

Inequalities for mixed p -affine surface area ^{*}

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Abstract

We prove new Alexandrov-Fenchel type inequalities and new affine isoperimetric inequalities for mixed p -affine surface areas. We introduce a new class of bodies, the illumination surface bodies, and establish some of their properties. We show, for instance, that they are not necessarily convex. We give geometric interpretations of L_p affine surface areas, mixed p -affine surface areas and other functionals via these bodies. The surprising new element is that not necessarily convex bodies provide the tool for these interpretations.

1 Introduction

This article deals with affine isoperimetric inequalities and Alexandrov-Fenchel type inequalities for mixed p -affine surface area. Mixed p -affine surface area was introduced by Lutwak for $p \geq 1$ in [28]. It has the dual mixed volume [25] and the L_p affine surface area [28] as special cases. L_p affine surface area is at the core of the rapidly developing L_p Brunn-Minkowski theory. Contributions here include the study of solutions of nontrivial ordinary and, respectively, partial differential equations (see e.g. Chen [10], Chou and Wang [11], Stancu [39, 40]), the study of the L_p Christoffel-Minkowski problem by Hu, Ma and Shen [18], extensions of L_p affine surface area to all p (see e.g., [34, 37, 38, 45]), a new proof by Fleury, Guédon and Paouris [12] of a result by Klartag [19] on concentration of volume, results on approximation of convex bodies by polytopes (e.g., [16, 24, 38]), results on valuations (e.g., Alesker [2, 3], and Ludwig and Reitzner [22, 23]) and the affine Plateau problem solved in \mathbb{R}^3 by Trudinger and Wang [41], and Wang [43].

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The classical affine isoperimetric inequality, which gives an upper bound for the affine surface area in terms of volume, is fundamental in many problems (e.g. [14, 15, 29, 36]). In particular, it was used to show the uniqueness of self-similar solutions of the affine curvature flow and to study its asymptotic behavior by Andrews [4, 5], Sapiro and Tannenbaum [35]. More general L_p affine isoperimetric inequalities were proved in [28] for $p > 1$ and in [45] for all p . These L_p affine isoperimetric inequalities generalize the celebrated Blaschke-Santaló inequality and inverse Santaló inequality due to Bourgain and Milman [6] (see also Kuperberg [20]). We also refer to related works by Lutwak, Yang and Zhang [30] and Campi and Gronchi [9].

For mixed p -affine surface area, Alexandrov-Fenchel type inequalities (for $p = 1, \pm\infty$) and affine isoperimetric inequalities (for $1 \leq p \leq n$) were first established by Lutwak in [25, 26, 28]. Here we derive new Alexandrov-Fenchel type inequalities for mixed p -affine surface area for all $p \in [-\infty, \infty]$ and new mixed p -affine isoperimetric inequalities for all $p \in [0, \infty]$. Classification of the equality cases for all p in the Alexandrov-Fenchel type inequalities for mixed p -affine surface area is related to the uniqueness of solutions of the L_p Minkowski problem (e.g., [10, 11, 27, 29, 31, 32, 39, 40]), which is unsolved for many cases. This is similar to the classical Alexandrov-Fenchel inequalities for mixed volume, where the complete classification of the equality cases is also an unsolved problem.

We also give new geometric interpretations for functionals on convex bodies. In particular, for L_p affine surface area, mixed p -affine surface area, and i -th mixed p -affine surface area (see below for the definitions). To do so, we construct a new class of bodies, the illumination surface bodies, and study the asymptotic behavior of their volumes. We show that the illumination surface bodies are not necessarily convex, thus introducing a novel idea in the theory of geometric characterizations of functionals on convex bodies, where to date only convex bodies were used (e.g. [34, 37, 38, 45]).

From now on, we will always assume that the centroid of a convex body K in \mathbb{R}^n is at the origin. We write $K \in C_+^2$, if K has C^2 boundary with everywhere strictly positive Gaussian curvature. For real $p \geq 1$, the mixed p -affine surface area, $as_p(K_1, \dots, K_n)$, of n convex bodies $K_i \in C_+^2$ was introduced in [28] by

$$as_p(K_1, \dots, K_n) = \int_{S^{n-1}} \left[h_{K_1}(u)^{1-p} f_{K_1}(u) \cdots h_{K_n}^{1-p} f_{K_n}(u) \right]^{\frac{1}{n+p}} d\sigma(u). \quad (1.1)$$

Here S^{n-1} is the boundary of the Euclidean unit ball B_2^n in \mathbb{R}^n , σ is the usual surface area measure on S^{n-1} , $h_K(u)$ is the support function of the convex body K at $u \in S^{n-1}$, and $f_K(u)$ is the curvature function of K at u , i.e., the reciprocal of the Gauss curvature $\kappa_K(x)$ at this point $x \in \partial K$, the boundary of K , that has u as its outer normal.

We propose here to extend the definition (1.1) for mixed p -affine surface area to all $p \neq -n$. We also propose a definition for the $(-n)$ - mixed affine surface area (see Section 2).

We show that mixed p -affine surface areas are affine invariants for all p . Note that for $p = \pm\infty$,

$$\begin{aligned} as_{\pm\infty}(K_1, \dots, K_n) &= \int_{S^{n-1}} \frac{1}{h_{K_1}(u)} \cdots \frac{1}{h_{K_n}(u)} d\sigma(u) \\ &= n\tilde{V}(K_1^\circ, \dots, K_n^\circ) \end{aligned} \quad (1.2)$$

where $K^\circ = \{y \in \mathbb{R}^n, \langle x, y \rangle \leq 1, \forall x \in K\}$ is the polar body of K , and $\tilde{V}(K_1^\circ, \dots, K_n^\circ)$ is the dual mixed volume of $K_1^\circ, \dots, K_n^\circ$, introduced by Lutwak in [25].

When all K_i coincide with K , then for all $p \neq -n$

$$as_p(K, \dots, K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u) = as_p(K). \quad (1.3)$$

$as_p(K)$ is the L_p affine surface area of K , which is defined for a general convex body K as in [28] ($p > 1$) and in [38] ($p < 1$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x). \quad (1.4)$$

$N_K(x)$ is the outer unit normal vector at x to ∂K , μ_K denotes the usual surface area measure on ∂K , and $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{R}^n which induces the Euclidian norm $\|\cdot\|$. If $K \in C_+^2$, (1.4) can be rewritten as (1.3). We show in Section 2 that the corresponding formula (1.3) for $p = -n$ also holds, where $as_{-n}(K)$ is the L_{-n} affine surface area of K introduced in [34].

Note further that the surface area of K can be written as (-1) -th mixed 1-affine surface area of K and the Euclidean ball B_2^n (see Section 2).

Thus, mixed p -affine surface area is an extension of dual mixed volume, surface area, and L_p affine surface area.

Further notations. For sets A and B , $[A, B] = \text{conv}(A, B) := \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1], x, y \in A \cup B\}$ is the convex hull of $A \cup B$. A subset K of \mathbb{R}^n is star convex if there exists $x_0 \in K$ such that the line segment $[x_0, x]$, from x_0 to any point x in K , is contained in K . A convex body K is said to be strictly convex if ∂K does not contain any line segment.

For a convex body K in \mathbb{R}^n , $|K|$ stands for the n -dimensional volume of K . More generally, for a set M , $|M|$ denotes the Hausdorff content of its appropriate dimension.

For $u \in S^{n-1}$, $H(x, u)$ is the hyperplane through x with outer normal vector u , $H(x, u) = \{y \in \mathbb{R}^n, \langle y, u \rangle = \langle x, u \rangle\}$. The two half-spaces generated by $H(x, u)$ are $H^-(x, u) = \{y \in \mathbb{R}^n, \langle y, u \rangle \geq \langle x, u \rangle\}$ and $H^+(x, u) = \{y \in \mathbb{R}^n, \langle y, u \rangle \leq \langle x, u \rangle\}$. For $f : \partial K \rightarrow \mathbb{R}_+ \cup \{0\}$, μ_f is the measure on ∂K defined by $\mu_f(A) = \int_A f d\mu_K$.

The paper is organized as follows. In Section 2, we prove new Alexandrov-Fenchel type inequalities and new isoperimetric inequalities for mixed p -affine surface areas. We show monotonicity behaviour of the quotients

$$\left(\frac{as_p(K_1, \dots, K_n)}{as_\infty(K_1, \dots, K_n)} \right)^{n+p} \quad \text{and} \quad \left(\frac{as_p(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+p}{p}}.$$

We prove Blaschke-Santaló type inequalities for mixed p -affine surface areas. Similar results for the i -th mixed p -affine surface areas are also proved in Section 2. In Section 3, we introduce the illumination surface body and describe some of its properties. In Section 4, we derive the asymptotic behavior of the volume of the illumination surface body, and geometric interpretations of L_p affine surface areas, mixed p -affine surface areas, and other functionals on convex bodies.

2 Mixed p -affine surface area and related inequalities

2.1 Inequalities for mixed p -affine surface area

We begin by proving that mixed p -affine surface area is affine invariant for all p . For $p \geq 1$, this was proved by Lutwak [28]. We will first treat the case $p \neq -n$. All the results concerning the case $p = -n$ are at the end of this subsection.

It will be convenient to use the notation

$$f_p(K, u) = h_K^{1-p}(u) f_K(u) \tag{2.5}$$

for a convex body K in \mathbb{R}^n and $u \in S^{n-1}$. We will also write $as_p^m(K_1, \dots, K_n)$ for $[as_p(K_1, \dots, K_n)]^m$, and $|\det(T)|$ for the absolute value of the determinant of linear transform T .

Lemma 2.1 *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transform. Then for all $p \neq -n$,*

$$as_p(TK_1, \dots, TK_n) = |\det(T)|^{\frac{n-p}{n+p}} as_p(K_1, \dots, K_n).$$

In particular, if $|\det(T)| = 1$, then $as_p(K_1, \dots, K_n)$ is affine invariant:

$$as_p(TK_1, \dots, TK_n) = as_p(K_1, \dots, K_n).$$

Proof.

Since $K \in C_+^2$, for any $u \in S^{n-1}$, there exists a unique $x \in \partial K$ such that $u = N_K(x)$ and $f_K(u) = \frac{1}{\kappa_K(x)}$. By Lemma 12 of [38]

$$f_K(u) = \frac{1}{\kappa_K(x)} = \frac{f_{TK}(v)}{\det^2(T) \|T^{-1t}(u)\|^{n+1}}, \quad (2.6)$$

where $v = \frac{T^{-1t}(u)}{\|T^{-1t}(u)\|} \in S^{n-1}$ and where for an operator A , A^t denotes its usual adjoint. On the other hand,

$$h_K(u) = \langle x, u \rangle = \langle Tx, T^{-1t}(u) \rangle = \|T^{-1t}(u)\| \langle Tx, v \rangle = \|T^{-1t}(u)\| h_{TK}(v).$$

Thus, with notation (2.5), for all p ,

$$f_p(K, u) = \frac{f_{TK}(v) h_{TK}^{1-p}(v) \|T^{-1t}(u)\|^{1-p}}{\det^2(T) \|T^{-1t}(u)\|^{n+1}} = \frac{f_p(TK, v)}{\det^2(T) \|T^{-1t}(u)\|^{n+p}}. \quad (2.7)$$

Lemma 10 and its proof in [38] show that -up to a small error-

$$f_{TK}(v) d\sigma(v) = |\det(T)| \|T^{-1t}(u)\| f_K(u) d\sigma(u).$$

Together with (2.6), one gets that (again up to a small error) $\|T^{-1t}(u)\|^{-n} d\sigma(u) = |\det(T)| d\sigma(v)$. Therefore, up to a small error,

$$[f_p(K_1, u) \cdots f_p(K_n, u)]^{\frac{1}{n+p}} d\sigma(u) = |\det(T)|^{\frac{p-n}{n+p}} [f_p(TK_1, v) \cdots f_p(TK_n, v)]^{\frac{1}{n+p}} d\sigma(v).$$

The lemma then follows by integrating over S^{n-1} .

A general version of the classical Alexandrov-Fenchel inequalities for mixed volumes (see [1, 8, 36]) can be written as

$$\prod_{i=0}^{m-1} V(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m) \leq V^m(K_1, \dots, K_n).$$

Here we prove the analogous inequalities for mixed p -affine surface area. For $p = \pm\infty$ and $p = 1$, the inequalities were proved by Lutwak [25, 26]. For $p \geq 1$, inequality (2.8) was proved by Lutwak in [28], with equality if and only if the associated K_i are dilates of each other.

Proposition 2.1 *Let all K_i be convex bodies in C_+^2 with centroid at the origin. If $p \neq -n$, then for $1 \leq m \leq n$*

$$as_p^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} as_p(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m).$$

Equality holds if the K_k , for $k = n - m + 1, \dots, n$ are dilates of each other. If $m = 1$, equality holds trivially.

In particular, if $m = n$,

$$as_p^n(K_1, \dots, K_n) \leq as_p(K_1) \cdots as_p(K_n). \quad (2.8)$$

Proof. Put $g_0(u) = [f_p(K_1, u) \cdots f_p(K_{n-m}, u)]^{\frac{1}{n+p}}$ and for $i = 0, \dots, m - 1$, put $g_{i+1}(u) = [f_p(K_{n-i}, u)]^{\frac{1}{n+p}}$. By Hölder's inequality (see [17])

$$\begin{aligned} as_p(K_1, \dots, K_n) &= \int_{S^{n-1}} g_0(u) g_1(u) \cdots g_m(u) d\sigma(u) \\ &\leq \prod_{i=0}^{m-1} \left(\int_{S^{n-1}} g_0(u) g_{i+1}^m(u) d\sigma(u) \right)^{\frac{1}{m}} \\ &= \prod_{i=0}^{m-1} as_p^{\frac{1}{m}}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i} \cdots K_{n-i}}_m). \end{aligned}$$

As $K_i \in C_+^2$, $f_p(K_i, u) > 0$ for all i and all $u \in S^{n-1}$. Therefore, equality in Hölder's inequality holds if and only if $g_0(u) g_{i+1}^m(u) = \lambda^m g_0(u) g_{j+1}^m(u)$ for some $\lambda > 0$ and all $0 \leq i \neq j \leq m - 1$. This is equivalent to $h_{K_{n-i}}(u)^{1-p} f_{K_{n-i}}(u) = \lambda h_{K_{n-j}}(u)^{1-p} f_{K_{n-j}}(u)$ for all $0 \leq i \neq j \leq m - 1$. This condition holds true if the K_k , for $k = n - m + 1, \dots, n$ are dilates of each other.

Remark. It is an unsolved problem for many p whether $f_p(K, u) = \lambda f_p(L, u)$ guarantees that K and L are dilates of each other. This is equivalent to the uniqueness of the solution of the L_p Minkowski problem: for fixed $\alpha \in \mathbb{R}$, under which conditions on a continuous function $\gamma : S^{n-1} \rightarrow (0, \infty)$, there exists a (unique) convex body K such that $h_K(u)^\alpha f_K(u) = \gamma(u)$ for all $u \in S^{n-1}$. In many cases, the uniqueness of the solution is an open problem. We refer to e.g., [11, 27, 29, 32, 39, 40] for detailed information and more references on the subject. For $p \geq 1, p \neq n$, the solution to the L_p Minkowski problem is known to be unique and for $p = n$, the solution is unique modulo dilates [27]. Therefore, we have the characterization of equality in Proposition 2.1 for $p \geq 1$.

Next, we prove affine isoperimetric inequalities for mixed p -affine surface areas.

Proposition 2.2 *Let all K_i be convex bodies in C_+^2 with centroid at the origin.*

(i) *For $p \geq 0$,*

$$\frac{as_p^n(K_1, \dots, K_n)}{as_p^n(B_2^n, \dots, B_2^n)} \leq \left(\frac{|K_1|}{|B_2^n|} \dots \frac{|K_n|}{|B_2^n|} \right)^{\frac{n-p}{n+p}},$$

with equality if the K_i are ellipsoids that are dilates of one another.

(ii) *For $0 \leq p \leq n$,*

$$\frac{as_p(K_1, \dots, K_n)}{as_p(B_2^n, \dots, B_2^n)} \leq \left(\frac{V(K_1, \dots, K_n)}{V(B_2^n, \dots, B_2^n)} \right)^{\frac{n-p}{n+p}}$$

with equality if the K_i are ellipsoids that are dilates of one another.

In particular, for $p = n$

$$as_n(K_1, \dots, K_n) \leq as_n(B_2^n, \dots, B_2^n),$$

with equality if and only if the K_i are ellipsoids that are dilates of one another.

(iii) *For $p \geq n$,*

$$\frac{as_p(K_1, \dots, K_n)}{as_p(B_2^n, \dots, B_2^n)} \leq \left(\frac{\tilde{V}(K_1, \dots, K_n)}{\tilde{V}(B_2^n, \dots, B_2^n)} \right)^{\frac{n-p}{n+p}},$$

with equality if and only if the K_i are ellipsoids that are dilates of one another.

In particular, for $p = \pm\infty$

$$\tilde{V}(K_1, \dots, K_n) \tilde{V}(K_1^\circ, \dots, K_n^\circ) \leq |B_2^n|^2,$$

with equality if and only if K_i are ellipsoids that are dilates of one another.

Remark. For $1 \leq p \leq n$, inequality (ii) (with equality if and only if the K_i are ellipsoids that are dilates of one another) was proved by Lutwak in [28]. If $K_i = K$ for all i , one recovers the L_p affine isoperimetric inequality proved in [45].

Remark. We cannot expect to get strictly positive lower bounds in Proposition 2.2. As in [45], we consider the convex body $K(R, \varepsilon) \subset \mathbb{R}^2$, obtained as the intersection of four Euclidean balls with radius R centered at $(\pm(R-1), 0)$, $(0, \pm(R-1))$, R arbitrarily large. We then “round” the corners by putting there arcs of Euclidean balls of radius ε , ε arbitrarily small. To obtain a body in C_+^2 , we “bridge” between the R -arcs and ε -arcs by C_+^2 -arcs on a set of arbitrarily small measure. Then $as_p(K(R, \varepsilon)) \leq \frac{16}{R^{2+p}} + 4\pi \varepsilon^{\frac{2}{2+p}}$, which goes to 0 as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$. Choose now R_i and ε_i , $1 \leq i \leq n$, such that $R_i \rightarrow \infty$ and $\varepsilon_i \rightarrow 0$, and let $K_i = K(R_i, \varepsilon_i)$ for $i = 1, 2, \dots, n$. By inequality (2.8), $as_p^n(K_1, \dots, K_n) \leq \prod_{i=1}^n as_p(K(R_i, \varepsilon_i))$ and thus $as_p(K_1, \dots, K_n) \rightarrow 0$ for $p > 0$. A similar construction can be done in higher dimensions.

Proof of Proposition 2.2.

(i) Clearly $as_p(B_2^n, \dots, B_2^n) = as_p(B_2^n) = n|B_2^n|$ for all $p \neq -n$. By inequality (2.8), one gets for all $p \geq 0$

$$\frac{as_p^n(K_1, \dots, K_n)}{as_p^n(B_2^n, \dots, B_2^n)} \leq \frac{as_p(K_1)}{as_p(B_2^n)} \dots \frac{as_p(K_n)}{as_p(B_2^n)} \leq \left(\frac{|K_1|}{|B_2^n|} \dots \frac{|K_n|}{|B_2^n|} \right)^{\frac{n-p}{n+p}}. \quad (2.9)$$

The second inequality follows, for $p \geq 0$, from the L_p affine isoperimetric inequality in [45]. Equality holds true in the L_p isoperimetric inequality [45] if and only if the K_i are all ellipsoids, and equality holds true in inequality (2.8) if the K_i are dilates of one another. Thus, equality holds in (2.9) if the K_i are ellipsoids that are dilates of one another.

(ii) A direct consequence of the classical Alexandrov-Fenchel inequality for mixed volume (see e.g. [7, 21]) is that

$$|K_1| \dots |K_n| \leq V^n(K_1, \dots, K_n).$$

If $0 \leq p \leq n$, then $\frac{n-p}{n+p} \geq 0$. Thus

$$(|K_1| \cdots |K_n|)^{\frac{n-p}{n+p}} \leq [V^n(K_1, \dots, K_n)]^{\frac{n-p}{n+p}}.$$

As $V(B_2^n, \dots, B_2^n) = |B_2^n|$, one gets together with (2.9)

$$\frac{as_p(K_1, \dots, K_n)}{as_p(B_2^n, \dots, B_2^n)} \leq \left(\frac{V(K_1, \dots, K_n)}{V(B_2^n, \dots, B_2^n)} \right)^{\frac{n-p}{n+p}},$$

with equality if the K_i are ellipsoids that are dilates of one another.

(iii) The analogous inequality for dual mixed volume [25] is

$$|K_1| \cdots |K_n| \geq \tilde{V}^n(K_1, \dots, K_n),$$

with equality if and only the K_i are dilates of one another. $p > n$ implies $\frac{n-p}{n+p} < 0$.

Thus

$$(|K_1| \cdots |K_n|)^{\frac{n-p}{n+p}} \leq [\tilde{V}^n(K_1, \dots, K_n)]^{\frac{n-p}{n+p}}.$$

Together with (2.9) and $\tilde{V}(B_2^n, \dots, B_2^n) = |B_2^n|$, one gets

$$\frac{as_p(K_1, \dots, K_n)}{as_p(B_2^n, \dots, B_2^n)} \leq \left(\frac{\tilde{V}(K_1, \dots, K_n)}{\tilde{V}(B_2^n, \dots, B_2^n)} \right)^{\frac{n-p}{n+p}}.$$

As for $p \geq 1$ equality in (2.8) holds if and only if the K_i are dilates of one another, equality holds true here if and only if the K_i are ellipsoids that are dilates of one another.

Proposition 2.3 *Let \mathcal{E} be a centered ellipsoid. If either all $K_i \in C_+^2$ are subsets of \mathcal{E} , for $0 \leq p < n$, or \mathcal{E} is subset of all K_i for $p > n$, then*

$$as_p(K_1, \dots, K_n) \leq as_p(\mathcal{E}).$$

For $p = n$, the inequality holds for all K_i in C_+^2 by Proposition 2.2 (ii).

Remark. This proposition was proved by Lutwak [28] if $K_i = K$ for all i .

Proof. It is enough to prove the proposition for $\mathcal{E} = B_2^n$. For $0 \leq p < n$, one has $\frac{n-p}{n+p} > 0$ and hence $\left(\frac{|K_i|}{|B_2^n|} \right)^{\frac{n-p}{n+p}} \leq 1$ as $K_i \subset B_2^n$. Similarly, $p > n$ implies $\frac{n-p}{n+p} < 0$ and therefore $\left(\frac{|K_i|}{|B_2^n|} \right)^{\frac{n-p}{n+p}} \leq 1$ as $B_2^n \subset K_i$ for all i . In both cases, the proposition follows by inequality (2.9).

The next proposition gives a Blaschke-Santaló type inequality for p -mixed affine surface area. When $K_i = K$ for all i , the proposition was proved in [45].

Proposition 2.4 *Let all K_i be convex bodies in C_+^2 with centroid at the origin. For all $p \geq 0$,*

$$as_p^n(K_1, \dots, K_n) as_p^n(K_1^\circ, \dots, K_n^\circ) \leq n^{2n} |K_1| |K_1^\circ| \cdots |K_n| |K_n^\circ|. \quad (2.10)$$

Furthermore, $as_p(K_1, \dots, K_n) as_p(K_1^\circ, \dots, K_n^\circ) \leq as_p^2(B_2^n)$ with equality if the K_i are ellipsoids that are dilates of one another.

Proof. It follows from (2.8) that for all $p \neq -n$,

$$as_p^n(K_1, \dots, K_n) as_p^n(K_1^\circ, \dots, K_n^\circ) \leq as_p(K_1) as_p(K_1^\circ) \cdots as_p(K_n) as_p(K_n^\circ),$$

with equality if the K_i are dilates of one another. By Corollary 4.1 in [45], for $p \geq 0$,

$$as_p^n(K_1, \dots, K_n) as_p^n(K_1^\circ, \dots, K_n^\circ) \leq n^{2n} |K_1| |K_1^\circ| \cdots |K_n| |K_n^\circ|.$$

Blaschke-Santaló inequality states that $|K| |K^\circ| \leq |B_2^n|^2$ with equality if and only if K is a 0-centered ellipsoid. We apply it to inequality (2.10), and obtain that for $p \geq 0$,

$$as_p(K_1, \dots, K_n) as_p(K_1^\circ, \dots, K_n^\circ) \leq as_p^2(B_2^n, \dots, B_2^n).$$

Equality holds if the K_i are ellipsoids that are dilates of one another.

Theorem 2.1 *Let $s \neq -n, r \neq -n, p \neq -n$ be real numbers. Let all K_i be convex bodies in C_+^2 with centroid at the origin.*

(i) *If $\frac{(n+p)(r-s)}{(n+r)(p-s)} > 1$, then*

$$as_p(K_1, \dots, K_n) \leq (as_r(K_1, \dots, K_n))^{\frac{(p-s)(n+r)}{(r-s)(n+p)}} (as_s(K_1, \dots, K_n))^{\frac{(r-p)(n+s)}{(r-s)(n+p)}}.$$

(ii) *If $\frac{n+p}{n+r} > 1$, then*

$$as_p(K_1, \dots, K_n) \leq (as_r(K_1, \dots, K_n))^{\frac{n+r}{n+p}} \left(n\tilde{V}(K_1^\circ, \dots, K_n^\circ) \right)^{\frac{p-r}{n+p}}. \quad (2.11)$$

Remark. When all K_i coincide with K , (i) of Theorem 2.1 was proved in [45].

Proof.

(i) By Hölder's inequality -which enforces the condition $\frac{(n+p)(r-s)}{(n+r)(p-s)} > 1$,

$$\begin{aligned}
as_p(K_1, \dots, K_n) &= \int_{S^{n-1}} [f_p(K_1, u) \cdots f_p(K_n, u)]^{\frac{1}{n+p}} d\sigma(u) \\
&= \int_{S^{n-1}} \left([f_r(K_1, u) \cdots f_r(K_n, u)]^{\frac{1}{n+r}} \right)^{\frac{(n+r)(p-s)}{(n+p)(r-s)}} \\
&\quad \left([f_s(K_1, u) \cdots f_s(K_n, u)]^{\frac{1}{n+s}} \right)^{\frac{(n+s)(r-p)}{(n+p)(r-s)}} d\sigma(u) \\
&\leq (as_r(K_1, \dots, K_n))^{\frac{(p-s)(n+r)}{(r-s)(n+p)}} (as_s(K_1, \dots, K_n))^{\frac{(r-p)(n+s)}{(r-s)(n+p)}}.
\end{aligned}$$

(ii) Similarly, again using Hölder's inequality -which now enforces the condition $\frac{n+p}{n+r} > 1$

$$\begin{aligned}
as_p(K_1, \dots, K_n) &= \int_{S^{n-1}} [f_p(K_1, u) \cdots f_p(K_n, u)]^{\frac{1}{n+p}} d\sigma(u) \\
&= \int_{S^{n-1}} \left([f_r(K_1, u) \cdots f_r(K_n, u)]^{\frac{1}{n+r}} \right)^{\frac{n+r}{n+p}} \\
&\quad \left[\frac{1}{h_{K_1}(u) \cdots h_{K_n}(u)} \right]^{\frac{p-r}{n+p}} d\sigma(u) \\
&\leq (as_r(K_1, \dots, K_n))^{\frac{n+r}{n+p}} (as_\infty(K_1, \dots, K_n))^{\frac{p-r}{n+p}}.
\end{aligned}$$

Together with (1.2), this completes the proof.

Remark. The condition $\frac{(n+p)(r-s)}{(n+r)(p-s)} > 1$ implies 8 cases: $-n < s < p < r$, $s < -n < r < p$, $p < r < -n < s$, $r < p < s < -n$, $s < p < r < -n$, $p < s < -n < r$, $r < -n < s < p$ and $-n < r < p < s$.

In [45], we proved monotonicity properties of $\left(\frac{as_r(K)}{n|K|} \right)^{\frac{n+r}{r}}$. Here we prove similar results for mixed p -affine surface area.

Proposition 2.5 *Let all $K_i \in C_+^2$ be convex bodies with centroid at the origin.*

(i) *If $-n < r < p$ or $r < p < -n$, one has*

$$\left(\frac{as_p(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+p} \leq \left(\frac{as_r(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+r}.$$

(ii) If $0 < p < r$, or $p < r < -n$, or $r < -n < 0 < p$, or $-n < p < r < 0$, one has

$$\left(\frac{as_p(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+p}{p}} \leq \left(\frac{as_r(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+r}{r}}.$$

Proof.

(i) We divide both sides of inequality (2.11) by $n\tilde{V}(K_1^\circ, \dots, K_n^\circ)$, and get for $\frac{n+p}{n+r} > 1$,

$$\frac{as_p(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \leq \left(\frac{as_r(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{\frac{n+r}{n+p}}. \quad (2.12)$$

Condition $\frac{n+p}{n+r} > 1$ implies that $-n < r < p$ or $p < r < -n$. If $-n < r < p$, $n+p > 0$ and therefore, inequality (2.12) implies inequality (i). If $p < r < -n$, $n+p < 0$ and therefore,

$$\left(\frac{as_p(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+p} \geq \left(\frac{as_r(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+r}.$$

Switching r and p , one obtains the inequality in (i): for $r < p < -n$,

$$\left(\frac{as_p(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+p} \leq \left(\frac{as_r(K_1, \dots, K_n)}{n\tilde{V}(K_1^\circ, \dots, K_n^\circ)} \right)^{n+r}.$$

(ii) Let $s = 0$ in inequality (i) of Theorem 2.1. Then for $\frac{r(n+p)}{p(n+r)} > 1$,

$$as_p(K_1, \dots, K_n) \leq (as_r(K_1, \dots, K_n))^{\frac{p(n+r)}{r(n+p)}} (as_0(K_1, \dots, K_n))^{\frac{(r-p)n}{r(n+p)}}.$$

We divide both sides of the inequality by $as_0(K_1, \dots, K_n)$ and get

$$\frac{as_p(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \leq \left(\frac{as_r(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{p(n+r)}{r(n+p)}}.$$

The condition $\frac{r(n+p)}{p(n+r)} > 1$ implies that $0 < p < r$, or $p < r < -n$, or $-n < r < p < 0$, or $r < -n < 0 < p$. In the cases $0 < p < r$, or $p < r < -n$, or $r < -n < 0 < p$, one has $\frac{n+p}{p} > 0$ and therefore inequality (ii) holds true. On the other hand, if $-n < r < p < 0$, then $\frac{n+p}{p} < 0$ and hence,

$$\left(\frac{as_r(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+r}{r}} \leq \left(\frac{as_p(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+p}{p}}.$$

Switching r and p , one gets inequality (ii): if $-n < p < r < 0$, then

$$\left(\frac{as_p(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+p}{p}} \leq \left(\frac{as_r(K_1, \dots, K_n)}{as_0(K_1, \dots, K_n)} \right)^{\frac{n+r}{r}}.$$

Now we treat the case $p = -n$. The mixed $(-n)$ -affine surface area of K_1, \dots, K_n is defined as

$$as_{-n}(K_1, \dots, K_n) = \max_{u \in S^{n-1}} \left[f_{K_1}(u)^{\frac{1}{2n}} h_{K_1}(u)^{\frac{n+1}{2n}} \cdots f_{K_n}(u)^{\frac{1}{2n}} h_{K_n}(u)^{\frac{n+1}{2n}} \right]. \quad (2.13)$$

It is easy to verify that $as_{-n}(K, \dots, K)$ equals to $as_{-n}(K)$, the L_{-n} affine surface area of K [34]. We have the following proposition.

Proposition 2.6 *Let all K_i be convex bodies in C_+^2 with centroid at the origin. Let $p \neq -n$ and $s \neq -n$ be real numbers.*

(i) *Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an invertible linear transform. Then*

$$as_{-n}(TK_1, \dots, TK_n) = |\det(T)| as_{-n}(K_1, \dots, K_n).$$

In particular, if $|\det(T)| = 1$, then $as_{-n}(K_1, \dots, K_n)$ is affine invariant:

$$as_{-n}(TK_1, \dots, TK_n) = as_{-n}(K_1, \dots, K_n).$$

(ii) *Alexandrov-Fenchel type inequalities*

$$as_{-n}^m(K_1, \dots, K_n) \leq \prod_{i=0}^{m-1} as_{-n}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m),$$

with equality if the K_j , for $j = n - m + 1, \dots, n$ are dilates.

In particular, if $m = n$,

$$as_{-n}^n(K_1, \dots, K_n) \leq as_{-n}(K_1) \cdots as_{-n}(K_n).$$

(iii) *If $\frac{n(s-p)}{(n+p)(n+s)} \geq 0$, then*

$$as_p(K_1, \dots, K_n) \leq (as_{-n}(K_1, \dots, K_n))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K_1, \dots, K_n).$$

(iv) *If $\frac{n(s-p)}{(n+p)(n+s)} \leq 0$, then*

$$as_p(K_1, \dots, K_n) \geq (as_{-n}(K_1, \dots, K_n))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K_1, \dots, K_n).$$

Proof.

(i) By formula (2.7), $f_{-n}(TK, v) = \det(T)^2 f_{-n}(K, u)$. Therefore

$$\begin{aligned} as_{-n}(TK_1, \dots, TK_n) &= \max_{v \in S^{n-1}} [f_{-n}(TK_1, v) \cdots f_{-n}(TK_n, v)]^{\frac{1}{2n}} \\ &= |\det(T)| \max_{u \in S^{n-1}} [f_{-n}(K_1, u) \cdots f_{-n}(K_n, u)]^{\frac{1}{2n}} \\ &= |\det(T)| as_{-n}(K_1, \dots, K_n). \end{aligned}$$

(ii) Let $\tilde{g}_0(u) = f_{K_1}(u)^{\frac{1}{2n}} h_{K_1}(u)^{\frac{n+1}{2n}} \cdots f_{K_{n-m}}(u)^{\frac{1}{2n}} h_{K_{n-m}}(u)^{\frac{n+1}{2n}}$ and $\tilde{g}_{i+1}(u) = f_{K_{n-i}}(u)^{\frac{1}{2n}} h_{K_{n-i}}(u)^{\frac{n+1}{2n}}$, for $i = 0, \dots, m-1$. Then

$$\begin{aligned} as_{-n}(K_1, \dots, K_n) &= \max_{u \in S^{n-1}} \tilde{g}_0(u) \tilde{g}_1(u) \cdots \tilde{g}_m(u) \\ &\leq \prod_{i=0}^{m-1} \left(\max_{u \in S^{n-1}} \tilde{g}_i(u) \tilde{g}_{i+1}^m(u) \right)^{\frac{1}{m}} \\ &= \prod_{i=0}^{m-1} as_{-n}^{\frac{1}{m}}(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_m). \end{aligned}$$

Equality holds if and only if for all i , $0 \leq i \leq m-1$, $\tilde{g}_i(u) \tilde{g}_{i+1}^m(u)$ attain their maximum at the same direction u_0 . This condition holds true if the K_j , for $j = n-m+1, \dots, n$, are dilates.

(iii) and (iv)

$$\begin{aligned} as_p(K_1, \dots, K_n) &= \int_{S^{n-1}} [f_p(K_1, u) \cdots f_p(K_n, u)]^{\frac{1}{n+p}} d\sigma(u) \\ &= \int_{S^{n-1}} [f_s(K_1, u) \cdots f_s(K_n, u)]^{\frac{1}{n+s}} \\ &\quad \left(h_{K_1}^{\frac{n+1}{2n}}(u) f_{K_1}^{\frac{1}{2n}}(u) \cdots h_{K_n}^{\frac{n+1}{2n}}(u) f_{K_n}^{\frac{1}{2n}}(u) \right)^{\frac{2n(s-p)}{(n+p)(n+s)}} d\sigma(u) \end{aligned}$$

which is

$$\leq (as_{-n}(K_1, \dots, K_n))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K_1, \dots, K_n), \quad \text{if } \frac{n(s-p)}{(n+p)(n+s)} \geq 0,$$

and

$$\geq (as_{-n}(K_1, \dots, K_n))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_s(K_1, \dots, K_n), \quad \text{if } \frac{n(s-p)}{(n+p)(n+s)} \leq 0.$$

2.2 i -th mixed p -affine surface area and related inequalities

For all $p \geq 1$ and all real i , the i -th mixed p -affine surface area of $K, L \in C_+^2$ is defined as [26, 42]

$$as_{p,i}(K, L) = \int_{S^{n-1}} f_p(K, u)^{\frac{n-i}{n+p}} f_p(L, u)^{\frac{i}{n+p}} d\sigma(u).$$

Recall that $f_p(K, u) = f_K(u)h_K^{1-p}(u)$. Here we further generalize this definition to all $p \neq -n$ and all i . An analogous definition for the i -th mixed $(-n)$ -affine surface area of K and L is

$$as_{-n,i}(K, L) = \max_{u \in S^{n-1}} \left[f_K(u)^{\frac{n-i}{2n}} h_K(u)^{\frac{(n+1)(n-i)}{2n}} f_L(u)^{\frac{i}{2n}} h_L(u)^{\frac{(n+1)i}{2n}} \right].$$

When $i \in \mathbb{N}$, $0 \leq i \leq n$, then, for all p , the i -th mixed p -affine surface area of K and L is

$$as_{p,i}(K, L) = as_p(\underbrace{K, \dots, K}_{n-i}, \underbrace{L, \dots, L}_i).$$

Clearly, for all p , $as_{p,0}(K, L) = as_p(K)$, and $as_{p,n}(K, L) = as_p(L)$. When $L = B_2^n$, we write $as_{p,i}(K)$ for $as_{p,i}(K, B_2^n)$. Thus

$$\begin{aligned} as_{p,i}(K) &= \int_{S^{n-1}} f_p(K, u)^{\frac{n-i}{n+p}} d\sigma(u), \quad \text{for } p \neq -n, \\ as_{p,i}(K) &= \max_{u \in S^{n-1}} \left[f_K(u)^{\frac{n-i}{2n}} h_K(u)^{\frac{(n+1)(n-i)}{2n}} \right], \quad \text{for } p = -n. \end{aligned}$$

In particular, $as_{1,-1}(K) = \int_{S^{n-1}} f_K(u) d\sigma(u)$ is the surface area of K .

The next proposition and its proof is similar to Proposition 2.1 and its proof. Therefore we omit it.

Proposition 2.7 *Let K and L be convex bodies in C_+^2 with centroid at the origin. Let $i \in \mathbb{R}$ and $s \neq -n$, $r \neq -n$, and $p \neq -n$ be real numbers.*

(i) *If $\frac{(n+p)(r-s)}{(n+r)(p-s)} > 1$, then*

$$as_{p,i}(K, L) \leq (as_{r,i}(K, L))^{\frac{(p-s)(n+r)}{(r-s)(n+p)}} (as_{s,i}(K, L))^{\frac{(r-p)(n+s)}{(r-s)(n+p)}}.$$

(ii) *If $\frac{n+p}{n+r} > 1$, then*

$$as_{p,i}(K, L) \leq (as_{r,i}(K, L))^{\frac{n+r}{n+p}} \left(n\tilde{V}_i(K^\circ, L^\circ) \right)^{\frac{p-r}{n+p}}$$

where $\tilde{V}_i(K^\circ, L^\circ) = \frac{1}{n} \int_{S^{n-1}} \frac{1}{h_K(u)^{n-i} h_L(u)^i} d\sigma(u)$ for all i .

(iii) If $\frac{n(s-p)}{(n+p)(n+s)} \geq 0$, then

$$as_{p,i}(K, L) \leq (as_{-n,i}(K, L))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_{s,i}(K, L).$$

(iv) If $\frac{n(s-p)}{(n+p)(n+s)} \leq 0$, then

$$as_{p,i}(K, L) \geq (as_{-n,i}(K, L))^{\frac{2n(s-p)}{(n+p)(n+s)}} as_{s,i}(K, L).$$

The following proposition was proved in [26, 42] for $p \geq 1$.

Proposition 2.8 *Let K and L be convex bodies in C_+^2 with centroid at the origin. If $j < i < k$ or $k < i < j$ (equivalently, $\frac{k-i}{k-i} > 1$), then for all p ,*

$$as_{p,i}(K, L) \leq as_{p,j}(K, L)^{\frac{k-i}{k-j}} as_{p,k}(K, L)^{\frac{i-j}{k-j}},$$

with equality if K and L are dilates of each other.

In particular,

$$as_{p,i}(K) \leq as_{p,j}(K)^{\frac{k-i}{k-j}} as_{p,k}(K)^{\frac{i-j}{k-j}},$$

with equality if K is a ball.

For $p \neq -n$, the proof is the same as the proof in [26, 42]. For $p = -n$, it is similar to the proof of Proposition 2.1. Note that for $i \in \mathbb{N}$, $0 < i < m$, $m = j$ and $k = 0$, the proposition is a direct consequence of Proposition 2.1.

In Proposition 2.8, if $j = 0$ and $k = n$, then for all p and $0 \leq i \leq n$

$$as_{p,i}^n(K, L) \leq as_p^{n-i}(K) as_p^i(L). \quad (2.14)$$

If we let $i = 0$ and $j = n$, then for all $k \leq 0$ and for all p

$$as_{p,k}^n(K, L) \geq as_p^{n-k}(K) as_p^k(L). \quad (2.15)$$

Let $i = n$, $j = 0$ and $k > n$. Then inequality (2.15) also holds true for $k \geq n$ and all p . In both (2.14) and (2.15), equality holds for all p if K and L are dilates.

From inequality (2.14) and Corollary 4.1 in [45], one gets that

$$\begin{aligned} as_{p,i}^n(K, L)as_{p,i}^n(K^\circ, L^\circ) &\leq (as_p(K)as_p(K^\circ))^{n-i}(as_p(L)as_p(L^\circ))^i \\ &\leq n^{2n}(|K||K^\circ|)^{n-i}(|L||L^\circ|)^i \end{aligned} \quad (2.16)$$

holds true for all $p \geq 0$ and $0 < i < n$. The inequality also holds if $i = 0$ and $i = n$ [45]. We apply Blaschke-Santaló inequality to inequality (2.16) and get

$$as_{p,i}(K, L)as_{p,i}(K^\circ, L^\circ) \leq as_p^2(B_2^n)$$

for all $p \geq 0$ and $0 \leq i \leq n$. Equality holds true if K and L are ellipsoids that are dilates of each other. Hence we have proved the following proposition, which, for $p \geq 1$, was proved in [42].

Proposition 2.9 *Let K and L be convex bodies in C_+^2 with centroid at the origin. If $p \geq 0$ and $0 \leq i \leq n$, then*

$$as_{p,i}(K, L)as_{p,i}(K^\circ, L^\circ) \leq as_p^2(B_2^n),$$

with equality if K and L are ellipsoids that are dilates of each other.

We now establish isoperimetric inequalities for $as_{p,i}(K)$.

Proposition 2.10 *Let $K \in C_+^2$ be a convex body with centroid at the origin.*

(i) If $p \geq 0$ and $0 \leq i \leq n$, then

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)^n}}$$

with equality if K is a ball. Moreover, $as_{p,i}(K)as_{p,i}(K^\circ) \leq as_p^2(B_2^n)$ with equality if K is a ball.

(ii) If $p \geq 0$ and $i \geq n$, then

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)^n}}$$

with equality if K is a ball. Moreover, $as_{p,i}(K)as_{p,i}(K^\circ) \geq as_p^2(B_2^n)$ with equality if K is a ball.

(iii) If $-n < p < 0$ and $i \leq 0$, then

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}}$$

with equality if K is a ball. Moreover, $as_{p,i}(K)as_{p,i}(K^\circ) \geq c^{n-i}as_p^2(B_2^n)$ where c is the universal constant in the inverse Santaló inequality [6, 20].

(iv) If $p < -n$ and $i \leq 0$, then

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq c^{\frac{(n-i)p}{n+p}} \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}}.$$

Moreover, $as_{p,i}(K)as_{p,i}(K^\circ) \geq c^{n-i}as_p^2(B_2^n)$ where c is the same constant as in (iii).

(v) If $i \leq 0$, then

$$\frac{as_{-n,i}(K)}{as_{-n,i}(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{n-i}{n}}.$$

Moreover, $as_{-n,i}(K)as_{-n,i}(K^\circ) \geq as_{-n,i}^2(B_2^n)$.

Proof.

(i) For $i = n$, the equality holds trivially. For $i = 0$, the inequality was proved in [45]. We now prove the case $0 < i < n$. $L = B_2^n$ in inequality (2.14) gives

$$\left(\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \right)^n \leq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{n-i} \quad (2.17)$$

for all $p \neq -n$ and $0 \leq i \leq n$. We also use that $as_{p,i}(B_2^n) = as_p(B_2^n)$. Then, as $as_p(B_2^n) = n|B_2^n|$, we get for all $p \geq 0$ and $0 \leq i \leq n$, the following isoperimetric inequality as a consequence of the L_p affine isoperimetric inequality in [45]

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \leq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{\frac{n-i}{n}} \leq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}},$$

with equality if K is a ball. The inequality $as_{p,i}(K)as_{p,i}(K^\circ) \leq as_{p,i}^2(B_2^n)$ follows from Proposition 2.9 with $L = B_2^n$.

(ii) For $i = n$, the equality holds trivially. Similarly, let $L = B_2^n$ in inequality (2.15), then for all $p \neq -n$, and $i \geq n$ or $i \leq 0$,

$$\left(\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \right)^n \geq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{n-i}. \quad (2.18)$$

Hence for $i \geq n$ and $p \geq 0$, the L_p affine isoperimetric inequality in [45] implies that

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{\frac{n-i}{n}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}}$$

with equality if K is a ball. Moreover, by Corollary 4.1 (i) in [45] and the remark after it, one has for all $i \geq n$

$$\left(\frac{as_{p,i}(K)as_{p,i}(K^\circ)}{as_{p,i}^2(B_2^n)} \right)^n \geq \left(\frac{as_p(K)as_p(K^\circ)}{as_p^2(B_2^n)} \right)^{n-i} \geq 1,$$

or equivalently, $as_{p,i}(K)as_{p,i}(K^\circ) \geq as_{p,i}^2(B_2^n)$, with equality if K is a ball.

(iii) If $i \leq 0$ and $-n < p < 0$, inequality (2.18) and Theorem 4.2 (ii) of [45] imply that

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{\frac{n-i}{n}} \geq \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}}$$

with equality if K is a ball. By Corollary 4.1 (ii) of [45] and the remark after it,

$$\left(\frac{as_{p,i}(K)as_{p,i}(K^\circ)}{as_{p,i}^2(B_2^n)} \right)^n \geq \left(\frac{as_p(K)as_p(K^\circ)}{as_p^2(B_2^n)} \right)^{n-i} \geq c^{n(n-i)},$$

or equivalently, $as_{p,i}(K)as_{p,i}(K^\circ) \geq c^{n-i}as_{p,i}^2(B_2^n)$ where c is the constant in the inverse Santaló inequality [6, 20].

(iv) If $i \leq 0$ and $p < -n$ inequality (2.18) and Theorem 4.2 (iii) of [45] imply that

$$\frac{as_{p,i}(K)}{as_{p,i}(B_2^n)} \geq \left(\frac{as_p(K)}{as_p(B_2^n)} \right)^{\frac{n-i}{n}} \geq c^{\frac{(n-i)p}{n+p}} \left(\frac{|K|}{|B_2^n|} \right)^{\frac{(n-p)(n-i)}{(n+p)n}}.$$

The proof of $as_{p,i}(K)as_{p,i}(K^\circ) \geq c^{n-i}as_{p,i}^2(B_2^n)$ is same as in (iii).

(v) Inequality (2.15) implies that $as_{-n,i}(K)^n \geq as_{-n}(K)^{n-i}$ for $i \leq 0$. As $as_{-n,i}(B_2^n) = 1$ for all i ,

$$\left(\frac{as_{-n,i}(K)}{as_{-n,i}(B_2^n)} \right)^n \geq \left(\frac{as_{-n}(K)}{as_{-n}(B_2^n)} \right)^{n-i} \geq \left(\frac{|K|}{|B_2^n|} \right)^{n-i}.$$

The second inequality follows from the L_{-n} affine isoperimetric inequality in [45].

Moreover, by Corollary 4.2 in [45], for $i \leq 0$,

$$\left(\frac{as_{-n,i}(K)as_{-n,i}(K^\circ)}{as_{-n,i}^2(B_2^n)} \right)^n \geq \left(\frac{as_{-n}(K)as_{-n}(K^\circ)}{as_{-n}^2(B_2^n)} \right)^{n-i} \geq 1,$$

or equivalently, $as_{-n,i}(K)as_{-n,i}(K^\circ) \geq as_{-n,i}^2(B_2^n)$.

Remark. The example $K(R, \varepsilon)$ mentioned in the remarks after Proposition 2.2 shows that we cannot expect to get strictly positive lower bounds in (i) of Proposition 2.10 for $p > 0$ and $0 \leq i < n$. In fact, by inequality (2.17), one has

$$\left(\frac{as_{p,i}(K(R, \varepsilon))}{as_{p,i}(B_2^n)} \right)^n \leq \left(\frac{as_p(K(R, \varepsilon))}{as_p(B_2^n)} \right)^{n-i}.$$

As in [45], $as_p(K(R, \varepsilon)) \rightarrow 0$ for $p > 0$ as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$. $0 \leq i < n$ implies that $n - i > 0$, and therefore $as_{p,i}(K(R, \varepsilon)) \rightarrow 0$.

This example also shows that, likewise, we cannot expect finite upper bounds in (ii) (for $p > 0$ and $i > n$), (iii) (for $-n < p < 0$ and $i \leq 0$), and (iv) (for $p < -n$ and $i \leq 0$), of Proposition 2.10. For instance, if $i \leq 0$, by inequality (2.18), one has

$$\left(\frac{as_{p,i}(K(R, \varepsilon))}{as_{p,i}(B_2^n)} \right)^n \geq \left(\frac{as_p(K(R, \varepsilon))}{as_p(B_2^n)} \right)^{n-i}.$$

For $-2 < p < 0$, one has $as_p(K(R, \varepsilon)) \rightarrow \infty$ as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$. Therefore, if $i \leq 0$, we obtain that $as_{p,i}(K(R, \varepsilon)) \rightarrow \infty$ as $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$, i.e., there are no finite upper bounds in (iii).

Remark. In (iv), if $p = -\infty$, then for all $i \leq 0$,

$$\int \int_{S^{n-1} \times S^{n-1}} (h_K(u)h_{K^\circ}(v))^{i-n} d\sigma(u) d\sigma(v) \geq c^{n-i} as_{p,i}^2(B_2^n)$$

or equivalently, for all $i \leq 0$,

$$\int \int_{S^{n-1} \times S^{n-1}} (\rho_K(u)\rho_{K^\circ}(v))^{n-i} d\sigma(u) d\sigma(v) \geq c^{n-i} as_{p,i}^2(B_2^n).$$

In particular, if $i = 0$, this is equivalent to the inverse Santaló inequality [6].

3 Illumination surface bodies

We now define a new family of bodies associated with a given convex body K . These new bodies are a variant of the illumination bodies [44] (compare also [38]).

Definition 3.1 (Illumination surface body) Let $s \geq 0$ and $f : \partial K \rightarrow \mathbb{R}$ be a nonnegative, integrable function. The illumination surface body $K^{f,s}$ is defined as

$$K^{f,s} = \left\{ x : \mu_f(\partial K \cap \overline{[x, K] \setminus K}) \leq s \right\}.$$

Obviously, $K \subseteq K^{f,s}$ for any $s \geq 0$ and any nonnegative, integrable function f . Moreover, $K^{f,s} \subseteq K^{f,t}$ for any $0 \leq s \leq t$.

Notice also that $K^{f,s}$ needs to be neither bounded nor convex:

Example 3.1 Let $K = B_\infty^2 = \{x \in \mathbb{R}^2 : \max_{1 \leq i \leq 2} |x_i| \leq 1\}$ and

$$f(x) = \begin{cases} \frac{1}{12}, & x \in [(-1, 1), (1, 1)] \cup [(1, 1), (1, -1)] \\ \frac{1}{6}, & \text{otherwise} \end{cases}$$

$K^{f,s}$ is calculated as follows. If $s < \frac{1}{6}$, then $K^{f,s} = K$.

If $s \in [\frac{1}{6}, \frac{1}{3})$, then

$$K^{f,s} = \{(x_1, x_2) : x_1 \geq -1, x_2 \in [-1, 1]; \text{ or } x_2 \geq -1, x_1 \in [-1, 1]\}.$$

If $s \in [\frac{1}{3}, \frac{1}{2})$, then

$$K^{f,s} = \{(x_1, x_2) : x_1, x_2 \geq -1; \text{ or } x_1 \leq -1, x_2 \in [-1, 1]; \text{ or } x_2 \leq -1, x_1 \in [-1, 1]\}.$$

If $s \in [\frac{1}{2}, \frac{2}{3})$, then $K^{f,s} = \{(x_1, x_2) : x_1 \geq -1 \text{ or } x_2 \geq -1\}$.

Except for $s < 1/6$, all of them are neither bounded nor convex.

If $s \geq \frac{2}{3}$, then $K^{f,s} = \mathbb{R}^2$

The following lemmas describe some of the properties of the bodies $K^{f,s}$.

Lemma 3.1 Let $s \geq 0$ and $f : \partial K \rightarrow \mathbb{R}$ be a nonnegative, integrable function. Then

(i) $K^{f,s} = \bigcap_{\delta > 0} K^{f,s+\delta}$.

(ii) $K^{f,s}$ is star convex, i.e., for all $x \in K^{f,s}$: $[0, x] \subset K^{f,s}$.

Proof.

(i) We only need to show that $K^{f,s} \supseteq \bigcap_{\delta > 0} K^{f,s+\delta}$. Let $x \in \bigcap_{\delta > 0} K^{f,s+\delta}$. Then for all $\delta > 0$, $\mu_f(\partial K \cap \overline{[x, K] \setminus K}) \leq s + \delta$. Thus, letting $\delta \rightarrow 0$, $\mu_f(\partial K \cap \overline{[x, K] \setminus K}) \leq s$.

(ii) Let $x \in K^{f,s}$. We claim that $[0, x] \subseteq K^{f,s}$. Let $y \in [0, x]$. Since $y \in [0, x] \subset [x, K]$, we have $[y, K] \setminus K \subseteq [x, K] \setminus K$ and thus $\partial K \cap \overline{[y, K] \setminus K} \subseteq \partial K \cap \overline{[x, K] \setminus K}$. This implies that

$$\mu_f(\partial K \cap \overline{[y, K] \setminus K}) \leq \mu_f(\partial K \cap \overline{[x, K] \setminus K}) \leq s$$

and hence $y \in K^{f,s}$.

Remark. We *can not* expect $K^{f,s}$ to be convex, even for $K = B_2^n$ and f smooth. Indeed, let $K = B_2^2$ and $s = \frac{1}{64}$. Define

$$f(x) = \begin{cases} \frac{1}{4\pi}, & x \text{ is in the first and third quadrant} \\ \frac{1}{16\pi}, & x \text{ is in the fourth quadrant} \\ \frac{23}{16\pi}, & x \text{ is in the second quadrant} \end{cases}$$

Then $K^{f,s}$ is not convex. In fact, $K^{f,s}$ contains the arc from the point $(\tan(\frac{\pi}{32}), 1)$ to the point $(1, \tan(\frac{\pi}{32}))$ of the Euclidean ball centered at 0 with radius $r = \sec(\frac{\pi}{32})$. Moreover, the point $(\sec(\frac{\pi}{20}), 0)$ is on the boundary of $K^{f,s}$. The tangent line at $(1, \tan(\frac{\pi}{32}))$ of $B_2^2(0, r)$ is $y = \frac{-x+r^2}{\sqrt{r^2-1}}$. This tangent line intersects the x -axis at $(r^2, 0) = (\sec^2(\frac{\pi}{32}), 0)$. Since $\sec^2(\frac{\pi}{32}) \sim 1.009701 < \sec(\frac{\pi}{20}) \sim 1.01264$, $K^{f,s}$ is not convex.

We can modify f so that it becomes smooth also at the points $(\pm 1, 0)$ and $(0, \pm 1)$ and $\partial K^{f,s}$ still intersects the positive x -axis at the point $(\sec(\frac{\pi}{20}), 0)$. Therefore, $K^{f,s}$ is not convex, even if f is smooth.

Lemma 3.2 *Let $s \geq 0$ and $f : \partial K \rightarrow \mathbb{R}$ be an integrable, μ_K -almost everywhere strictly positive function. Then*

(i) $K = K^{f,0}$.

(ii) There exists $s_0 > 0$, such that for all $0 \leq s \leq s_0$, $K^{f,s}$ is bounded.

Proof. (i) It is enough to prove that $K^{f,0} \subseteq K$. Suppose this is not the case. Then there is $x \in K^{f,0}$ but $x \notin K$. Since $0 \in \text{int}(K)$, there is $\alpha > 0$ such that

$$B_2^n(0, \alpha) \subseteq K \subseteq B_2^n(0, 1/\alpha). \quad (3.19)$$

Let $y \in [0, x] \cap \partial K$ and $\text{Con}(x, \alpha) = [x, B_2^n(0, \alpha)]$ be the convex hull of x and $B_2^n(0, \alpha)$. $H(y, N_K(y)) \cap \text{Con}(x, \alpha)$ contains a $(n-1)$ -dimensional Euclidean ball with radius (at least) $r_1 = \alpha \frac{\|x-y\|}{\|x\|} > 0$.

Hence $\mu_K(\partial K \cap \overline{[x, K]} \setminus K) \geq |H(y, N_K(y)) \cap \text{Con}(x, \alpha)| \geq r_1^{n-1} |B_2^{n-1}| > 0$. Let

$$E_j = \left\{ z \in \partial K \cap \overline{[x, K]} \setminus K : f(z) \geq \frac{1}{j} \right\}, \quad j = 1, 2, \dots$$

As $\mu_K(\{z \in \partial K : f(z) = 0\}) = 0$ and $E_j \subseteq E_{j+1}$ for all j ,

$$\mu_K(\partial K \cap \overline{[x, K]} \setminus K) = \mu_K\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \rightarrow \infty} \mu_K(E_j).$$

Therefore there exists j_1 such that $\mu_K(E_{j_1}) > 0$. Thus

$$\mu_f(\partial K \cap \overline{[x, K] \setminus K}) \geq \mu_f(E_{j_1}) \geq \frac{\mu_K(E_{j_1})}{j_1} > 0$$

which contradicts that $x \in K^{f,0}$.

(ii) is an immediate consequence of Lemma 3.1 (i) and Lemma (3.2) (i). Indeed, these lemmas imply that $K = K^{f,0} = \bigcap_{s>0} K^{f,s}$. So, also using (3.19), there exists $s_0 > 0$ such that for all $0 \leq s \leq s_0$, $K^{f,s} \subset 2K \subset B_2^n(0, \frac{2}{\alpha})$. In particular, $K^{f,s_0} \subset B_2^n(0, \frac{2}{\alpha})$.

Remark. The assumption that f is μ_K -almost everywhere strictly positive is necessary in order that $K^{f,0} = K$. To see that, let $K = B_2^2$ and

$$f(x, y) = \begin{cases} 0 & x = \sqrt{1 - y^2}, y \in [-1, 1], \\ \frac{1}{\pi} & \text{otherwise.} \end{cases}$$

Then $K^{f,0} = K \cup \{(x, y) : x \geq 0, |y| \leq 1\}$.

This example also shows that there is no s_0 such that $K^{f,s}$ is bounded for all $0 \leq s \leq s_0$ unless f is μ_K -almost everywhere strictly positive.

Let K be a convex body with $0 \in \text{int}(K)$. Let $f : \partial K \rightarrow \mathbb{R}$ be an integrable, μ_K -almost everywhere strictly positive function. For $x \notin K$, let $t_0 = t_0(x)$ be the strictly positive real number such that $t_0 x = \partial K \cap [0, x]$. Define $h_x(t)$ to be

$$h_x(t) = \mu_f(\partial K \cap \overline{[tx, K] \setminus K}), \quad t \geq t_0.$$

Clearly $h_x(t_0) = 0$. Moreover $h_x(t) \leq s$ if $tx \in K^{f,s}$, and $h_x(t) > s$ if $tx \notin K^{f,s}$.

Lemma 3.3 *Let K be a convex body in \mathbb{R}^n and $f : \partial K \rightarrow \mathbb{R}$ be an integrable, μ_K -almost everywhere strictly positive function.*

(i) $h_x(t)$ is increasing and left continuous on $[t_0, \infty)$.

(ii) $K^{f,s}$ is closed for all $s \geq 0$. In particular, it is compact for all $0 \leq s \leq s_0$.

If K is in addition strictly convex, then

(iii) $h_x(t)$ is continuous on $[t_0, \infty)$.

(iv) For any $0 \leq s \leq s_0$ and $x \in \partial K^{f,s}$, one has $\mu_f(\partial K \cap \overline{[x, K] \setminus K}) = s$.

Proof.

(i) If $t_1 \leq t_2$, then $\partial K \cap \overline{[t_1x, K]} \setminus K \subseteq \partial K \cap \overline{[t_2x, K]} \setminus K$. Thus $h_x(t_1) \leq h_x(t_2)$.

Let now $t > t_0$ and $(t_m)_{m \in \mathbb{N}}$ be a sequence, increasing to t . Then, by monotonicity of h_x , $h_x(t_m) \leq h_x(t)$ for all m and thus $\lim_m h_x(t_m) \leq h_x(t)$. We have to show that $\lim_m h_x(t_m) \geq h_x(t)$. Let $y \in \text{relint}_{\partial K}(\partial K \cap \overline{[tx, K]} \setminus K)$, where $\text{relint}_B(A)$ is the relative (with respect to B) interior of a set $A \subseteq B$, i.e., $\text{relint}_B(A) = \{x \in A : \text{there is a } \delta > 0, \text{ such that } B(x, \delta) \cap B \subset A\}$. Then $y \in \text{int}([tx, K])$, and therefore there exists $m_0(y) \in \mathbb{N}$, such that $y \in \text{int}([t_{m_0(y)}x, K])$. This implies that $y \in \overline{[t_{m_0(y)}x, K]} \setminus K \cap \partial K$ and thus

$$\text{relint}_{\partial K}(\overline{[tx, K]} \setminus K \cap \partial K) \subset \bigcup_{m \geq 1} \overline{[t_mx, K]} \setminus K \cap \partial K.$$

By continuity of the measure μ_f from below, one has

$$\begin{aligned} h_x(t) &= \mu_f(\overline{[tx, K]} \setminus K \cap \partial K) = \mu_f\left(\text{relint}_{\partial K}(\overline{[tx, K]} \setminus K \cap \partial K)\right) \\ &\leq \mu_f\left(\bigcup_{m \geq 1} (\overline{[t_mx, K]} \setminus K \cap \partial K)\right) \\ &= \lim_m \mu_f\left(\partial K \cap \overline{[t_mx, K]} \setminus K\right) = \lim_m h_x(t_m) \end{aligned}$$

(ii) It will follow from Lemma 3.2 (ii) that $K^{f,s}$ is compact for $0 \leq s \leq s_0$, once we have proved that $K^{f,s}$ is closed.

To that end, we show that $(K^{f,s})^c$, the complement of $K^{f,s}$ in \mathbb{R}^n , is open for all $s \geq 0$. Suppose this is not the case. Then there exists $x \in (K^{f,s})^c$ and a sequence $(x_m)_{m \in \mathbb{N}}$, such that $x_m \rightarrow x$ as $m \rightarrow \infty$ but $x_m \in K^{f,s}$ for all m . Without loss of generality, we can assume that x_m are not in the ray of $\{tx : t \geq 0\}$. Otherwise, if $x_m \in K^{f,s}$ are in the ray, then $h_x\left(\frac{\|x_m\|}{\|x\|}\right) \leq s$ and by (i), $\lim_m h_x\left(\frac{\|x_m\|}{\|x\|}\right) = h_x(1) \leq s$. This contradicts with $h_x(1) > s$.

Now we let $K_m = [x_m, K]$. For sufficiently big m , $\partial K_m \cap [0, x] \neq \emptyset$. Suppose not, then $x \in K_m$ implies that $[x, K] \subset K_m$, and hence $\overline{[x, K]} \setminus K \cap \partial K \subset \overline{K_m} \setminus K \cap \partial K$. Since $\mu_f(\overline{[x, K]} \setminus K \cap \partial K) > s$, one gets $\mu_f(\overline{K_m} \setminus K \cap \partial K) > s$, a contradiction with $x_m \in K^{f,s}$. Let $y_m = \partial K_m \cap [0, x]$. Thus $\mu_f(\overline{[y_m, K]} \setminus K \cap \partial K) \leq s$. Let α be as in (3.19). Similarly, $\partial([x_m, B_2^n(0, \alpha)]) \cap [0, x] \neq \emptyset$ for sufficiently big m and we denote $z_m = \partial([x_m, B_2^n(0, \alpha)]) \cap [0, x]$.

It is easy to check that $0 \leq \|x\| - \|y_m\| \leq \|x\| - \|z_m\|$ for any m . As $\alpha \leq \|z_m\| \leq \|x\|$ and $\frac{\alpha}{\|z_m\|} \leq \frac{\|x-x_m\|}{\|x\|-\|z_m\|}$, one has $\|x\| - \|z_m\| \leq \frac{\|x\| \|x-x_m\|}{\alpha}$. Thus $z_m \rightarrow x$, and hence also $y_m \rightarrow x$, as $m \rightarrow \infty$. Therefore we can choose a subsequence $(y_{m_k})_{k \in \mathbb{N}}$ that is monotone increasing to x . By (i) with $t_{m_k} = \frac{\|y_{m_k}\|}{\|x\|}$, $h_x(t_{m_k}) \nearrow h_x(1)$ as $k \rightarrow \infty$. Since for all k , $h_x(t_{m_k}) \leq s$, one has $h_x(1) \leq s$, a contradiction.

(iii) It is enough to prove that $h_x(t)$ is right continuous on $[t_0, \infty)$. To do so, let $t \geq t_0$ and let $(t_m)_{m \in \mathbb{N}}$ be a sequence decreasing to t . By (i), $h_x(t_m) \geq h_x(t)$ for all m , thus $\lim_m h_x(t_m) \geq h_x(t)$ and we have to show that $\lim_m h_x(t_m) \leq h_x(t)$. We claim that if K is strictly convex, then

$$\partial K \cap \overline{[tx, K] \setminus K} = \bigcap_{m=1}^{\infty} [\partial K \cap \overline{[t_m x, K] \setminus K}]. \quad (3.20)$$

We only need to prove that $\bigcap_{m=1}^{\infty} [\partial K \cap \overline{[t_m x, K] \setminus K}] \subseteq \partial K \cap \overline{[tx, K] \setminus K}$. Let $z_0 \in \bigcap_{m=1}^{\infty} [\partial K \cap \overline{[t_m x, K] \setminus K}]$. Thus $z_0 \in \partial K$. Let $l(z_0, tx)$ be the line passing through tx and z_0 . We have two cases.

Case 1: $l(z_0, tx)$ is in a tangent hyperplane of K . Then $l(z_0, tx) \cap \partial K = \{z_0\}$ by strict convexity of K . Therefore, $\{z_0\} = \overline{[z_0, tx] \setminus K} \cap \partial K \subseteq \overline{[tx, K] \setminus K} \cap \partial K$.

Case 2: $l(z_0, tx) \cap \partial K$ consists of two points, z_0 and z_1 . As $z_0 \in \bigcap_{m=1}^{\infty} [\partial K \cap \overline{[t_m x, K] \setminus K}]$, we must have $\|tx - z_0\| < \|tx - z_1\|$. Therefore, $\{z_0\} = \overline{[z_0, tx] \setminus K} \cap \partial K \subseteq \overline{[tx, K] \setminus K} \cap \partial K$.

Hence by (3.20) and continuity of the measure μ_f from above,

$$\begin{aligned} h_x(t) &= \mu_f \left(\bigcap_{m=1}^{\infty} [\partial K \cap \overline{[t_m x, K] \setminus K}] \right) \\ &= \lim_m \mu_f \left(\partial K \cap \overline{[t_m x, K] \setminus K} \right) = \lim_m h_x(t_m). \end{aligned}$$

(iv) Let $0 \leq s \leq s_0$, and $x \in \partial K^{f,s}$ which implies that $h_x(1) \leq s$. Define $\Phi_x(s) = \{t : h_x(t) = s\}$. Then $\Phi_x(s) \neq \emptyset$. Indeed, let $t_\alpha x = \partial B_2^n(0, \frac{3}{\alpha}) \cap T_x$, where α is as in (3.19) and $T_x = \{tx : t \geq t_0(x) > 0\}$. The proof of Lemma 3.2 (ii) shows that $K^{f,s_0} \subset B_2^n(0, \frac{2}{\alpha})$. It is clear that $t_\alpha x \notin K^{f,s_0}$, and hence $h_x(t_\alpha) > s_0$. In fact, if $t_\alpha x \in K^{f,s_0}$, then $t_\alpha x \in B_2^n(0, \frac{2}{\alpha})$, but by definition of t_α , $t_\alpha x \in \partial B_2^n(0, \frac{3}{\alpha})$. This is a contradiction.

By continuity of $h_x(\cdot)$, there must exist $t \in [t_0, t_\alpha]$, such that $h_x(t) = s$. This also shows that $\bar{t} = \sup \Phi_x(s) \leq t_\alpha$. Clearly $h_x(\bar{t}) = s$ and thus $\bar{t}x \in K^{f,s}$. This implies that $\bar{t} \leq 1$ because $x \in \partial K^{f,s}$. Suppose $\bar{t} < 1$. Then $s = h_x(\bar{t}) \leq h_x(1) \leq s$

by monotonicity of $h_x(\cdot)$, a contradiction with $\bar{t} = \sup \Phi(s)$. Thus $\bar{t} = 1$ and $h_x(1) = s$.

Remark. Strict convexity is needed in (iii) and (iv). Indeed, let $x = (0, 2)$ and

$$K = \text{conv}(\{(1, 1), (-1, 1), (-2, 0), (2, 0)\}).$$

Then $\partial K \cap \overline{[x, K] \setminus K} = [(-1, 1), (1, 1)]$. However for any point tx with $t > 1$,

$$\overline{[tx, K] \setminus K} \cap \partial K = \partial K \setminus [(-2, 0), (2, 0)] \supsetneq \partial K \cap \overline{[x, K] \setminus K}.$$

Thus, for any function f with $f > 0$ on $[(-2, 0), (-1, 1)]$ and/or $[(1, 1), (2, 0)]$, $h_x(\cdot)$ is not right continuous on $[1, \infty)$.

To see that strict convexity is needed also in (iv), observe that $K^{f, 1/12} = K$ in Example 3.1. Thus, for $x \in \partial K^{f, 1/12} = \partial K$, we have

$$\mu_f(\overline{[x, K] \setminus K} \cap \partial K) = 0 \neq \frac{1}{12}.$$

4 Geometric interpretation of functionals on convex bodies

We now give geometric interpretations of functionals on convex bodies, such as L_p affine surface area and mixed p -affine surface area for all $p \neq -n$ using the *non convex* illumination surface bodies. While there are no geometric interpretations for mixed p -affine surface area, many geometric interpretations of L_p affine surface area have been discovered in the last years, all based on using *convex* bodies (e.g., [33, 37, 38, 45]). The remarkable new fact here is that now the bodies involved in the geometric interpretation are not necessarily convex.

Theorem 4.1 *Let K be a convex body in C_+^2 . Let $c > 0$ be a constant, and $f : \partial K \rightarrow \mathbb{R}$ be an integrable function such that $f \geq c \mu_K$ -almost everywhere. Then*

$$\lim_{s \rightarrow 0} c_n \frac{|K^{f, s}| - |K|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu_K(x), \quad (4.21)$$

where $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$.

Remark. As $d\mu_K = f_K d\sigma$, we also have

$$\lim_{s \rightarrow 0} c_n \frac{|K^{f,s}| - |K|}{s^{\frac{2}{n-1}}} = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n-2}{n-1}}}{f(N_K^{-1}(u))^{\frac{2}{n-1}}} d\sigma(u), \quad (4.22)$$

where N_K^{-1} is the inverse of the Gauss map $N_K(\cdot)$.

The geometric interpretation of L_p affine surface area is then a corollary to Theorem 4.1. The theorem also gives geometric interpretations of other known functionals on convex bodies, e.g. the surface area and the mixed p -affine surface area. Notice that these geometric interpretations can also be obtained using e.g. the (convex) surface body [38, 45].

Define

$$\tilde{f}(N_K^{-1}(u)) = f_K(u)^{\frac{n-2}{2}} [f_p(K_1, u) \cdots f_p(K_n, u)]^{\frac{1-n}{2(n+p)}},$$

where $f_p(K, u) = h_K(u)^{1-p} f_K(u)$.

Corollary 4.1 *Let K and K_i , $i = 1, \dots, n$, be convex bodies in C_+^2 . Then*

$$\lim_{s \rightarrow 0} c_n \frac{|K^{\tilde{f},s}| - |K|}{s^{\frac{2}{n-1}}} = as_p(K_1, \dots, K_n).$$

In particular, if all K_i coincide with K , then $as_p(K_1, \dots, K_n) = as_p(K)$ and we get a geometric interpretation of $as_p(K)$

$$c_n \lim_{s \rightarrow 0} \frac{|K^{g_p,s}| - |K|}{s^{\frac{2}{n-1}}} = as_p(K),$$

where $g_p : \partial K \rightarrow \mathbb{R}$ is defined by $g_p(x) = \kappa_K(x)^{\frac{n+2p-np}{2(n+p)}} \langle x, N_K(x) \rangle^{\frac{n(n-1)(p-1)}{2(n+p)}}$.

Corollary 4.2 *Let K be a convex body in C_+^2 and $g(x) = \sqrt{\kappa_K(x)}$. Then*

$$\lim_{s \rightarrow 0} c_n \frac{|K^{g,s}| - |K|}{s^{\frac{2}{n-1}}} = \mu_K(\partial K).$$

The proof of the corollaries follows immediately from Theorem 4.1. To prove Theorem 4.1, we need several other concepts and lemmas.

As K is in C_+^2 , for any $x \in \partial K$, the indicatrix of Dupin is an ellipsoid. As in [38], we apply an affine transform $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to K so that the indicatrix of Dupin

is transformed into an $(n - 1)$ -dimensional Euclidean ball. T has the following properties:

$$T(x) = x \quad T(N_K(x)) = N_K(x) \quad \det(T) = 1 \quad (4.23)$$

and T maps a measurable subset of a hyperplane orthogonal to $N_K(x)$ onto a subset of the same $(n - 1)$ -dimensional measure. It was also shown in [38] that for any $\epsilon > 0$ there is $\Delta_1 = \Delta_1(\epsilon) > 0$ such that for all measurable subsets A of $\partial K \cap H^-(x - \Delta_1 N_K(x), N_K(x))$

$$(1 - \epsilon) \mu_K(A) \leq |T(A)| \leq (1 + \epsilon) \mu_K(A). \quad (4.24)$$

$T(K)$ can be approximated at $x = T(x)$ by a n -dimensional Euclidean ball: For any $\epsilon > 0$ there is $\Delta_2 = \Delta_2(\epsilon)$ such that

$$\begin{aligned} & B_2^n(x - rN_K(x), r) \cap H^-(x - \Delta_2 N_K(x), N_K(x)) \\ & \subseteq T(K) \cap H^-(x - \Delta_2 N_K(x), N_K(x)) \\ & \subseteq B_2^n(x - RN_K(x), R) \cap H^-(x - \Delta_2 N_K(x), N_K(x)), \end{aligned} \quad (4.25)$$

where $r = r(x) = \kappa_K(x)^{-\frac{1}{n-1}}$ and $R = R(x)$ with $r \leq R \leq (1 + \epsilon)r$. We put

$$\Delta = \Delta(\epsilon) = \min\{\Delta_1, \Delta_2\}. \quad (4.26)$$

Moreover, for $x \in \partial K$, let

$$x_s \in \partial K^{f,s} \text{ be such that } x \in [0, x_s] \cap \partial K \quad (4.27)$$

and define \tilde{x}_s to be the orthogonal projection of x_s onto the ray $\{y : y = x + tN_K(x), t \geq 0\}$. Clearly $T(\tilde{x}_s) = \tilde{x}_s$, and the distance from $T(x_s)$ to the hyperplane $H(x, N_K(x))$ is the same as the distance from x_s to this hyperplane.

We say that a family of sets $E_s \subseteq \partial K$, $0 < s \leq s_0$ shrinks nicely to a point $x \in \partial K$ (see [13]) if

(SN1) $\text{diam}E_s \rightarrow 0$, as $s \rightarrow 0$.

(SN2) There is a constant $\beta > 0$ such that for all $s \leq s_0$ there exists t_s with

$$\mu_K(\partial K \cap B(x, t_s)) \geq \mu_K(E_s) \geq \beta \mu_K(\partial K \cap B(x, t_s)).$$

Lemma 4.1 *Let K be a convex body in C_+^2 and $f : \partial K \rightarrow \mathbb{R}$ an integrable, μ_K -almost everywhere strictly positive function. Let $x \in \partial K$ and let x_s and \tilde{x}_s be as above (4.27). Then*

(i) *The family $\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K$, $0 < s \leq s_0$ shrinks nicely to x .*

(ii) *The family $\partial K \cap \overline{[x_s, K]} \setminus K$, $0 < s \leq s_0$ shrinks nicely to x .*

(iii)

$$\lim_{s \rightarrow 0} \frac{\mu_f(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K)}{\mu_K(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K)} = f(x) \quad \mu_K\text{-a.e.} \quad (4.28)$$

(iv)

$$\lim_{s \rightarrow 0} \frac{\mu_f(\partial K \cap \overline{[x_s, K]} \setminus K)}{\mu_K(\partial K \cap \overline{[x_s, K]} \setminus K)} = f(x) \quad \mu_K\text{-a.e.} \quad (4.29)$$

Proof. Formulas (4.28) and (4.29) in (iii) and (iv) follow from the Lebesgue differentiation theorem (see [13]) once we have proved that $\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K$ and $\partial K \cap \overline{[x_s, K]} \setminus K$ shrink nicely to x . Therefore it is enough to prove (i) and (ii).

(i) For $x \in \partial K$, let $r = r(x)$ and $R = R(x)$ be as in (4.25). We abbreviate $B(r) = B_2^n(x - rN_K(x), r)$ and $B(R) = B_2^n(x - RN_K(x), R)$. Let

$$\Delta(x, s) = \left\langle \frac{x}{\|x\|}, N_K(x) \right\rangle \|x_s - x\| = \langle x_s - x, N_K(x) \rangle$$

be the distance from x_s to $H(x, N_K(x))$. This is the same as the distance from \tilde{x}_s (defined after formula (4.27)) to $H(x, N_K(x))$.

Let $h_R = h_R(s) = \frac{R \Delta(x, s)}{R + \Delta(x, s)}$ be the height of the cap of $B(R)$ that is ‘‘illuminated’’ by \tilde{x}_s . Then

$$H^-(x - h_R N_K(x), N_K(x)) \cap \partial B(R) = \overline{[T(\tilde{x}_s), B(R)] \setminus B(R)} \cap \partial B(R). \quad (4.30)$$

Let Δ be as in (4.26). Since $\Delta(x, s) \rightarrow 0$ as $s \rightarrow 0$, one can choose $s_1 \leq s_0$, such that for all $0 < s \leq s_1$, $h = 2h_R < \Delta$. Therefore (4.25) holds:

$$\begin{aligned} H^-(x - hN_K(x), N_K(x)) \cap B(r) &\subset H^-(x - hN_K(x), N_K(x)) \cap T(K) \\ &\subset H^-(x - hN_K(x), N_K(x)) \cap B(R). \end{aligned} \quad (4.31)$$

(4.31) and (4.30) imply that for all small enough $s \leq s_2 \leq s_1$

$$T(\overline{[\tilde{x}_s, K]} \setminus K \cap \partial K) = \overline{[T(\tilde{x}_s), T(K)] \setminus T(K)} \cap \partial T(K)$$

$$\subseteq H^-(x - hN_K(x), N_K(x)) \cap \partial T(K) \subseteq B(R) \cap H^-(x - hN_K(x), N_K(x)).$$

Let $t_s = \|x - z\| = \sqrt{\frac{4R^2 \Delta(x,s)}{R + \Delta(x,s)}}$ where z is any point in $H(x - hN_K(x), N_K(x)) \cap \partial B(R)$. As $B(R) \cap H^-(x - hN_K(x), N_K(x)) \subseteq B_2^n(x, t_s)$,

$$T(\overline{[\tilde{x}_s, K] \setminus K}) \cap \partial T(K) = T(\overline{[\tilde{x}_s, K] \setminus K} \cap \partial K) \subseteq B_2^n(x, t_s) \cap \partial T(K)$$

and $t_s \rightarrow 0$ as $s \rightarrow 0$.

This shows that condition (SN1) is satisfied for $T(\overline{[\tilde{x}_s, K] \setminus K}) \cap \partial T(K)$ to shrink nicely to $T(x) = x$.

We now show that condition (SN2) also holds true.

First, $\overline{[T(\tilde{x}_s), B(r)] \setminus B(r)} \cap H(x, N_K(x))$ is a $(n - 1)$ -dimensional Euclidean ball with radius $\frac{r\Delta(x,s)}{\sqrt{2r\Delta(x,s) + \Delta^2(x,s)}}$. Then for any $0 < s < s_2$,

$$\begin{aligned} |T(\overline{[\tilde{x}_s, K] \setminus K} \cap \partial K)| &\geq |\overline{[T(\tilde{x}_s), B(r)] \setminus B(r)} \cap H(x, N_K(x))| \\ &\geq |B_2^{n-1}| \left(\frac{r\Delta(x,s)}{\sqrt{2r\Delta(x,s) + \Delta^2(x,s)}} \right)^{n-1}. \end{aligned}$$

We can choose (a new, smaller) s_2 such that $\frac{\Delta(x,s)}{r} \leq 2$. Then

$$|T(\overline{[\tilde{x}_s, K] \setminus K} \cap \partial K)| \geq 2^{1-n} |B_2^{n-1}| (r\Delta(x,s))^{\frac{n-1}{2}}. \quad (4.32)$$

On the other hand, for ε small enough, there exists $s_3 < s_2$, such that, for all $0 < s \leq s_3$ and for any subset A of $H^-(x - 2hN_K(x), N_K(x)) \cap \partial T(K)$ [38]

$$|P_{H(x-2hN_K(x), N_K(x))}(A)| \leq |A| \leq (1 + \varepsilon) |P_{H(x-2hN_K(x), N_K(x))}(A)| \quad (4.33)$$

where $P_H(A)$ is the orthogonal projection of A onto the hyperplane H . We apply this to $A = B_2^n(x, t_s) \cap \partial T(K)$:

$$\begin{aligned} |B_2^n(x, t_s) \cap \partial T(K)| &\leq (1 + \varepsilon) |P_{H(x-2hN_K(x), N_K(x))}(B_2^n(x, t_s) \cap \partial T(K))| \\ &\leq (1 + \varepsilon) |B(R) \cap H(x - 2hN_K(x), N)| \\ &\leq (1 + \varepsilon) |B_2^{n-1}| \left(\frac{2\sqrt{2}R\sqrt{R\Delta(x,s)}}{R + \Delta(x,s)} \right)^{n-1} \\ &\leq 4^n |B_2^{n-1}| (r\Delta(x,s))^{\frac{n-1}{2}} \end{aligned} \quad (4.34)$$

The last inequality follows as $r \leq R < (1 + \varepsilon)r$. It now follows from (4.32) and (4.34) that also condition (SN2) holds true for e.g. $\beta = 8^{-n}$.

Hence the family $T(\overline{[\tilde{x}_s, K]} \setminus \overline{K}) \cap \partial T(K)$ shrinks nicely to $T(x) = x$ and therefore, as T^{-1} exists, the family $\overline{[\tilde{x}_s, K]} \setminus \overline{K} \cap \partial K = T^{-1}(T(\overline{[\tilde{x}_s, K]} \setminus \overline{K}) \cap \partial T(K))$ shrinks nicely to x .

(ii) Let $v_1 = x_s - (x - rN_K(x))$ and $v_2 = x_s - (x - RN_K(x))$. θ denotes the angle between $N_K(x)$ and x and $\phi_i = \phi_i(x, s), i = 1, 2$ is the angle between $N_K(x)$ and $v_i, i = 1, 2$. These angles can be computed as follows

$$\begin{aligned}\tan(\phi_1) &= \frac{\Delta(x, s) \tan(\theta)}{r + \Delta(x, s)} \\ \tan(\phi_2) &= \frac{\Delta(x, s) \tan(\theta)}{R + \Delta(x, s)}.\end{aligned}$$

Then, for $i = 1, 2$, $\phi_i \rightarrow 0$ as $s \rightarrow 0$. Since K is in C_+^2 , this means that for any $\varepsilon > 0$ there is $\bar{s}_\varepsilon \leq s_0$ such that for all $s \leq \bar{s}_\varepsilon$

$$1 - \varepsilon \leq \frac{\mu_K(\overline{[x_s, K]} \setminus \overline{K} \cap \partial K)}{\mu_K(\overline{[\tilde{x}_s, K]} \setminus \overline{K} \cap \partial K)} \leq 1 + \varepsilon. \quad (4.35)$$

By (4.25) and as $\phi_i \rightarrow 0, i = 1, 2$ as $s \rightarrow 0$, one can choose $\tilde{h}_R = \tilde{h}_R(s) = \frac{3R \Delta(x, s)}{R + \Delta(x, s)}$ so small that

$$\begin{aligned}T\left(\overline{[x_s, K]} \setminus \overline{K} \cap \partial K\right) &= \overline{[T(x_s), T(K)]} \setminus \overline{T(K)} \cap \partial T(K) \\ &\subset H^-\left(x - \tilde{h}_R N_K(x), N_K(x)\right) \cap B(R).\end{aligned}$$

Let $\tilde{t}_s = \sqrt{\frac{6R^2 \Delta(x, s)}{R + \Delta(x, s)}}$ be the distance from x to any point in $H\left(x - \tilde{h}_R N_K(x), N_K(x)\right) \cap \partial B(R)$. Then

$$T\left(\overline{[x_s, K]} \setminus \overline{K} \cap \partial K\right) \subseteq B_2^n(x, \tilde{t}_s) \cap \partial T(K).$$

(4.24), (4.35) and Lemma 4.1 (i) then give

$$\begin{aligned}|T\left(\overline{[x_s, K]} \setminus \overline{K} \cap \partial K\right)| &\geq (1 - \varepsilon)^3 |T\left(\overline{[\tilde{x}_s, K]} \setminus \overline{K} \cap \partial K\right)| \\ &\geq (1 - \varepsilon)^3 \beta |B_2^n(x, \tilde{t}_s) \cap \partial T(K)|.\end{aligned} \quad (4.36)$$

Furthermore, by (4.33), one has

$$\begin{aligned}|B_2^n(x, \tilde{t}_s) \cap \partial T(K)| &\geq |H(x - hN_K(x), N_K(x)) \cap T(K)| \\ &\geq |H(x - hN_K(x), N_K(x)) \cap B(r)| \\ &= \left(\frac{4R^2 r \Delta(x, s) + 4Rr \Delta(x, s)^2 - 4R^2 \Delta(x, s)^2}{(R + \Delta(x, s))^2}\right)^{\frac{n-1}{2}} |B_2^{n-1}|.\end{aligned}$$

Since $\Delta(x, s) \rightarrow 0$ as $s \rightarrow 0$, for \bar{s}_ε small enough, and any $0 < s < \bar{s}_\varepsilon$, one get

$$\begin{aligned} |B_2^n(x, t_s) \cap \partial T(K)| &\geq \left(\frac{2R}{R + \Delta(x, s)} \sqrt{r\Delta(x, s) - \Delta(x, s)^2} \right)^{n-1} |B_2^{n-1}| \\ &\geq 2^{-n} (r\Delta(x, s))^{\frac{n-1}{2}} |B_2^{n-1}|. \end{aligned} \quad (4.37)$$

A computation similar to (4.34) shows that for all $0 < s \leq \bar{s}_\varepsilon$ with (a possibly new) \bar{s}_ε small enough

$$\begin{aligned} |B_2^n(x, \tilde{t}_s) \cap \partial T(K)| &\leq (1 + \varepsilon) |B(R) \cap H(x - \tilde{h}N_K(x), N)| \\ &= (1 + \varepsilon) |B_2^{n-1}| \left(\frac{\sqrt{6}R\sqrt{R\Delta(x, s)}}{R + \Delta(x, s)} \right)^{n-1} \\ &\leq 3^n |B_2^{n-1}| (r\Delta(x, s))^{\frac{n-1}{2}} \end{aligned} \quad (4.38)$$

(4.36), (4.37) and (4.38) imply that

$$|T(\overline{[x_s, K] \setminus K} \cap \partial K)| \geq (48)^{-n-1} \beta |B_2^n(x, \tilde{t}_s) \cap \partial T(K)|.$$

This shows that $T(\overline{[x_s, K] \setminus K} \cap \partial K)$ shrinks nicely to x . Therefore also $\overline{[x_s, K] \setminus K} \cap \partial K$ shrinks nicely to x .

Lemma 4.2 *Let K be a convex body in C_+^2 and $f : \partial K \rightarrow \mathbb{R}$ an integrable, μ_K -almost everywhere strictly positive function. Then for μ_K -almost all $x \in \partial K$ one has*

$$\lim_{s \rightarrow 0} c_n \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{s^{\frac{2}{n-1}}} = \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}, \quad (4.39)$$

where $x_s \in \partial K^{f,s}$ is such that $x \in [0, x_s]$.

Proof. It is enough to consider $x \in \partial K$ such that $f(x) > 0$. As x and x_s are collinear and as $(1+t)^n \geq 1+tn$ for $t \in [0, 1)$, one has for small enough s ,

$$\frac{\langle x, N_K(x) \rangle}{n} \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] = \frac{\langle x, N_K(x) \rangle}{n} \left[\left(1 + \frac{\|x_s - x\|}{\|x\|} \right)^n - 1 \right] \geq \Delta(x, s).$$

Recall that $\Delta(x, s) = \left\langle \frac{x}{\|x\|}, N_K(x) \right\rangle \|x_s - x\| = \langle x_s - x, N_K(x) \rangle$ is the distance from x_s to $H(x, N_K(x))$.

Similarly, as $(1+t)^n \leq 1+nt+2^nt^2$ for $t \in [0,1)$, one has for s small enough,

$$\frac{\langle x, N_K(x) \rangle}{n} \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] \leq \Delta(x, s) \left[1 + \frac{2^n}{n} \left(\frac{\|x_s - x\|}{\|x\|} \right) \right]. \quad (4.40)$$

Hence for $\varepsilon > 0$ there exists $s_\varepsilon \leq s_0$ such that for all $0 < s \leq s_\varepsilon$

$$1 \leq \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{n\Delta(x, s)} \leq 1 + \varepsilon.$$

K is strictly convex as $K \in C_+^2$. Thus, $\mu_f(\partial K \cap \overline{[x_s, K]} \setminus K) = s$ by Lemma 3.3 (iv). Therefore

$$1 \leq \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] \left(\mu_f(\partial K \cap \overline{[x_s, K]} \setminus K) \right)^{\frac{2}{n-1}}}{n s^{\frac{2}{n-1}} \Delta(x, s)} \leq 1 + \varepsilon.$$

By Lemma 4.1 (iv) and (4.35), it then follows that we can choose (a new) s_ε so small that we have for all $s \leq s_\varepsilon$

$$1 - c_1\varepsilon \leq \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] \left(f(x) \mu_K(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) \right)^{\frac{2}{n-1}}}{n s^{\frac{2}{n-1}} \Delta(x, s)} \leq 1 + c_2\varepsilon. \quad (4.41)$$

with absolute constants c_1 and c_2 .

Let T be as in (4.23) and let $r = r(x)$ and $R = R(x)$ be as in (4.25). We abbreviate again $B(r) = B_2^n(x - rN_K(x), r)$ and $B(R) = B_2^n(x - RN_K(x), R)$. Let $h_r = h_r(s) = \frac{r\Delta(x, s)}{r + \Delta(x, s)}$. As $h_r \rightarrow 0$ as $s \rightarrow 0$, we have for all s sufficiently small that $h_r < \Delta$ where Δ is as in (4.26). Hence by (4.25)

$$H^-(x - h_r N_K(x), N_K(x)) \cap B(r) \subset H^-(x - h_r N_K(x), N_K(x)) \cap T(K).$$

If we denote by P_H the orthogonal projection onto the hyperplane $H^-(x - h_r N_K(x), N_K(x))$, this then implies that

$$\begin{aligned} P_H(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) &\supset P_H(\partial T^{-1}(B(r)) \cap \overline{[\tilde{x}_s, T^{-1}(B(r))] \setminus T^{-1}(B(r))}) \\ &= H(x - h_r N_K(x), N_K(x)) \cap T^{-1}(B(r)). \end{aligned}$$

Hence for s sufficiently small

$$\begin{aligned}
\mu_K(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) &\geq |P_H(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K)| \\
&\geq |H(x - h_r N_K(x), N_K(x)) \cap T^{-1}(B(r))| \\
&= |T(H(x - h_r N_K(x), N_K(x)) \cap T^{-1}(B(r)))| \\
&= |H(x - h_r N_K(x), N_K(x)) \cap B(r)| \\
&= \frac{\left(1 - \frac{\Delta(x, s)}{2r}\right)^{\frac{n-1}{2}}}{\left(1 + \frac{\Delta(x, s)}{r}\right)^{n-1}} (2r\Delta(x, s))^{\frac{n-1}{2}} |B_2^{n-1}| \\
&\geq (1 - c_3\varepsilon) \frac{(2\Delta(x, s))^{\frac{n-1}{2}}}{\sqrt{\kappa_K(x)}} |B_2^{n-1}| \tag{4.42}
\end{aligned}$$

where $c_3 > 0$ is an absolute constant.

By (4.25), one has

$$\begin{aligned}
P_H(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) &\subseteq P_H(\partial(T^{-1}(B(R))) \cap \overline{[\tilde{x}_s, T^{-1}(B(R))] \setminus T^{-1}(B(R))}) \\
&= H(x - h_R N_K(x), N_K(x)) \cap T^{-1}(B(R))
\end{aligned}$$

where $H = H(x - h_R N_K(x), N_K(x))$. The equality follows as $h_R = \frac{R\Delta(x, s)}{R + \Delta(x, s)}$.

Together with (4.33), for s_ε small enough, whenever $0 < s < s_\varepsilon$, one has

$$\begin{aligned}
\mu_K(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) &\leq (1 + \varepsilon) |P_H(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K)| \\
&\leq (1 + \varepsilon) |H(x - h_R N_K(x), N_K(x)) \cap T^{-1}(B(R))|.
\end{aligned}$$

A calculation similar to (4.42) then shows that with an absolute constant c_4

$$\mu_K(\partial K \cap \overline{[\tilde{x}_s, K]} \setminus K) \leq (1 + c_4\varepsilon) \frac{(2\Delta(x, s))^{\frac{n-1}{2}}}{\sqrt{\kappa_K(x)}} |B_2^{n-1}|. \tag{4.43}$$

Combining (4.41), (4.42) and (4.43), we prove the formula (4.39), i.e.,

$$\lim_{s \rightarrow 0} c_n \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{n s^{\frac{2}{n-1}}} = \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}.$$

Lemma 4.3 *Let K be a convex body in C_+^2 . Let $c > 0$ be a constant and $f : \partial K \rightarrow \mathbb{R}$ an integrable function with $f \geq c$ μ_K -almost everywhere. Then there*

exists $\bar{s} \leq s_0$, such that for all $s \leq \bar{s}$,

$$\frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{s^{\frac{2}{n-1}}} \leq c(K, n),$$

where $c(K, n)$ is a constant (depending on K and n only), and x and x_s are as in Lemma 4.2.

Proof.

As $K \in C_+^2$, by the Blaschke rolling theorem [36], there exists $r_0 > 0$ such that for all $x \in \partial K$, $B_2^n(x - r_0 N_K(x), r_0) \subseteq K$. Let γ be such that $0 < \gamma \leq \min\{1, r_0\}$. By Lemmas 3.1 (i) and 3.2 (ii), $K = K^{f,0} = \bigcap_{s>0} K^{f,s}$. Therefore there exists $\bar{s} = s_\gamma \leq s_0$, such that for all $s \leq \bar{s}$, $K^{f,s} \subseteq (1 + \gamma)K$. Hence for $x_s \in \partial K^{f,s}$ and $x = [0, x_s] \cap \partial K$, $\frac{\|x_s\|}{\|x\|} \leq 1 + \gamma$, or equivalently -as x and x_s are collinear-

$$\frac{\|x_s\|}{\|x\|} - 1 = \frac{\|x_s - x\|}{\|x\|} \leq \gamma \leq 1. \quad (4.44)$$

Together with (4.40), one has for all $s \leq \bar{s}$ (with a possibly smaller \bar{s})

$$\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] \leq \Delta(x, s) [n + 2^n]. \quad (4.45)$$

As $K \in C_+^2$, K is strictly convex. Hence, by Lemma 3.3 (iv), (4.25) and as $f \geq c$ on ∂K

$$\begin{aligned} s &= \mu_f(\partial K \cap \overline{[x_s, K]} \setminus K) = \int_{\partial K \cap \overline{[x_s, K]} \setminus K} f d\mu_K \geq c \mu_K(\partial K \cap \overline{[x_s, K]} \setminus K) \\ &\geq c |H(x, N_K(x)) \cap [x_s, T^{-1}(B(r))]| = c |H(x, N_K(x)) \cap [T(x_s), B(r)]| \end{aligned}$$

where T is as in (4.23) and $r = r(x)$ is as in (4.25).

As in the proof of Lemma 3.2 (i), $H(x, N_K(x)) \cap [T(x_s), B(r)]$ contains a $(n-1)$ -dimensional Euclidean ball of radius at least

$$\frac{r \sqrt{2r\Delta(x, s) + \Delta^2(x, s)}}{2r + \Delta(x, s)} \geq \frac{\alpha}{1 + 2\alpha} \sqrt{2r_0\Delta(x, s)}.$$

The inequality follows as $\Delta(x, s) = \frac{\|x_s - x\|}{\|x\|} \langle x, N_K(x) \rangle \leq \frac{\gamma}{\alpha} \leq \frac{1}{\alpha}$, which is a direct consequence of (3.19) and (4.44).

Hence $s \geq c \left(\frac{\alpha}{1+2\alpha} \sqrt{2r_0 \Delta(x, s)} \right)^{n-1} |B_2^{n-1}|$ and with (4.45) we get that

$$\frac{\langle x, N_K(x) \rangle}{s^{\frac{2}{n-1}}} \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right] \leq (n+2^n) \left(\frac{1+2\alpha}{\alpha} \right)^2 (2r_0 c^{\frac{2}{n-1}} |B_2^{n-1}|^{\frac{2}{n-1}})^{-1}.$$

Finally, we also need the following lemma. It is well known and we omit the proof.

Lemma 4.4 *Let K be a convex body and L be a star-convex body in \mathbb{R}^n .*

(i) *If $0 \in \text{int}(L)$ and $L \subset K$, then*

$$|K| - |L| = \frac{1}{n} \int_{\partial K} \langle x, N_K(x) \rangle \left[1 - \left(\frac{\|x'\|}{\|x\|} \right)^n \right] d\mu_K(x)$$

where $x \in \partial K$ and $x' \in \partial L \cap [0, x]$.

(ii) *If $0 \in \text{int}(K)$ and $K \subset L$, then*

$$|L| - |K| = \frac{1}{n} \int_{\partial K} \langle x, N_K(x) \rangle \left[\left(\frac{\|x'\|}{\|x\|} \right)^n - 1 \right] d\mu_K(x)$$

where $x \in \partial K$, $x' \in \partial L$ and $x = \partial K \cap [0, x']$.

Proof of Theorem 4.1.

As $K \in C_+^2$, K is strictly convex. By Lemmas 4.4, 4.2, 4.3 and Lebegue's Dominated Convergence theorem

$$\begin{aligned} c_n \lim_{s \rightarrow 0} \frac{|K^{f,s}| - |K|}{s^{\frac{2}{n-1}}} &= c_n \lim_{s \rightarrow 0} \int_{\partial K} \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{n s^{\frac{2}{n-1}}} d\mu_K(x) \\ &= c_n \int_{\partial K} \lim_{s \rightarrow 0} \frac{\langle x, N_K(x) \rangle \left[\left(\frac{\|x_s\|}{\|x\|} \right)^n - 1 \right]}{n s^{\frac{2}{n-1}}} d\mu_K(x) \\ &= \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu_K(x). \end{aligned}$$

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