binom{n}{k} V_{n-k,k}(K, L) t^{k+1} (1-t)^{n-k} dt}{\int_0^1 \sum_{k=0}^n \binom{n}{k} V_{n-k,k}(K, L) t^k (1-t)^{n-k} dt} \\ &= \frac{c}{n+2} \frac{\sum_{k=0}^n (k+1) V_{n-k,k}(K, L)}{\sum_{k=0}^n V_{n-k,k}(K, L)}. \end{aligned}$$

□

The next lemma is well known ([18], p. 216, formula 54).

Lemma 5 For all $n \in \mathbb{N}$ and $t \geq 0$ one has $|B_2^n + tB_\infty^n| = \sum_{k=0}^n \binom{n}{k} 2^k |B_2^{n-k}| t^k$, with the convention that $\text{vol}_0(B_2^0) = 1$. Therefore, for $0 \leq k \leq n$,

$$V_{n-k,k}(B_2^n, B_\infty^n) = 2^k |B_2^{n-k}|.$$

Lemma 6 Let $a, b > 0$, $K = \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}}$, $L = \frac{B_\infty^n}{2}$ and $M_n = \text{co}[(K, -a), (L, b)]$. Then the center of gravity $g(M_n)$ of M_n satisfies $g(M_n) = \left(0, \dots, 0, g(M_n)(n+1) \right)$ and

$$\lim_{n \rightarrow \infty} g(M_n)(n+1) = \frac{1}{e} (-a) + \left(1 - \frac{1}{e} \right) b.$$

Proof. By symmetry, the centroid of M_n has coordinates $\left(0, \dots, 0, g(M_n)(n+1) \right)$. Instead of $M_n = \text{co}[(K, -a), (L, b)]$, we consider $\tilde{M}_n = \text{co}[(K, 0), (L, a+b)]$ with centroid

$g(\tilde{M}_n) = g(M_n) + ae_{n+1}$. Let $g(\tilde{M}_n) = (0, \dots, 0, g(\tilde{M}_n)(n+1)) = (0, \dots, 0, g(M_n)(n+1) + a)$. We use Lemma 4 with $c = a + b$ to get

$$g(\tilde{M}_n)(n+1) = \frac{a+b}{n+2} \frac{\sum_{k=0}^n (k+1) V_{n-k,k}(K, L)}{\sum_{k=0}^n V_{n-k,k}(K, L)}.$$

By Lemma 5 and the positive linearity of the mixed volumes in each component, we get

$$\begin{aligned} g(\tilde{M}_n)(n+1) &= \frac{a+b}{n+2} \frac{\sum_{k=0}^n (k+1) |B_2^n|^{\frac{k}{n}} |B_2^{n-k}|}{\sum_{k=0}^n |B_2^n|^{\frac{k}{n}} |B_2^{n-k}|} \\ &= \frac{a+b}{n+2} \frac{\sum_{k=0}^n \frac{k+1}{\Gamma(1+n/2)^{\frac{k}{n}} \Gamma(1+(n-k)/2)}}{\sum_{k=0}^n \frac{1}{\Gamma(1+n/2)^{\frac{k}{n}} \Gamma(1+(n-k)/2)}}. \end{aligned} \quad (13)$$

Now we apply Lemma 16 of the Appendix. \square

Eventually we will have to investigate expressions of the form

$$\left| \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \cap t \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right|, \text{ for } t \geq 0.$$

Schechtman and Zinn established asymptotic formulas for large n for the volumes of $B_p^n \cap tB_q^n$ [36]. To do so, they considered real independent random variables h_1^p, \dots, h_n^p with Weibull density ([3], p. 52)

$$\frac{e^{-|t|^p}}{2\Gamma(1 + \frac{1}{p})} \quad (14)$$

when $p > 0$. We denote by \mathbb{P}_p^n the probability measure on \mathbb{R}^n with density $f_p : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$f_p(x) = f_p(x_1, \dots, x_n) = \prod_{i=1}^n f_{h_i^p}(x_i) = \frac{1}{(2\Gamma(1 + \frac{1}{p}))^n} e^{-\sum_{i=1}^n |x_i|^p}.$$

Here, we need uniform estimates instead of asymptotic ones.

Lemma 7 *Let $0 < p, q < +\infty$ and let h^p be a random variable with density given by formula (14). Then*

$$\mathbb{E}|h^p|^q = \frac{1}{\Gamma(\frac{1}{p})} \Gamma\left(\frac{q+1}{p}\right). \quad (15)$$

In particular, we get

$$\mathbb{E}|h^1| = 1, \quad \mathbb{E}|h^1|^2 = 2 \quad \text{and} \quad \mathbb{E}|h^2|^2 = \frac{1}{2}. \quad (16)$$

The next lemma follows from the law of large numbers.

Lemma 8 Let (Ω, \mathbb{P}) be a probability space. Let $g_i : \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq n$, be independent $N(0, 1)$ -random variables and let $h_i^1 : \Omega \rightarrow \mathbb{R}$, $1 \leq i \leq n$, be independent random variables with density $e^{-|t|}$. Then, for every $\gamma > 0$ there is n_0 such that for all $n \geq n_0$

$$\mathbb{P} \left\{ \left| \frac{\frac{1}{n} \sum_{i=1}^n |g_i|}{\left(\frac{1}{n} \sum_{i=1}^n |g_i|^2\right)^{\frac{1}{2}}} - \sqrt{\frac{2}{\pi}} \right| \leq \gamma \right\} \geq \frac{1}{2} \quad (17)$$

and

$$\mathbb{P} \left\{ \left| \frac{\left(\frac{1}{n} \sum_{i=1}^n |h_i^1|^2\right)^{\frac{1}{2}}}{\frac{1}{n} \sum_{i=1}^n |h_i^1|} - \sqrt{2} \right| \leq \gamma \right\} \geq \frac{1}{2}. \quad (18)$$

Lemma 9 For every convex body K in \mathbb{R}^n with 0 in its interior, one has

$$|B_p^n \cap K| = n |B_p^n| \int_0^1 r^{n-1} \mathbb{P}_p^n \left\{ x \in \mathbb{R}^n \left| \frac{\|x\|_K}{\|x\|_p} \leq \frac{1}{r} \right. \right\} dr.$$

Proof. Since $|B_p^n| = (2\Gamma(1 + \frac{1}{p}))^n (\Gamma(1 + \frac{n}{p}))^{-1}$, one has

$$\begin{aligned} & n |B_p^n| \int_0^1 r^{n-1} \mathbb{P}_p^n \left\{ x \in \mathbb{R}^n \left| \frac{\|x\|_K}{\|x\|_p} \leq \frac{1}{r} \right. \right\} dr \\ &= \frac{n}{\Gamma(1 + \frac{n}{p})} \int_0^1 \left(\int_{\{x: r\|x\|_K \leq \|x\|_p\}} e^{-\|x\|_p^p} dx \right) r^{n-1} dr \\ &= \frac{n}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \left(\int_0^{\min(1, \frac{\|x\|_p}{\|x\|_K})} r^{n-1} dr \right) e^{-\|x\|_p^p} dx \\ &= \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} \frac{1}{(\max(1, \frac{\|x\|_K}{\|x\|_p}))^n} e^{-\|x\|_p^p} dx. \end{aligned}$$

Passing to polar coordinates and denoting by σ_{n-1} the surface measure on S^{n-1}

$$\begin{aligned} & n |B_p^n| \int_0^1 r^{n-1} \mathbb{P}_p^n \left\{ x \in \mathbb{R}^n; \frac{\|x\|_K}{\|x\|_p} \leq \frac{1}{r} \right\} dr \\ &= \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\theta \in S_{n-1}} \left(\int_0^{+\infty} e^{-r^p \|\theta\|_p^p r^{n-1}} dr \right) \frac{1}{(\max(1, \frac{\|\theta\|_K}{\|\theta\|_p}))^n} d\sigma_{n-1}(\theta) \\ &= \frac{1}{\Gamma(1 + \frac{n}{p})} \frac{1}{p} \int_0^{+\infty} e^{-s} s^{\frac{n}{p}-1} ds \int_{\theta \in S_{n-1}} \frac{1}{\|\theta\|_p^n} \frac{1}{(\max(1, \frac{\|\theta\|_K}{\|\theta\|_p}))^n} d\sigma_{n-1}(\theta) \\ &= \frac{1}{n} \int_{\theta \in S_{n-1}} \frac{1}{(\max(\|\theta\|_p, \|\theta\|_K))^n} d\sigma_{n-1}(\theta) \\ &= |B_p^n \cap K|. \quad \square \end{aligned}$$

Corollary 10 Let $1 \leq p, q < \infty$ and $s \geq 0$. Let $h_i^p : \Omega \rightarrow \mathbb{R}^n$, $1 \leq i \leq n$, be independent random variables with density $\frac{e^{-|t|^p}}{2\Gamma(1+\frac{1}{p})}$ and $h^p = (h_1^p, \dots, h_n^p)$. Then

$$\begin{aligned} |B_p^n \cap sB_q^n| &= n|B_p^n| \int_0^1 r^{n-1} \mathbb{P}_p^n \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \left| \frac{(\frac{1}{n} \sum_{i=1}^n |x_i|^q)^{\frac{1}{q}}}{(\frac{1}{n} \sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}} \leq n^{\frac{1}{p}-\frac{1}{q}} \frac{s}{r} \right. \right\} dr \\ &= n|B_p^n| \int_0^1 r^{n-1} \mathbb{P} \left\{ \frac{\|h^p\|_q}{\|h^p\|_p} \leq \frac{s}{r} \right\} dr \end{aligned}$$

A more general situation has also been explored in [36].

Lemma 11 For every $\gamma > 0$ there is n_0 such that for all $n \geq n_0$, one has

$$\begin{aligned} (i) \quad \frac{1}{2} &\leq \left| \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \cap s \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right| \leq 1 \text{ for all } s \geq \left(\sqrt{\frac{2}{\pi}} + \gamma \right) \sqrt{n} \frac{|B_2^n|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}} \sim \left(\sqrt{\frac{2}{\pi}} + \gamma \right) \sqrt{\frac{2e}{\pi}}. \\ (ii) \quad \frac{1}{2} &\leq \left| \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \cap s \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \right| \leq 1 \text{ for all } s \geq \frac{\sqrt{2}+\gamma}{\sqrt{n}} \frac{|B_2^n|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}} \sim (\sqrt{2} + \gamma) \sqrt{\frac{\pi}{2e}}. \end{aligned}$$

Proof. The right-hand side inequalities are obvious.

For (i), it follows from Corollary 10 with the substitution $s = t \left(\frac{|B_2^n|}{|B_1^n|} \right)^{1/n}$ that

$$\begin{aligned} &\left| \frac{B_2^n}{|B_2^n|^{1/n}} \cap s \frac{B_1^n}{|B_1^n|^{1/n}} \right| \\ &= n \int_0^1 r^{n-1} \mathbb{P} \left\{ \frac{\frac{1}{n} \sum_{i=1}^n |g_i|}{\left(\frac{1}{n} \sum_{i=1}^n |g_i|^2 \right)^{\frac{1}{2}}} \leq \frac{s}{r\sqrt{n}} \left(\frac{|B_2^n|}{|B_1^n|} \right)^{\frac{1}{n}} \right\} dr. \end{aligned} \quad (19)$$

By Lemma 8, for every $\gamma > 0$, there is n_0 such that for all $n \geq n_0$

$$\mathbb{P} \left\{ \frac{\frac{1}{n} \sum_{i=1}^n |g_i|}{\left(\frac{1}{n} \sum_{i=1}^n |g_i|^2 \right)^{\frac{1}{2}}} \leq \sqrt{\frac{2}{\pi}} + \gamma \right\} \geq \frac{1}{2}.$$

Therefore, for every $\gamma > 0$ there is n_0 such that for all $n \geq n_0$ and all s that satisfy

$$\sqrt{\frac{2}{\pi}} + \gamma \leq \frac{s}{\sqrt{n}} \frac{|B_2^n|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}} \text{ or, equivalently, } s \geq \left(\sqrt{\frac{2}{\pi}} + \gamma \right) \sqrt{n} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}},$$

we have

$$\left| \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \cap s \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right| \geq \frac{1}{2}.$$

For (ii), by Corollary 10 with $q = 2$ and $p = 1$,

$$\frac{|B_1^n \cap tB_2^n|}{|B_1^n|} = n \int_0^1 r^{n-1} \mathbb{P} \left\{ \frac{\|h^1\|_2}{\|h^1\|_1} \leq \frac{t}{r} \right\} dr.$$

We put $t = s \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}}$ and obtain

$$\left| \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \cap s \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \right| = n \int_0^1 r^{n-1} \mathbb{P} \left\{ \frac{\|h^1\|_2}{\|h^1\|_1} \leq \frac{s}{r} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}} \right\} dr. \quad (20)$$

By Lemma 8, for every $\gamma > 0$ there is n_0 such that for all $n \geq n_0$

$$\mathbb{P} \left\{ \frac{(\frac{1}{n} \sum_{i=1}^n |h_i^1|^2)^{\frac{1}{2}}}{\frac{1}{n} \sum_{i=1}^n |h_i^1|} \leq \sqrt{2} + \gamma \right\} \geq \frac{1}{2}.$$

Therefore, for $0 < r \leq 1$ and all s that satisfy

$$\frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}} s \geq \frac{\sqrt{2} + \gamma}{\sqrt{n}} \text{ or, equivalently, } s \geq \frac{\sqrt{2} + \gamma}{\sqrt{n}} \frac{|B_2^n|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}},$$

we have

$$\mathbb{P} \left\{ \frac{(\frac{1}{n} \sum_{i=1}^n |h_i^1|^2)^{\frac{1}{2}}}{\frac{1}{n} \sum_{i=1}^n |h_i^1|} \leq s \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}} \right\} \geq \frac{1}{2}.$$

Using (20), the result follows. \square

Now let $a, b > 0$. By (12), for $K = \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}}$, $L = \frac{B_\infty^n}{2}$ and $-\frac{1}{a} \leq s \leq \frac{1}{b}$, one has

$$M_n^\circ(s) = (|B_2^n|^{\frac{1}{n}}(1+sa)B_2^n) \cap (2(1-sb)B_1^n) \quad (21)$$

$$= (1+sa)|B_2^n|^{\frac{2}{n}} \left(\frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \cap 2 \frac{(1-sb)}{(1+sa)} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{2}{n}}} \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right). \quad (22)$$

Lemma 12 *For every $\gamma > 0$, there is n_0 such that for all $n \geq n_0$*

(i) *and all $\frac{1}{b} \geq s \geq -\frac{\sqrt{\pi e}-2}{\sqrt{\pi e} a+2b} + \gamma$ we have*

$$2^{n-1}(1-sb)^n |B_1^n| \leq |M_n^\circ(s)| \leq 2^n(1-sb)^n |B_1^n|.$$

(ii) *and all $-\frac{1}{a} \leq s \leq -\frac{\sqrt{e}-1}{b+a\sqrt{e}} - \gamma$ we have*

$$\frac{1}{2}(1+sa)^n |B_2^n|^2 \leq |M_n^\circ(s)| \leq (1+sa)^n |B_2^n|^2$$

Proof. We only need to prove the left-hand side inequalities. Assume that $-\frac{1}{a} \leq s \leq \frac{1}{b}$.

For (i), we have

$$M_n^\circ(s) = 2(1-sb) \left(\left(\frac{(1+sa)|B_2^n|^{\frac{1}{n}}}{2(1-sb)} B_2^n \right) \cap B_1^n \right).$$

Therefore

$$\begin{aligned} |M_n^c(s)| &= 2^n(1-sb)^n \left| \left(\frac{(1+sa)|B_2^n|^{\frac{1}{n}}}{2(1-sb)} B_2^n \right) \cap B_1^n \right| \\ &= 2^n(1-sb)^n |B_1^n| \left| \left(\frac{1+sa}{2(1-sb)} \frac{|B_2^n|^{\frac{2}{n}}}{|B_1^n|^{\frac{1}{n}}} \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \right) \cap \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right|. \end{aligned}$$

By Lemma 11, for all $\gamma > 0$ there is n_0 such that for every $n \geq n_0$

$$\frac{1}{2} \leq \left| \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \cap \frac{1+sa}{2(1-sb)} \frac{|B_2^n|^{\frac{2}{n}}}{|B_1^n|^{\frac{1}{n}}} \frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \right| \leq 1, \quad (23)$$

provided that

$$\frac{1+sa}{2(1-sb)} \frac{|B_2^n|^{\frac{2}{n}}}{|B_1^n|^{\frac{1}{n}}} \geq \frac{\sqrt{2}+\gamma}{\sqrt{n}} \frac{|B_2^n|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}}$$

or

$$\frac{1+sa}{1-sb} \geq 2 \frac{\sqrt{2}+\gamma}{\sqrt{n}} \frac{|B_2^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}}, \text{ which means that } s \geq -\frac{\sqrt{n} |B_2^n|^{\frac{1}{n}} - 2(\sqrt{2}+\gamma)}{a \sqrt{n} |B_2^n|^{\frac{1}{n}} + 2b(\sqrt{2}+\gamma)}.$$

Since $|B_2^n|^{\frac{1}{n}} \sim \sqrt{\frac{2e\pi}{n}}$, inequality (23) holds provided for a new $\gamma > 0$, one has

$$s \geq -\frac{\sqrt{2\pi e} - 2\sqrt{2}}{a\sqrt{2\pi e} + 2b\sqrt{2}} + \gamma = -\frac{\sqrt{\pi e} - 2}{a\sqrt{\pi e} + 2b} + \gamma.$$

For (ii), we have

$$M_n^c(s) = (1+sa)|B_2^n|^{\frac{2}{n}} \left(\frac{B_2^n}{|B_2^n|^{\frac{1}{n}}} \cap 2 \frac{1-sb}{1+sa} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{2}{n}}} \frac{B_1^n}{|B_1^n|^{\frac{1}{n}}} \right).$$

By Lemma 11, for every $\gamma > 0$ there is n_0 such that for every $n \geq n_0$ and all

$$t \geq \left(\sqrt{\frac{2}{\pi}} + \gamma \right) \sqrt{n} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}}$$

we have

$$\frac{1}{2} \leq \left| \frac{B_2^n}{|B_2^n|^{1/n}} \cap t \frac{B_1^n}{|B_1^n|^{1/n}} \right| \leq 1. \quad (24)$$

Therefore,

$$\frac{1}{2} \leq \left| \frac{B_2^n}{|B_2^n|^{1/n}} \cap 2 \frac{1-sb}{1+sa} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{2}{n}}} \frac{B_1^n}{|B_1^n|^{1/n}} \right| \leq 1$$

provided that

$$\left(\sqrt{\frac{2}{\pi}} + \gamma \right) \sqrt{n} \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{1}{n}}} \leq 2 \left(\frac{1-sb}{1+sa} \right) \frac{|B_1^n|^{\frac{1}{n}}}{|B_2^n|^{\frac{2}{n}}},$$

which is equivalent to

$$\left(\sqrt{\frac{2}{\pi}} + \gamma\right)\sqrt{n} |B_2^n|^{\frac{1}{n}} \leq 2 \frac{1 - sb}{1 + sa}.$$

As above, inequality (24) holds if for some new $\gamma > 0$ one has $s \leq -\frac{\sqrt{e}-1}{b+a\sqrt{e}} - \gamma$. \square

Lemma 13 *Let $s_0 = \frac{1 - \sqrt{e}}{b + a\sqrt{e}}$ and $s_1 = \frac{2 - \sqrt{\pi e}}{a\sqrt{\pi e} + 2b}$. Then, for every $\gamma > 0$, there is n_0 such that for all n with $n \geq n_0$,*

$$(i) \int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds \leq (1 + \gamma) \int_{s_0 - \gamma}^{s_1 + \gamma} |M_n^\circ(s)| ds$$

$$(ii) \int_{-\frac{1}{a}}^{s_0 - \gamma} |s| |M_n^\circ(s)| ds \leq \gamma \int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds \text{ and } \int_{s_1 + \gamma}^{\frac{1}{b}} |s| |M_n^\circ(s)| ds \leq \gamma \int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds.$$

Remarks.

a. Please note that the expression $\int_{s_0 - \gamma}^{s_1 + \gamma} s |M_n^\circ(s)| ds$ is negative for small $\gamma > 0$. This lemma means that the volume of M_n° is concentrated between the hyperplanes orthogonal to e_{n+1} through $s_0 e_{n+1}$ and $s_1 e_{n+1}$.

b. Although the inequality $s_0 < s_1 < 0$ follows from the above computations, it is comforting to verify it directly. Actually $s_0 < s_1$ is equivalent to $(a + b)\sqrt{e}(2 - \sqrt{\pi}) > 0$, which holds for all positive a and b .

Proof of Lemma 13.

Note that if the statements of the lemma hold for a sufficiently small $\gamma_0 > 0$, then they also hold for all $\gamma \geq \gamma_0$ with the same n_0 . Therefore, it is enough to prove the lemma for a small enough γ .

(i) By Lemma 12, for every $\gamma > 0$ there is n_0 such that for $n \geq n_0$ and all s such that

$$-\frac{1}{a} \leq s \leq \frac{1 - \sqrt{e}}{b + a\sqrt{e}} - \gamma = s_0 - \gamma$$

we have

$$\frac{1}{2}(1 + sa)^n |B_2^n|^2 \leq |M_n^\circ(s)| \leq (1 + sa)^n |B_2^n|^2.$$

Therefore

$$\begin{aligned} \int_{s_0 - 2\gamma}^{s_0 - \gamma} |M_n^\circ(s)| ds &\geq \frac{|B_2^n|^2}{2} \int_{s_0 - 2\gamma}^{s_0 - \gamma} (1 + sa)^n ds \\ &= \frac{1}{2a(n+1)} |B_2^n|^2 \left((1 + a(s_0 - \gamma))^{n+1} - (1 + a(s_0 - 2\gamma))^{n+1} \right) \\ &= \frac{1}{2a(n+1)} |B_2^n|^2 (1 + a(s_0 - \gamma))^{n+1} \left(1 - \left(\frac{1 + a(s_0 - 2\gamma)}{1 + a(s_0 - \gamma)} \right)^{n+1} \right). \end{aligned}$$

For sufficiently large n

$$\int_{s_0-2\gamma}^{s_0-\gamma} |M_n^\circ(s)| ds \geq \frac{1}{4a(n+1)} |B_2^n|^2 (1+a(s_0-\gamma))^{n+1}.$$

On the other hand, by Lemma 12,

$$\begin{aligned} \int_{-\frac{1}{a}}^{s_0-2\gamma} |M_n^\circ(s)| ds &\leq |B_2^n|^2 \int_{-\frac{1}{a}}^{s_0-2\gamma} (1+sa)^n ds = \frac{1}{a(n+1)} |B_2^n|^2 (1+a(s_0-2\gamma))^{n+1} \\ &= \frac{1}{a(n+1)} |B_2^n|^2 (1+a(s_0-\gamma))^{n+1} \left(1 - \frac{a\gamma}{1+a(s_0-\gamma)}\right)^{n+1}. \end{aligned}$$

Thus, for n large enough

$$\int_{-\frac{1}{a}}^{s_0-2\gamma} |M_n^\circ(s)| ds \leq 4 \left(1 - \frac{a\gamma}{1+a(s_0-\gamma)}\right)^{n+1} \int_{s_0-2\gamma}^{s_0-\gamma} |M_n^\circ(s)| ds, \quad (25)$$

and

$$\int_{-\frac{1}{a}}^{s_0-2\gamma} |M_n^\circ(s)| ds \leq 2\gamma \int_{s_0-2\gamma}^{s_0-\gamma} |M_n^\circ(s)| ds.$$

Now we consider the interval $[s_1, \frac{1}{b}]$. By Lemma 12,

$$\begin{aligned} \int_{s_1+\gamma}^{s_1+2\gamma} |M_n^\circ(s)| ds &\geq |B_1^n| \int_{s_1+\gamma}^{s_1+2\gamma} 2^{n-1} (1-sb)^n ds \\ &= 2^{n-1} \frac{|B_1^n|}{b(n+1)} \left((1-b(s_1+\gamma))^{n+1} - (1-b(s_1+2\gamma))^{n+1} \right) \\ &= 2^{n-1} |B_1^n| \frac{(1-b(s_1+\gamma))^{n+1}}{(n+1)b} \left(1 - \left(1 - \frac{b\gamma}{1-b(s_1+\gamma)}\right)^{n+1} \right) \end{aligned}$$

Therefore, for sufficiently large n

$$\int_{s_1+\gamma}^{s_1+2\gamma} |M_n^\circ(s)| ds \geq 2^{n-2} |B_1^n| \frac{(1-b(s_1+\gamma))^{n+1}}{(n+1)b}.$$

On the other hand, by Lemma 12,

$$\begin{aligned} \int_{s_1+2\gamma}^{\frac{1}{b}} |M_n^\circ(s)| ds &\leq |B_1^n| \int_{s_1+2\gamma}^{\frac{1}{b}} 2^n (1-sb)^n ds = 2^n \frac{1}{b(n+1)} (1-b(s_1+2\gamma))^{n+1} |B_1^n| \\ &= 2^n \frac{1}{b(n+1)} (1-b(s_1+\gamma))^{n+1} \left(1 - \frac{b\gamma}{1-b(s_1+\gamma)}\right)^{n+1} |B_1^n|. \end{aligned}$$

Thus

$$\int_{s_1+2\gamma}^{\frac{1}{b}} |M_n^\circ(s)| ds \leq 4 \left(1 - \frac{b\gamma}{1-b(s_1+\gamma)}\right)^{n+1} \int_{s_1+\gamma}^{s_1+2\gamma} |M_n^\circ(s)| ds. \quad (26)$$

Since $s_1 < 0$, for sufficiently big n ,

$$\int_{s_1+2\gamma}^{\frac{1}{b}} |M_n^\circ(s)| ds \leq 2\gamma \int_{s_1+\gamma}^{s_1+2\gamma} |M_n^\circ(s)| ds. \quad (27)$$

It is left to pass to a new γ .

(ii) By (25)

$$\int_{-\frac{1}{a}}^{s_0-2\gamma} |s| |M_n^\circ(s)| ds \leq 4 \max\left\{\frac{1}{a}, \frac{1}{b}\right\} \left(1 - \frac{a\gamma}{1+a(s_0-\gamma)}\right)^{n+1} \int_{s_0-2\gamma}^{s_0-\gamma} |M_n^\circ(s)| ds.$$

We choose n big enough. The other estimate is done in the same way using (26). \square

Proof of Proposition 3. (i) is proved in Lemma 6. We show (ii). By definition,

$$g(M_n^\circ)(n+1) = \frac{\int_{-\frac{1}{a}}^{\frac{1}{b}} s |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds}.$$

Therefore, one has by Lemma 13 that

$$\left| g(M_n^\circ)(n+1) - \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} \right| \leq \frac{\int_{-\frac{1}{a}}^{s_0-\gamma} |s| |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} + \frac{\int_{s_1+\gamma}^{\frac{1}{b}} |s| |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} \leq 2\gamma,$$

for every small $\gamma > 0$. Thus

$$g(M_n^\circ)(n+1) - 2\gamma \leq \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} \leq g(M_n^\circ)(n+1) + 2\gamma.$$

Since $s_1 < 0$, we may assume that $s_1 + \gamma < 0$. Therefore

$$\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds < 0$$

and by Lemma 13 (i)

$$g(M_n^\circ)(n+1) - 2\gamma \leq \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} \leq \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{(1+\gamma) \int_{s_0-\gamma}^{s_1+\gamma} |M_n^\circ(s)| ds} \leq \frac{s_1+\gamma}{1+\gamma}.$$

On the other hand

$$s_0 - \gamma \leq \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{\int_{s_0-\gamma}^{s_1+\gamma} |M_n^\circ(s)| ds} \leq \frac{\int_{s_0-\gamma}^{s_1+\gamma} s |M_n^\circ(s)| ds}{\int_{-\frac{1}{a}}^{\frac{1}{b}} |M_n^\circ(s)| ds} \leq g(M_n^\circ)(n+1) + 2\gamma.$$

Therefore, $s_0 - 3\gamma \leq g(M_n^\circ)(n+1) \leq \frac{s_1 + \gamma}{1 + \gamma} + 2\gamma$.

(iv) We apply these estimates to the convex body $M_n = \text{co} \left[(K, -1), \left(L, \frac{1}{e-1} \right) \right]$. The centroid of M_n is $(0, \delta_n)$ with $\lim_{n \rightarrow \infty} \delta_n = 0$. We get

$$M_n^{(0, \delta_n)} = \left\{ (z, s) \mid \forall (x, y) \in K \times L : \langle z, x \rangle - s(1 + \delta_n) \leq 1, \langle z, y \rangle + s \left(\frac{1}{e-1} - \delta_n \right) \leq 1 \right\}.$$

It is left to apply the above estimates to M_n° with $a = 1 + \delta_n$ and $b = \frac{1}{e-1} - \delta_n$. \square

5 Centers of John and Löwner ellipsoids.

We want to give another example of affine covariant points that are far apart. This example involves the *John* and *Löwner ellipsoids* of K .

Recall that the John [19] (respectively, Löwner) ellipsoid of K is the unique ellipsoid contained in K (respectively, containing K) with maximal (respectively, with minimal) volume. See e.g. [2, 10, 11, 13] for recent results concerning these ellipsoids.

The centers of the John and the Löwner ellipsoid of a convex body K are affine covariant points.

We also need the following well known fact. For the reader's convenience, we give its proof.

Lemma 14 *If the John (respectively, Löwner) ellipsoid of a convex body K is αB_2^n for some $\alpha > 0$, then, for every $s, t \geq 0$, the John (respectively, Löwner) ellipsoid of $sK + tB_2^n$ is $(s\alpha + t)B_2^n$.*

Proof. It is enough to establish the case when $\alpha = s = 1$. By a well known characterization (see e.g. [1]), B_2^n is the John ellipsoid of K if and only if $B_2^n \subset K$ and there exist $u_i \in S^{n-1} \cap \partial K$ and $c_i \geq 0$, $1 \leq i \leq m$, such that

$$x = \sum_{i=1}^m c_i \langle u_i, x \rangle u_i \text{ for every } x \in \mathbb{R}^n \text{ and } 0 = \sum_{i=1}^m c_i u_i.$$

Let $W = \frac{K + tB_2^n}{1+t}$. Then $B_2^n \subset W$ and $u_i \in \partial W$ for all $1 \leq i \leq m$. Hence B_2^n is also the John ellipsoid of W .

The case of the Löwner ellipsoid is treated accordingly. \square

An easy construction shows that the center j of the John ellipsoid of a convex body K can be far away from the center l of its Löwner ellipsoid.

Proposition 15 *Let $L = L_n$ be the convex body in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ defined by*

$$L = \text{co}[(B_2^n, 0), (\Delta_n, 1)],$$

where Δ_n is the regular simplex inscribed in B_2^n . Then the Löwner ellipsoid of L is centered at $(0, \frac{1}{2})$, while the John ellipsoid is centered at $(0, c_n)$ with $c_n \sim \frac{1}{n}$. Thus, the measure of symmetry $\phi_{l,j}(L) \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$.

Proof. By uniqueness of the John and Löwner ellipsoids of L , both have the form

$$\mathcal{E} = \left\{ (x, t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \mid \frac{\|x\|_2^2}{a^2} + \frac{(t-c)^2}{b^2} \leq 1 \right\}, \quad (28)$$

for some $a, b > 0$ and $c \in \mathbb{R}$. Let \mathcal{E} be such an ellipsoid. For $t \in [0, 1]$, let

$$L(t) = \{x \in \mathbb{R}^n \mid (x, t) \in L\} \text{ and } \mathcal{E}(t) = \{x \in \mathbb{R}^n \mid (x, t) \in \mathcal{E}\}.$$

If \mathcal{E} is the Löwner ellipsoid of L , then $L(0) = B_2^n \subset \mathcal{E}(0)$ and $L(1) = \Delta_n \subset \mathcal{E}(1)$, which is equivalent to $B_2^n \subset \mathcal{E}(0)$ and $B_2^n \subset \mathcal{E}(1)$: the latter inclusion holds, as the sections of the ellipsoids (28) are Euclidean balls. Since $L(t) = (1-t)L(0) + tL(1) \subset B_2^n$, \mathcal{E} is also the Löwner ellipsoid of

$$N = \text{co}[(B_2^n, 0), (B_2^n, 1)].$$

Since N is centrally symmetric about $p = (0, \dots, 0, \frac{1}{2})$, this p is the center of the Löwner ellipsoid of N and hence also the center of the Löwner ellipsoid of L .

If \mathcal{E} is the John ellipsoid of L , then $\mathcal{E}(t) \subset L(t)$ for $t \in [0, 1]$ or, more precisely, for $t \in [c-b, c+b]$, with $0 \leq c-b \leq c+b \leq 1$. This means that $b \in [0, 1/2]$ and $c \in [b, 1-b]$. Since $L(t) = (1-t)B_2^n + t\Delta_n$ and as the John ellipsoid of Δ_n is $\frac{1}{n}B_2^n$, it follows from the preceding Lemma 14 that the John ellipsoid of $L(t)$ is $(1-t)B_2^n + \frac{t}{n}B_2^n$. Thus for every $t \in [0, 1]$,

$$\mathcal{E}(t) \subset \left(1 - \frac{(n-1)}{n} t\right) B_2^n.$$

We maximize $\text{vol}_{n+1}(\mathcal{E}) = a^n b \text{vol}_{n+1}(B_2^{n+1})$ under the constraints $b \leq c \leq 1-b$, $b \in [0, \frac{1}{2}]$ and

$$a \sqrt{1 - \left(\frac{t-c}{b}\right)^2} \leq 1 - \frac{n-1}{n} t, \text{ for every } t \in [c-b, c+b].$$

To get maximum volume for \mathcal{E} , there should be equality in the preceding inequality for some $t \in [c-b, c+b]$. This gives the condition

$$\left(1 - \frac{n-1}{n} c\right)^2 = a^2 + \left(\frac{n-1}{n} b\right)^2$$

or, equivalently,

$$a = \left(\left(1 - \frac{n-1}{n} c\right)^2 - \left(\frac{n-1}{n} b\right)^2 \right)^{\frac{1}{2}}. \quad (29)$$

Thus, using (29), the volume of \mathcal{E} is maximal for fixed $b \in [0, \frac{1}{2}]$ if

$$ba^n = b \left(\left(1 - \left(\frac{n-1}{n} c\right)\right)^2 - \left(\frac{n-1}{n} b\right)^2 \right)^{\frac{n}{2}} =: f(c)$$

is maximal. As f is decreasing in c , this happens when $c = b$. It now remains to maximize the function $f(b) = b(1 - 2\frac{n-1}{n}b)^{\frac{n}{2}}$. An easy computation shows that f reaches its maximum at

$$b = c = \frac{n}{(n-1)(n+2)} \sim \frac{1}{n}.$$

If $d(l(L), j(L))$ is the distance between the centers $l(L)$ of the Löwner ellipsoid and $j(L)$ of the John ellipsoid of L , it follows that $d(l(L), j(L)) \rightarrow \frac{1}{2}$ when $n \rightarrow +\infty$. \square

6 Appendix.

Lemma 16 $\lim_{n \rightarrow \infty} \frac{1}{n+2} \frac{\sum_{k=0}^n \frac{k+1}{\Gamma(1+\frac{n}{2})^{\frac{k}{n}} \Gamma(1+\frac{n-k}{2})}}{\sum_{k=0}^n \frac{1}{\Gamma(1+\frac{n}{2})^{\frac{k}{n}} \Gamma(1+\frac{n-k}{2})}} = 1 - \frac{1}{e}.$

Proof. Note that

$$\begin{aligned} \frac{1}{n+2} \frac{\sum_{k=0}^n \frac{k+1}{\Gamma(1+\frac{n}{2})^{\frac{k}{n}} \Gamma(1+\frac{n-k}{2})}}{\sum_{k=0}^n \frac{1}{\Gamma(1+\frac{n}{2})^{\frac{k}{n}} \Gamma(1+\frac{n-k}{2})}} &= \frac{1}{n+2} \frac{\sum_{k=0}^n \frac{n-k+1}{\Gamma(1+\frac{n}{2})^{\frac{n-k}{n}} \Gamma(1+\frac{k}{2})}}{\sum_{k=0}^n \frac{1}{\Gamma(1+\frac{n}{2})^{\frac{n-k}{n}} \Gamma(1+\frac{k}{2})}} \\ &= \frac{1}{n+2} \frac{\sum_{k=0}^n \frac{(n-k+1)\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}}{\sum_{k=0}^n \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}} = \frac{n+1}{n+2} - \frac{1}{n+2} \frac{\sum_{k=0}^n k \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}}{\sum_{k=0}^n \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}}. \end{aligned}$$

It is thus needed to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \frac{\sum_{k=0}^n k \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}}{\sum_{k=0}^n \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}} = \frac{1}{e}. \quad (30)$$

For every $x \geq 0$,

$$\begin{aligned} \sum_{k=0}^n \frac{kx^k}{\Gamma(1+\frac{k}{2})} &= \sum_{k=1}^n \frac{kx^k}{\frac{k}{2}\Gamma(\frac{k}{2})} = 2x \sum_{k=1}^n \frac{x^{k-1}}{\Gamma(\frac{k}{2})} \\ &= 2x \left(\frac{1}{\Gamma(\frac{1}{2})} + \sum_{k=2}^n \frac{x^{k-1}}{\Gamma(\frac{k}{2})} \right) = 2x \left(\frac{1}{\Gamma(\frac{1}{2})} + x \sum_{k=0}^{n-2} \frac{x^k}{\Gamma(1+\frac{k}{2})} \right). \end{aligned}$$

Thus for $x = \Gamma(1+\frac{n}{2})^{\frac{1}{n}}$

$$\sum_{k=0}^n \frac{k\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})} = 2\Gamma\left(1+\frac{n}{2}\right)^{\frac{1}{n}} \left(\frac{1}{\Gamma(\frac{1}{2})} + \Gamma\left(1+\frac{n}{2}\right)^{\frac{1}{n}} \sum_{k=0}^{n-2} \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})} \right). \quad (31)$$

Let

$$A_n = \sum_{k=0}^n k \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})} \text{ and } B_n = \sum_{k=0}^n \frac{\Gamma(1+\frac{n}{2})^{\frac{k}{n}}}{\Gamma(1+\frac{k}{2})}.$$

Equality (30) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{A_n}{nB_n} = \frac{1}{e}.$$

Let $c_n := \Gamma(1 + \frac{n}{2})^{\frac{1}{n}} \sim \sqrt{\frac{n}{2e}}$. Then

$$B_n \geq \frac{\Gamma(1 + \frac{n}{2})^{\frac{2}{n}}}{\Gamma(2)} \sim \frac{n}{2e}.$$

In fact, it can be proved that $B_n \sim 2e^{c_n^2} \sim 2e^{\frac{n}{2e}}$, but for our purposes the above estimate is enough. Also, when $n \rightarrow +\infty$,

$$\frac{\Gamma(1 + \frac{n}{2})^{\frac{n-1}{n}}}{\Gamma(1 + \frac{n-1}{2})} \rightarrow \sqrt{e}.$$

Therefore, one has by (31)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{A_n}{nB_n} &= \lim_{n \rightarrow \infty} \frac{2c_n}{n} \left(\frac{1}{\Gamma(\frac{1}{2})B_n} + c_n \left(1 - \frac{\Gamma(1 + \frac{n}{2})^{\frac{n-1}{n}}}{\Gamma(1 + \frac{n-1}{2})} + \frac{\Gamma(1 + \frac{n}{2})^{\frac{n}{n}}}{\Gamma(1 + \frac{n}{2})} \right) \right) \\ &= \lim_{n \rightarrow \infty} \frac{2c_n^2}{n} = \lim_{n \rightarrow \infty} \frac{2}{n} \left(\sqrt{\frac{n}{2e}} \right)^2 = \frac{1}{e}. \quad \square \end{aligned}$$

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