On the approximation of a polytope by its dual L_p-centroid bodies *

Grigoris Paouris† and Elisabeth M. Werner ‡

Abstract

We show that the rate of convergence on the approximation of volumes of a convex symmetric polytope $P \in \mathbb{R}^n$ by its dual L_p-centroid bodies is independent of the geometry of P. In particular we show that if P has volume 1,

$$\lim_{p\to\infty} \frac{p}{\log p}\left(\frac{|Z_p^\circ(P)|}{|P^\circ|} - 1\right) = n^2.$$

We provide an application to the approximation of polytopes by uniformly convex sets.

1 Introduction

Let K be a convex body in $\mathbb{R}^n$ of volume 1 and, for $\delta \in (0,1)$, let K_δ be the convex floating body of K [22]. It is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume δ from K. Note that K_δ converges to K in the Hausdorff metric as $\delta \to 0$. C. Schütt and the second name author showed an exact formula for the convergence of volumes [22],

$$\lim_{\delta\to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}} = \mathrm{as}_1(K),$$

which involves the affine surface area of K, $\mathrm{as}_1(K)$. The same phenomenon (and similar formulas) has been observed for other types of approximation using instead of floating bodies, convolution bodies [21], illumination bodies [27] or Santaló bodies [18]. We refer to e.g. [2], [4]-[9], [12]-[17], [23]-[26], [28]-[30] for further details, extensions and applications. Another family of bodies that approximate a given

*Keywords: centroid bodies, floating bodies, polytopes, L_p Brunn Minkowski theory. 2000 Mathematics Subject Classification: 52A20, 53A15

†Partially supported by an NSF grant (DMS-0906150)

‡Partially supported by an NSF grant, a FRG-NSF grant and a BSF grant

convex body K are the L_p-centroid bodies of K introduced by Lutwak and Zhang [17]. For a symmetric convex body K of volume 1 in $\mathbb{R}^n$ and $1 \leqslant p \leqslant n$, the L_p-centroid body $Z_p(K)$ is the convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}}, \ \theta \in S^{n-1}.$$

Note that $Z_p(K)$ converges to K in the Hausdorff metric as $p \to \infty$. It has been shown in [19] that the family of L_p-centroid bodies is isomorphic to the family of the floating bodies: K_δ is isomorphic to $Z_{\log \frac{1}{\delta}}(K)$. However, it was proved in [19] that in the case of C^2_+ bodies, the convergence of volume of the L_p-centroid bodies is independent of the "geometry" of K: For any symmetric convex body in $\mathbb{R}^n$ of volume 1 that is C^2_+ (i.e. K has C^2 boundary with everywhere strictly positive Gaussian curvature),

$$\lim_{p \to \infty} \frac{p}{\log p} \left(|Z_p^\circ(K)| - |K^\circ| \right) = \frac{n(n+1)}{2} |K^\circ|.$$

In this work we show that the same phenomenon occurs also in the case of polytopes. We show the following

Theorem 1.1. *Let K be a symmetric polytope of volume 1 in $\mathbb{R}^n$. Then*

$$\lim_{p \to \infty} \frac{p}{\log p} \left(|Z_p^\circ(K)| - |K^\circ| \right) = n^2 |K^\circ|.$$

As an application of this result we get bounds for the approximation of a polytope by a uniformly convex body with respect to the symmetric difference metric:

Theorem 1.2. *Let P be a symmetric polytope in $\mathbb{R}^n$. Then there exists $p_0 = p_0(P)$ such that for every $p \geqslant p_0$, there exists a p-uniformly convex body K_p such that*

$$d_s(P, K_p) \leqslant 2n^2 |P| \frac{\log p}{p},$$

where d_s is the symmetric difference metric.

The statements and proofs are for symmetric convex bodies only. If K is not symmetric, then $Z_p(K)$ does not converge to K since the $Z_p(K)$ are centrally symmetric by definition. However, all results can be extended to the non-symmetric case with minor modifications of the proofs by using the non-symmetric version of the L_p-centroid bodies from [12] (see also [6]).

The paper is organized as follows. In section 2 we give some bounds for the approximation of volume in the case of a general convex body. In section 3 we consider the case of polytopes and we give the proof of Theorem 1.1. Finally, in section 4, we discuss approximation of a polytope by p-uniformly convex bodies (see [11]) and we give the proof of Theorem 1.2.

Notation.

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write σ for the rotationally invariant surface measure on S^{n-1}.
A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric, if $x \in C$ implies that $-x \in C$. We say that C has center of mass at the origin if $\int_C \langle x, \theta\rangle dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y\rangle : y \in C\}$. $C^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for all } x \in C\}$ is the polar body of C.
We refer to [1] and [20] for basic facts from the Brunn-Minkowski theory.

Acknowledgments. The authors would like to thank the American Institute of Mathematics. Part of this work has been carried out during a stay at AIM.

2 General Bounds

Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $\theta \in S^{n-1}$. We define the parallel section function $f_{K,\theta} : [-h_k(\theta), h_k(\theta)] \to \mathbb{R}_+$ by

$$f_{K,\theta}(t) := |K \cap (\theta^\perp + t\theta)|.$$

By Brunn's principle, $f_{K,\theta}^{\frac{1}{n-1}}$ is concave and attains its maximum at 0. So we have that

$$\left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_{K,\theta}(0) \leqslant f_{K,\theta}(t) \leqslant f_{K,\theta}(0). \tag{1}$$

The right-hand side inequality is sharp if and only if K is a cylinder in the direction of θ and the left-hand side inequality is sharp if and only if K is a double cone in the direction of θ.

The next proposition is well known. There, for $x, y > 0$, $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1} d\lambda = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $\Gamma(x) = \int_0^\infty \lambda^{x-1} e^{-\lambda} d\lambda$ is the Gamma function.

Proposition 2.1. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $1 \leqslant p < \infty$ and $\theta \in S^{n-1}$. Then*

$$B(p+1, n)^{\frac{1}{p}} \leqslant \frac{h_{Z_p(K)}(\theta)}{h_K(\theta)} \leqslant \left(\frac{n}{p+1}\right)^{\frac{1}{p}}.$$

Proof. As $|K| = 1$,

$$\frac{2}{n} h_K(\theta) f_{K,\theta}(0) \leqslant 1 \leqslant 2 h_K(\theta) f_{K,\theta}(0).$$

Hence, on the one hand, with (1),

$$\begin{aligned} h^p_{Z_p(K)}(\theta) &= 2\int_0^{h_K(\theta)} t^p f_{K,\theta}(t)dt \leqslant 2f_{K,\theta}(0)\int_0^{h_K(\theta)} t^p dt \\ &= \frac{2}{p+1} f_{K,\theta}(0)\ h_K^{p+1}(\theta) \leqslant \frac{n}{p+1} h^p_K(\theta). \end{aligned}$$

On the other hand, also with with (1),

$$\begin{aligned} h^p_{Z_p(K)}(\theta) &= 2\int_0^{h_K(\theta)} t^p f_{K,\theta}(t)dt \geqslant 2f_{K,\theta}(0)\ \int_0^{h_K(\theta)} t^p \left(1-\frac{t}{h_K(\theta)}\right)^{n-1} dt \\ &= 2f_{K,\theta}(0) h_K^{p+1}(\theta)\int_0^1 s^p(1-s)^{n-1} ds \geqslant B(p+1,n) h^p_K(\theta). \end{aligned}$$

The proof is complete. □

As it was mentioned in the introduction, it was proved in [19] that if K is a C^2_+ symmetric convex body of volume 1, then

$$\lim_{p\to\infty} \frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) = \frac{n(n+1)}{2}|K^\circ|.$$

Before we consider the case of polytopes, we show that for every convex body we have that $|Z_p^\circ(K)| - |K^\circ| = O(\frac{p}{\log p})$. In particular, the following proposition holds.

Proposition 2.2. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1. Then*

$$n|K^\circ| \leqslant \lim_{p\to\infty} \frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) \leqslant n^2|K^\circ|.$$

Proof. We have that

$$\begin{aligned} |Z_p^\circ(K)| - |K^\circ| &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{h^n_{Z_p(K)}(\theta)} - \frac{1}{h^n_K(\theta)} d\sigma(\theta) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{h^n_K(\theta)}\left(\frac{h^n_K(\theta)}{h^n_{Z_p(K)}(\theta)} - 1\right) d\sigma(\theta), \end{aligned}$$

where σ is the usual surface area measure on S^{n-1}. By Proposition 2.1,

$$\frac{h^n_K(\theta)}{h^n_{Z_p(K)}(\theta)} \geqslant \left(\frac{n}{p+1}\right)^{-\frac{n}{p}} = 1 + \frac{n\log p}{p} \pm o(\frac{p}{\log p})$$

and

$$\frac{h^n_K(\theta)}{h^n_{Z_p(K)}(\theta)} \leqslant B(p+1,n)^{-\frac{n}{p}} = 1 + \frac{n^2\log p}{p} \pm o(\frac{p}{\log p}).$$

For the last equality see e.g. [19], Lemma 4.3 - which is also stated here as Lemma 3.3. Lebesgue's convergence theorem completes the proof. □

3 Polytopes

Let K be a convex polytope in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_M$. For $0 \leq k \leq n-1$, let $\mathcal{A}_k = \{F_k : F_k$ is a k-dimensional face of $K\}$. For $\theta \in S^{n-1}$ and $0 \leq s \leq h_k(\theta)$ let

$$g(\theta, s) = \text{card}\left(\{v_i : v_i \in K \cap \{\langle v_i, \theta\rangle \geq s\}\right).$$

Let

$$\mathcal{B}_K = \{\theta \in S^{n-1} : \forall\ s \leq h_K(\theta) : g(\theta, s) > 1\} \tag{2}$$

and

$$\mathcal{G}_K = \{\theta \in S^{n-1} : \exists\ s < h_K(\theta) : g(\theta, s) = 1\} \tag{3}$$

Finally, for $\theta \in \mathcal{G}_K$, let

$$s_\theta = \min\{s > 0 : g(\theta, s) = 1\} \tag{4}$$

Remarks. Let $\theta \in \mathcal{G}_K$.

(i) Then there is a vertex v_i such that for all $s_\theta \leq s \leq h_K(\theta)$

$$\{x \in K : \langle x, \theta\rangle \geq s\} = \text{co}\big[K \cap (\theta^\perp + s\theta), v_i\big]$$

(ii) Recall that $f_{K,\theta}(s) = |K \cap (\theta^\perp + s\theta)|$. We have for all $s_\theta \leq s \leq h_K(\theta)$

$$f_{K,\theta}(s) = f_{K,\theta}(s_\theta)\left(\frac{1 - \frac{s}{h_K(\theta)}}{1 - \frac{s_\theta}{h_K(\theta)}}\right)^{n-1} \tag{5}$$

For a convex body K, let $H_K = \max_{\theta \in S^{n-1}} h_K(\theta)$.
For $1 \leq k \leq n$, let K be a k-dimensional convex body in a k-dimensional affine space of $\mathbb{R}^n$. Let

$$r(K) = \sup\{r > 0 : \exists\ x \in K \ \text{ such that } \ x + rB_2^k \subseteq K\} \tag{6}$$

be the inradius of K. Let

$$r_0 = \min_{1 \leq k \leq n-1} \min_{F_k \in \mathcal{A}_k} r(F_k)$$

Note that $r_0 > 0$. We also put $h_0 = \max_{u \in \mathcal{B}_K} h_K(u)$.

For $\delta > 0$, we define

$$A(\delta) = \{\theta \in S^{n-1} : \exists\ u \in \mathcal{B}_K : \|\theta - u\| < \delta\}. \tag{7}$$

and

$$s(\delta) = \sup_{\theta \in S^{n-1} \setminus A(\delta)} \frac{s_\theta}{h_K(\theta)} \tag{8}$$

Remark. $s(\delta) < 1$ and if $\theta \to \phi$ where $\phi \in \mathcal{B}_K$, then by continuity, $\frac{s_\theta}{h_K(\theta)} \to 1$. Hence we may assume that for $\delta > 0$ small enough, $s(\delta)$ is attained on the "boundary" of $S^{n-1} \setminus A(\delta)$.

Lemma 3.1. *Let K be a 0-symmetric polytope in $\mathbb{R}^n$ of volume 1. Then for δ small enough,*

$$s(\delta) = \sup_{\theta \in S^{n-1} \setminus A(\delta)} \frac{s_\theta}{h_K(\theta)} \leq 1 - \frac{\delta r_0}{2h_0}$$

Proof. Let $\delta \leq \frac{h_0}{H_K}$. By the above Remark, for $\delta > 0$ small enough, there exists $\phi \in S^{n-1} \setminus A(\delta)$ such that $s(\delta) = \frac{s_\phi}{h_K(\phi)}$.
As $\phi \in S^{n-1} \setminus A(\delta)$, there exists $u \in \mathcal{B}_K$, such that $\|u - \phi\| = \delta$. Let $v \in \partial K$ be that vertex of K such that $\langle \phi, v \rangle = \max_{x \in K} \langle \phi, x \rangle$. Let

$$x_0 = \{\alpha\phi : \alpha \geq 0\} \cap \partial K, \qquad z_0 = \{\alpha u : \alpha \geq 0\} \cap \partial K,$$

and

$$d_1 = \|x_0 - z_0\|, \qquad d_2 = \|x_0 - v\|.$$

x_0, v and z_0 lie in the $n-1$-dimensional face F orthogonal to u. As $\phi \in \mathcal{G}_K$, we may also assume that δ is small enough such that $s_\phi = \|x_0\|$, and hence $s(\delta) = \frac{\|x_0\|}{h_K(\phi)}$.

Let ω be the angle between ϕ and u. Then

$$\tan \omega = \frac{d_1}{h_K(u)} \quad \text{and} \quad \sin \omega = \frac{h_K(\phi) - s_\phi}{d_2}.$$

Hence

$$\frac{h_K(\phi) - s_\phi}{d_2} = \frac{d_1 \cos \omega}{h_K(u)}$$

and thus

$$\frac{s_\phi}{h_K(\phi)} = 1 - \frac{d_1 d_2 \cos \omega}{h_K(u) h_K(\phi)}.$$

As $d_2 \geq r_0$ and as $\delta \leq \frac{d_1 \cos \omega}{h_K(u)}$, we get that

$$\frac{s_\phi}{h_K(\phi)} \leq 1 - \frac{\delta \ r_0}{h_K(\phi)}.$$

Now observe that

$$h_k(\phi) = h_K(\phi - u) + h_K(u) \leq \delta H_K + h_K(u) \leq 2h_0.$$

Therefore,

$$\frac{s_\phi}{h_K(\phi)} \leq 1 - \frac{\delta r_0}{2h_0}.$$

□

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 log-concave function with $\int_{\mathbb{R}_+} f(t)dt < \infty$ and let $p \geq 1$. Let $g_p(t) = t^p f(t)$ and let $t_p = t_p(f)$ the unique point such that $g'(t_p) = 0$. We make use of the following Lemma due to B. Klartag [10] (Lemma 4.3 and Lemma 4.5).

Lemma 3.2. *Let f be as above. For every $\varepsilon \in (0,1)$,*

$$\int_0^\infty t^p f(t)dt \le \left(1 + Ce^{-cp\varepsilon^2}\right) \int_{t_p(1-\varepsilon)}^{t_p(1+\varepsilon)} t^p f(t)dt$$

where $C > 0$ and $c > 0$ are universal constants.

We will use Lemma 3.2 for the function $f_{K,\theta}(s) = |K \cap (\theta^\perp + s\theta)|$ in the proof of the next lemma. First we oberve

Remark 1. Let $\theta \in \mathcal{G}_K$. As above, let $g_p(t) = t^p f_{K,\theta}(t)$ and let t_p be the unique point such that $g_p'(t_p) = 0$. Note that, since $t_p \to h_K(\theta)$, as $p \to \infty$ (see e.g. [19], Lemma 4.5), for p large enough - namely p so large that $t_p \ge s_\theta$ - we can use (5) and compute t_p.

$$t_p = \frac{p}{p+n-1} h_K(\theta) \tag{9}$$

We will also use (see e.g. [19], Lemma 4.3).

Lemma 3.3. *Let $p > 0$. Then*

$$\begin{aligned} (B(p+1,n))^{\frac{n}{p}} &= 1 - \frac{n^2}{p}\log p + \frac{n}{p}\log(\Gamma(n)) + \frac{n^4}{2p^2}(\log p)^2 - \frac{n^3}{p^2}\log(\Gamma(n))\log p \\ &\pm o(p^2). \end{aligned}$$

Lemma 3.4. *Let K be a 0-symmetric polytope in $\mathbb{R}^n$ of volume 1. For all sufficiently small δ, for all $\theta \in S^{n-1} \setminus A(\delta)$ and for all $p \ge \frac{\alpha_n(K)}{\delta}$, we have*

$$\left(\frac{h_{Z_p(K)}(\theta)}{h_K(\theta)}\right)^n \le 1 - n^2 \, \frac{\log p}{p} + (n-1)n \, \frac{\log \frac{1}{\delta}}{p} + \frac{c_{K,n}}{p}.$$

$\alpha_n(K) = \frac{4(n-1)h_0}{r_0}$ and $c_{K,n}$ are constants that depend on K and n only.

Proof. Let $0 < \delta \le \frac{h_0}{H_K}$ be as in Lemma 3.1. Let $\theta \in S^{n-1} \setminus A(\delta)$. Hence, in particular, $\theta \in \mathcal{G}_K$. By Lemma 3.2 we have for all $\varepsilon \in (0,1)$

$$\begin{aligned} h^p_{Z_p(K)}(\theta) &= 2\int_0^{h_K(\theta)} t^p f_{K,\theta}(t)dt \\ &\le 2\left(1 + Ce^{-cp\varepsilon^2}\right) \int_{(1-\varepsilon)t_p}^{h_K(\theta)} t^p f_{K,\theta}(t)dt \end{aligned}$$

Since $t_p \to h_K(\theta)$, as $p \to \infty$ (see e.g. [19], Lemma 4.5), there exists $p_\varepsilon > 0$ (which we will now determine), such that for all $p \ge p_\varepsilon$,

$$(1-\varepsilon)t_p \ge s_\theta. \tag{10}$$

By (9), (10) holds for all $p \geq p_\varepsilon$ with

$$p_\varepsilon \geq \frac{(n-1)\frac{s_\theta}{h_K(\theta}}{1-\varepsilon-\frac{s_\theta}{h_K(\theta}}.$$

By Lemma 3.1, $\frac{s(\theta)}{h_K(\theta)} \leq 1 - \frac{\delta r_0}{2h_0}$ and thus (10) holds for all $p \geq p_\varepsilon$ with

$$p_\varepsilon \geq \frac{n-1}{\delta}\ \frac{2h_0 - \delta r_0}{r_0 - 2h_0\varepsilon/\delta}.$$

We choose $\varepsilon = \frac{r_0 \delta}{4h_0}$. Then for

$$p_\varepsilon \geq \frac{n-1}{\delta}\ \frac{4h_0}{r_0}$$

the estimate (10) holds for all $p \geq p_\varepsilon$ uniformly for all $\theta \in S^{n-1} \setminus A(\delta)$. Thus, using also (5),

$$\begin{aligned}
h^p_{Z_p(K)}(\theta) &\leq 2\left(1+Ce^{-cp\varepsilon^2}\right)\int_{(1-\varepsilon)t_p}^{h_K(\theta)} t^p f_{K,\theta}(t)dt \\
&\leq 2\left(1+Ce^{-cp\varepsilon^2}\right)\int_{s_\theta}^{h_K(\theta)} t^p f_{K,\theta}(t)dt \\
&= 2\left(1+Ce^{-cp\varepsilon^2}\right)\ \frac{h_K^{p+1}(\theta) f_{K,\theta}(s_\theta)}{\left(1-\frac{s_\theta}{h_K(\theta)}\right)^{n-1}}\ \int_{\frac{s_\theta}{h_K(\theta)}}^1 u^p\,(1-u)^{n-1}\,du \\
&\leq 2\left(1+Ce^{-cp\varepsilon^2}\right)\ \frac{h_K^{p+1}(\theta) f_{K,\theta}(0)}{\left(1-\frac{s_\theta}{h_K(\theta)}\right)^{n-1}}\ \int_{\frac{s_\theta}{h_K(\theta)}}^1 u^p\,(1-u)^{n-1}\,du \\
&\leq n\ \left(1+Ce^{-cp\varepsilon^2}\right)\ B(p+1,n)\ h_K^p(\theta)\left(\frac{2h_0}{\delta r_0}\right)^{n-1}. \qquad (11)
\end{aligned}$$

In the last inequality we have used that $1-\frac{s_\theta}{h_K(\theta)} \geq \frac{\delta r_0}{2h_0}$ and that $\frac{2}{n}h_K(\theta)f_{K,\theta}(0) \leq |K| = 1$. Equivalently, (11) becomes

$$\left(\frac{h_{Z_p(K)}(\theta)}{h_K(\theta)}\right)^n \leq n^{\frac{n}{p}}\left(1+Ce^{-cp\varepsilon^2}\right)^{\frac{n}{p}}\left(\frac{2h_0}{\delta r_0}\right)^{\frac{(n-1)n}{p}} B(p+1,n)^{\frac{n}{p}}.$$

With Lemma 3.3, we then get

$$\left(\frac{h_{Z_p(K)}(\theta)}{h_K(\theta)}\right)^n \leq 1 - n^2\frac{\log p}{p} + (n-1)n\frac{\log\frac{1}{\delta}}{p} + \frac{c_{K,n}}{p}.$$

□

Let $\delta \in [0,1)$ and $\theta \in S^{n-1}$. We define the cap $C(\theta,\delta)$ of the sphere S^{n-1} around θ by

$$C(\theta,\delta) := \{\phi \in S^{n-1} : \|\phi-\theta\|_2 \leqslant \delta\}.$$

We will estimate the surface area of a cap, and to do so we will make use of the following fact which follows immediately from e.g. Lemma 1.3 in [23].

Lemma 3.5. *Let $\theta \in S^{n-1}$ and $\delta < 1$. Then*

$$\begin{aligned} vol_{n-1}\big(B_2^{n-1}\big)\left(1-\frac{\delta^2}{4}\right)^{\frac{n-1}{2}} \delta^{n-1} &\leqslant \\ \sigma(C(\theta,\delta)) &\leq \\ & vol_{n-1}\big(B_2^{n-1}\big)\left(1-\frac{\delta^2}{4}\right)^{\frac{n-1}{2}} \frac{\left(1+\frac{\delta^4}{4}\right)^{\frac12}}{\left(1-\frac{\delta^2}{2}\right)} \delta^{n-1}. \end{aligned}$$

Proof of Theorem 1.1.
For p given, let $\delta = \frac{1}{\log p}$. Let $A(\delta)$ as defined in (2.10). Let p_0 be such that p_0 and $\delta = \frac{1}{\log p}$ satisfy the assumptions of Lemma 3.4, i.e. $\frac{p_0}{\log p_0} \geq \frac{4(n-1)h_0}{r_0}$. By Lemma 3.4, we have for all $p \geq p_0$,

$$\begin{aligned} &|Z_p^\circ(K)| - |K^\circ| \geq \frac{1}{n}\int_{S^{n-1}\setminus A(\delta)} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(1-\frac{h^n_{Z_p(K)}(\theta)}{h^n_K(\theta)}\right) d\sigma(\theta) \\ &\geq \frac{1}{n}\int_{S^{n-1}\setminus A(\delta)} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(\frac{n^2\log p}{p} - (n-1)n\frac{\log\log p}{p} + \frac{c_{K,n}}{p}\right) d\sigma(\theta) \\ &= \frac{1}{n}\int_{S^{n-1}} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(\frac{n^2\log p}{p} - (n-1)n\frac{\log\log p}{p} + \frac{c_{K,n}}{p}\right) d\sigma(\theta) \\ &-\frac{1}{n}\int_{A(\delta)} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(\frac{n^2\log p}{p} - (n-1)n\frac{\log\log p}{p} + \frac{c_{K,n}}{p}\right) d\sigma(\theta). \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) \geq \\ &\frac{1}{n}\int_{S^{n-1}} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(n^2 - \frac{(n-1)n\log\log p}{\log p} + \frac{c_{K,n}}{\log p}\right) d\sigma(\theta) \\ &-\frac{1}{n}\int_{A(\delta)} \frac{1}{h^n_{Z_p(K)}(\theta)}\left(n^2 - \frac{(n-1)n\log\log p}{\log p} + \frac{c_{K,n}}{\log p}\right) d\sigma(\theta). \end{aligned}$$

Note that, since K is centrally symmetric, $r(K) = \inf_{\theta\in S^{n-1}} h_K(\theta)$. Also, since $Z_p(K)$ converges to K, for p sufficiently large, $h^n_{Z_p(K)}(\theta) \geq \left(\frac{r(K)}{2}\right)^n$ for every

$\theta \in S^{n-1}$. Together with Lemma 3.5 we thus get

$$\frac{1}{n}\int_{A(\delta)} \frac{1}{h^n_{Z_p(K)}(\theta)} d\sigma(\theta) \leq$$

$$\frac{2^{n+1}}{n\ r(K)^n} \operatorname{card}\left(\mathcal{B}_K\right) \operatorname{vol}_{n-1}\left(B_2^{n-1}\right) \delta^{n-1} \left(1-\frac{\delta^2}{4}\right)^{\frac{n-1}{2}} \frac{\left(1+\frac{\delta^4}{4}\right)^{\frac{1}{2}}}{\left(1-\frac{\delta^2}{2}\right)}$$

$$\leq \frac{2^{n+1}\operatorname{card}\left(\mathcal{B}_K\right)}{n\ r(K)^n} \frac{\operatorname{vol}_{n-1}\left(B_2^{n-1}\right)}{(\log p)^{n-1}}.$$

By Proposition 2.2 and Lebesgue's convergence theorem we can interchange integration and limit and get

$$\lim_{p\to\infty} \frac{p}{\log p}\left(|Z_p^\circ(K)| - |K^\circ|\right) \geq$$

$$\frac{1}{n}\int_{S^{n-1}} \lim_{p\to\infty} \frac{1}{h^n_{Z_p(K)}(\theta)} \left(n^2 - \frac{(n-1)n\log\log p}{\log p} + \frac{c_{K,n}}{\log p}\right) d\sigma(\theta)$$

$$-\frac{2^{n+1}\operatorname{card}\left(\mathcal{B}_K\right)\operatorname{vol}_{n-1}\left(B_2^{n-1}\right)}{n\ r(K)^n} \lim_{p\to\infty}\left(\frac{n^2}{(\log p)^{n-1}} - \frac{(n-1)n\log\log p}{(\log p)^n} + \frac{c_{K,n}}{(\log p)^n}\right)$$

$$= n^2|K^\circ|.$$

Here, we have also used that $\lim_{p\to\infty} h_{Z_p(K)}(\theta) = h_K(\theta)$.

The inequality from above follows by Proposition 2.2. □

4 Approximation with uniformly convex bodies

Let K be a symmetric convex body in $\mathbb{R}^n$ and $2 \leqslant p < \infty$. We say that K is p-uniformly convex (with constant C_p) (see e.g. [3, 11]), if for every $x, y \in \partial K$,

$$\|\frac{x+y}{2}\|_K \leqslant 1 - C_p\|x-y\|_K^p.$$

We will need the following Proposition. The proof is based on Clarkson inequalities and can be found in e.g. ([3], pp. 148).

Proposition 4.1. *Let K be a compact set in $\mathbb{R}^n$ of volume 1. Then for $p \geqslant 2$, $Z_p^\circ(K)$ is p-uniformly convex with constant $C_p = \frac{1}{p2^p}$.*

The symmetric difference metric between two convex bodies K and C is

$$d_s(C,K) = |(C\setminus K)\cup(K\setminus C)|.$$

Proof of Theorem 1.2.

Let $P_1 = \frac{P^\circ}{|P^\circ|^{\frac{1}{n}}}$. Then $P_1^\circ = |P^\circ|^{\frac{1}{n}} P$ and $|P_1^\circ| = |P||P^\circ|$. Let $K_p = |P^\circ|^{-\frac{1}{n}} Z_p^\circ(P_1)$. Then by Proposition 4.1 we have that K_p is uniformly convex. Note that $P \subseteq K_p$. By Theorem 1.1 we have that

$$\lim_{p\to\infty} \frac{p}{\log p} \left(|Z_p^\circ(P_1)| - |P_1^\circ|\right) = n^2 |P_1^\circ|.$$

So, for every $\varepsilon > 0$, there exists $p_0(\varepsilon, P)$ such that

$$d_s(P, K_p) = |K_p| - |P| = \frac{1}{|P^\circ|} \left(|Z_p^\circ(P_1)| - |P_1^\circ|\right) \leqslant$$

$$(1+\varepsilon) n^2 \frac{|P_1^\circ|}{|P^\circ|} \frac{\log p}{p} = (1+\varepsilon) n^2 |P| \frac{\log p}{p}.$$

We choose $\varepsilon = 1$ and the proof is complete. □

References

[1] R.J. Gardner, *Geometric tomography*, Encyclopedia of Mathematics and its Applications, **58**. Cambridge University Press, Cambridge, 1995.

[2] R. J. Gardner and G. Zhang, *Affine inequalities and radial mean bodies.* Amer. J. Math. **120** no.3 (1998), 505-528.

[3] D. J. H. Garling, *Inequalities, A journey into Linear Analysis*, Cambridge University Press (2007).

[4] E. Grinberg and G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. **78** no.3 (1999), 77-115.

[5] C. Haberl, *Blaschke valuations*, Amer. J. of Math., in press

[6] C. Haberl and F. Schuster, *General Lp affine isoperimetric inequalities.* J. Differential Geometry **83** (2009), 1-26.

[7] C. Haberl, E. Lutwak, D. Yang and G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math. **224** (2010), 2485-2510

[8] D. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80-101.

[9] D. Klain, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), 95-113.

[10] B. Klartag, *A central limit theorem for convex sets*, Invent. Math. **168** (2007) 91-131.

[11] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I, II*, reprint, Springer, 1996.

[12] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191-4213.

[13] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geometry, **86** (2010), 133-162.

[14] M. Ludwig and M. Reitzner, *A Characterization of Affine Surface Area*, Adv. Math. **147** (1999), 138-172.

[15] M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations.* Ann. of Math. **172** (2010), 1223-1271.

[16] E. Lutwak, *The Brunn-Minkowski-Firey theory II : Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244-294.

[17] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[18] M. Meyer and E. Werner, *The Santalo-regions of a convex body*, Transactions of the AMS **350** (1998), 4569–4591.

[19] G. Paouris and E. Werner, *Relative entropy of cone measures and L_p-centroid bodies*, Proc. London Math. Soc., in press (available at arXiv:0909.436).

[20] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, Cambridge (1993).

[21] M. Schmuckenschläger, *The distribution function of the convolution square of a convex symmetric body in $\mathbb{R}^n$*, Israel Journal of Math. **78** (1992), 309–334.

[22] C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66**, (1990), 275–290.

[23] C. Schütt and E. Werner, *Polytopes with Vertices Chosen Randomly from the Boundary of a Convex Body*, Geom. Aspects of Funct. Analysis, Lecture Notes in Math. **1807** (2003), 241–422.

[24] C. Schütt and E. Werner, *Surface bodies and p-affine surface area.* Adv. Math. **187** (2004), 98-145.

[25] A. Stancu, *The Discrete Planar L_0-Minkowski Problem.* Adv. Math. **167** (2002), 160-174.

[26] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem.* Adv. Math. **180** (2003), 290-323.

[27] E. Werner, *Illumination bodies and affine surface area*, Studia Math. **110** (1994), 257–269 .

[28] E. Werner and D. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218** no.3 (2008), 762-780.

[29] E. Werner and D. Ye, *Inequalities for mixed p-affine surface area*, Math. Ann. **347** (2010), 703-737

[30] G. Zhang, *New Affine Isoperimetric Inequalities*, ICCM 2007, Vol. II, 239-267.

Grigoris Paouris
Department of Mathematics
Texas A & M University
College Station, TX, , U. S. A.

Elisabeth Werner
Department of Mathematics
Case Western Reserve University
Cleveland, Ohio 44106, U. S. A.
elisabeth.werner@case.edu

Université de Lille 1
UFR de Mathématique
59655 Villeneuve d'Ascq, France