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Relative entropy of cone measures and L_p centroid bodies

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Abstract

Let K be a convex body in $\mathbb{R}^n$. We introduce a new affine invariant, which we call Ω_K, that can be found in three different ways:

(a) as a limit of normalized L_p-affine surface areas;
(b) as the relative entropy of the cone measure of K and the cone measure of K°;
(c) as the limit of the volume difference of K and L_p-centroid bodies.

We investigate properties of Ω_K and of related new invariant quantities. In particular, we show new affine isoperimetric inequalities and we show an 'information inequality' for convex bodies.

1. *Introduction*

An important affine invariant quantity in convex geometric analysis is the *L_p-affine surface area*, which, for a convex body K in $\mathbb{R}^n$ and $-\infty \leqslant p \leqslant \infty$, $p \neq -n$, is defined by

$$\mathrm{as}_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{p/(n+p)}}{\langle x, N_K(x)\rangle^{n(p-1)/(n+p)}}\, d\mu_K(x). \tag{1.1}$$

We see that $\kappa(x) = \kappa_K(x)$ is the generalized Gaussian curvature at the boundary point x of K, $N_K(x)$ is the outer unit normal vector at x to ∂K, the boundary of K and $\mu = \mu_K$ is the surface area measure on the boundary ∂K.

We denote by $|K|$ the n-dimensional volume of the convex body K and by $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1\}$ the polar body of K. We use the L_p-affine surface area to introduce a new affine invariant Ω_K as a limit of normalized L_p-affine surface areas:

$$\Omega_K = \lim_{p\to\infty} \left(\frac{\mathrm{as}_p(K)}{n|K^\circ|}\right)^{n+p}. \tag{1.2}$$

This is a first way how Ω_K appears.

The second way how Ω_K appears is as the exponential of the *relative entropy* or *Kullback–Leibler divergence* D_{KL} of the cone measures cm_K and cm_{K° of a convex body K and its polar body K°:

$$\Omega_K^{1/n} = \frac{|K^\circ|}{|K|} \exp\left(-D_{\mathrm{KL}}(N_K N_{K^\circ}^{-1} \mathrm{cm}_{\partial K^\circ} \| \mathrm{cm}_{\partial K})\right). \tag{1.3}$$

Here N_K^{-1} is the inverse of the Gauss map. We refer to Section 3 for its definition and that of the relative entropy and the cone measures.

For a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \leqslant p \leqslant \infty$, the L_p *centroid body* $Z_p(K)$ is this convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta\rangle|^p\, dx\right)^{1/p}. \tag{1.4}$$

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The study of the asymptotic behavior of the volume of L_p centroid bodies as p tends to infinity resulted in the discovery that, for a symmetric convex body K of volume 1,

$$\lim_{p\to\infty}\frac{2p}{n}\left(\frac{(1-n(n+1)\log p/2p)|Z_p^\circ(K)|}{|K^\circ|}-1\right)=-\frac{1}{2}\log\frac{\Omega_K^{1/n}}{2^{n+1}\pi^{n-1}}. \tag{1.5}$$

This is the third way how Ω_K appears.

Thus, the invariant Ω_K introduces a novel idea (relative entropy) into the theory of convex bodies and links concepts from classical convex geometry, like L_p centroid bodies and L_p-affine surface area, with concepts from information theory. Such links have already been established. Guleryuz, Lutwak, Yang and Zhang [**18**, **35**–**38**]) use L_p Brunn–Minkowski theory to develop certain entropy inequalities. Also, classical Brunn–Minkowski theory is related to information theoretic concepts (see, for example, [**3**, **4**, **13**, **14**]).

An important affine invariant quantity in convex geometric analysis is the affine surface area, which, for a convex body $K\in\mathbb{R}^n$, is defined as

$$\mathrm{as}_1(K)=\int_{\partial K}\kappa^{1/(n+1)}(x)\,d\mu(x). \tag{1.6}$$

Originally, a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention (for example, [**5**, **32**, **40**, **49**, **56**]). It is fundamental in the theory of valuations (see, for example, [**1**, **2**, **22**, **29**]), in approximation of convex bodies by polytopes (for example, [**17**, **30**, **50**]) and it is the subject of the affine Plateau problem solved in $\mathbb{R}^3$ by Trudinger and Wang [**54**, **55**].

The definition (1.6), at least for convex bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$ with sufficiently smooth boundary, goes back to Blaschke [**8**] and was extended to arbitrary convex bodies by, for example, [**27**, **32**, **40**, **49**]. Schütt and Werner showed in [**49**] that the affine surface area equals

$$\mathrm{as}_1(K)=\lim_{\delta\to 0}c_n\frac{|K|-|K_\delta|}{\delta^{2/(n+1)}},$$

where c_n is a constant depending only on n and K_δ is the *convex floating body* of K (see [**49**]): the intersection of all half-spaces H^+ whose defining hyperplanes H cut off a set of volume δ from K.

It was shown by Milman and Pajor [**42**] that if K is a symmetric convex body, then, for large δ, the floating body K_δ is always uniformly, up to a factor $c(\delta)$ depending on δ, isomorphic to the dual of the Binet ellipsoid from classical mechanics and consequently K_δ° is isomorphic (up to a factor $c(\delta)$) to the Binet ellipsoid.

Lutwak and Zhang [**39**] generalized the notion of Binet ellipsoid and introduced the L_p centroid bodies defined by their support function $h_{Z_p(K)}$ as given in (1.4).

Note that in [**39**] a different notation and normalization was used for the centroid body. In the present paper, we follow the notation and normalization that appeared in [**45**].

The results of this paper deal mostly with centrally symmetric convex bodies K. Symmetry is assumed mainly because the L_p centroid bodies are symmetric by definition (1.4) and used to approximate the convex bodies K. There exists a non-symmetric definition of L_p centroid bodies in [**28**] (see also [**19**]). Using this definition, we feel the results of the paper can be carried over to non-symmetric convex bodies.

In Theorem 2.2, we generalize the result by Milman and Pajor mentioned above and show that the floating body K_δ is, up to a universal constant, homothetic to the centroid body $Z_{\log(1/\delta)}(K)$.

The L_p-affine surface area, an extension of affine surface area, was introduced by Lutwak in the ground-breaking paper [**33**] for $p>1$, and by Schütt and Werner [**51**] for general p. It is now at the core of the rapidly developing L_p Brunn–Minkowski theory. Contributions here include new interpretations of L_p-affine surface areas [**41**, **50**, **51**, **56**, **57**], the study of solutions of non-trivial ordinary and partial differential equations (see, for example, Chen [11], Chou and Wang [12], Stancu [52, 53]), the study of the L_p Christoffel–Minkowski problem by Hu, Ma and Shen [20], characterization theorems by Ludwig and Reitzner [29] and the study of L_p-affine isoperimetric inequalities by Lutwak [33] and Werner and Ye [56, 57].

From now on we shall always assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. We write $K \in C^2_+$, if K has C^2 boundary with everywhere strictly positive Gaussian curvature κ_K. For real $p \neq -n$ we define the L_p-affine surface area $\mathrm{as}_p(K)$ of K as in [33] $(p > 1)$ and [51] $(p < 1, p \neq -n)$ as in (1.1) by

$$\mathrm{as}_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{p/(n+p)}}{\langle x, N_K(x)\rangle^{n(p-1)/(n+p)}}\, d\mu_K(x)$$

and

$$\mathrm{as}_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n}}\, d\mu_K(x), \tag{1.7}$$

provided the integrals exist. In particular, for $p = 0$,

$$\mathrm{as}_0(K) = \int_{\partial K} \langle x, N_K(x)\rangle\, d\mu_K(x) = n|K|.$$

For $p = 1$ we get the classical affine surface area (1.6) which is independent of the position of K in space.

In Section 3, we introduce the new affine invariant

$$\Omega_K = \lim_{p\to\infty} \left(\frac{\mathrm{as}_p(K)}{n|K^\circ|}\right)^{n+p},$$

and describe properties of this new invariant. For example, in Corollary 3.9 we prove the remarkable identity (1.3), which shows that the invariant Ω_K is the exponential of the relative entropy or Kullback–Leibler divergence D_{KL} of the cone measures cm_K and cm_{K° of K and K°.

We show that the information inequality [13] for the relative entropy of the cone measures implies an 'information inequality' for convex bodies

$$\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$$

with equality if and only if K is an ellipsoid. Independently, we can derive this inequality from properties of the L_p-affine surface areas.

The next proposition gives a sample of some inequalities that hold for the affine invariant Ω_K, among them an isoperimetric inequality. More can be found in Proposition 3.5.

Proposition. *Let K be a convex body with its centroid at the origin.*

(i) *For all $p \geqslant 0$, $\Omega_K \leqslant (\mathrm{as}_p(K)/n|K^\circ|)^{n+p}$.*
(ii) *We have $\Omega_K \leqslant (|K|/|K^\circ|)^n$.*
(iii) *If in addition $|K| = 1$, then $\Omega_{K^\circ} \leqslant \Omega_{(B^n_2/|B^n_2|^{1/n})^\circ}$.*

If K is in addition in C^2_+, then equality holds in (i) *and* (ii) *if and only if K is an ellipsoid and in* (iii) *if and only if K is a normalized ellipsoid.*

Theorem 2.2 states that the floating body K_δ is, up to a universal constant, homothetic to the centroid body $Z_{\log_{(e/2\delta)}}(K)$. This, and the geometric interpretations of L_p-affine surface areas in terms of variants of the floating bodies [51, 56, 57], led us to investigate the L_p centroid bodies also in the context of affine surface area. Note the similarities in behavior of the floating body and the L_p centroid body. Both 'approximate' K as $\delta \to 0$, and $p \to \infty$, respectively: If K is symmetric and of volume 1, then $Z_p(K) \to K$ as $p \to \infty$.

We found an amazing connection between the L_p centroid bodies and the new invariant Ω_K. The precise statement is given in Theorem 4.1 for convex bodies in C^2_+. A forthcoming paper will address general convex bodies.

In view of Theorem 2.2, the first part of Theorem 4.1 came as a surprise to us because it reveals a different behavior of the bodies K_δ and $Z_{\log(1/\delta)}(K)$ when $\delta \to 0$. Indeed, it was shown in [**41**] that, with a constant c_n that depends on n only,

$$\lim_{\delta\to 0} c_n \frac{|(K_\delta)^\circ| - |K^\circ|}{\delta^{2/(n+1)}} = \mathrm{as}_{-n(n+2)}(K) = \mathrm{as}_{-n/(n+2)}(K^\circ),$$

whereas

$$\lim_{p\to\infty} \frac{p}{\log p}(|Z_p^\circ(K)| - |K^\circ|) = \frac{n(n+1)}{2}|K^\circ|.$$

Even more surprising is the second part of Theorem 4.1, which, combined with Proposition 3.6, shows how the new invariant and the L_p centroid bodies are related via the formula (1.5). The details are given in Section 4.

Further notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write σ for the rotationally invariant surface measure on S^{n-1}.

A convex body is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is 0-symmetric, if $x \in C$ implies that $-x \in C$. We say that C has center of mass at the origin if $\int_C \langle x,\theta\rangle\,dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x,y\rangle : y \in C\}$. The polar body C° of C is $C^\circ = \{y \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1$ for all $x \in C\}$.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. The letters c, c', c_1, c_2 and so on. denote absolute positive constants which may change from line to line. We refer the reader to the books [**47**, **48**] for basic facts from the Brunn–Minkowski theory and the asymptotic theory of finite-dimensional normed spaces.

2. *Comparison of floating bodies and L_p centroid bodies*

It is well known from mechanics that the body $Z_2(K)$ is an ellipsoid. Its polar body $Z_2^\circ(K)$ is called the Binet ellipsoid of inertia. We see that $Z_1(K) = Z(K)$ is the classical centroid body and it is a zonoid by definition (see [**15**, **48**]).

The isotropic constant L_K of a convex body $K \in \mathbb{R}^n$ is defined as

$$L_K = \left(\frac{|Z_2(K)|}{|B_2^n|}\right)^{1/n}.$$

Here L_K is an affine invariant and $L_K \geqslant L_{B_2^n}$.

A major open problem in convex geometry asks if there exists a universal constant $C > 0$ such that $L_K \leqslant C$. The best known result up to date is due to Klartag [**23**] and states that $L_K \leqslant Cn^{1/4}$, improving by a factor of logarithm an earlier result by Bourgain [**9**].

Let us briefly state some of the known properties of the L_p centroid bodies. For the proofs and further references, see [**45**].

Let $T \in \mathrm{SL}(n)$, that is, $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator with determinant 1. Let T^* denote its adjoint. Then

$$h_{Z_p(TK)}(\theta) = \left(\int_{TK} |\langle x,\theta\rangle|^p\,dx\right)^{1/p} = \left(\int_K |\langle x,T^\star(\theta)\rangle|^p\,dx\right)^{1/p} = h_{Z_p(K)}(T^\star(\theta))$$

or

$$h_{Z_p(TK)}(\theta) = h_{T(Z_p(K))}(\theta).$$

By Hölder's inequality, we have for $1 \leqslant p \leqslant q \leqslant \infty$ and convex bodies K in $\mathbb{R}^n$ with $|K| = 1$, that

$$Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K) = K. \tag{2.1}$$

As an application of the Brunn–Minkowski inequality, one has for $1 \leqslant p \leqslant q < \infty$ that

$$Z_q(K) \subseteq c\frac{q}{p} Z_p(K). \tag{2.2}$$

Here $c > 0$ is a universal constant.

Inequality (2.2) is sharp with the right constant for the l_n^1-ball [7].

By Brunn's principle we get, for $p \geqslant n$ and a (new) absolute constant $c > 0$ (for example, [44]),

$$Z_p(K) \supseteq cK. \tag{2.3}$$

Lutwak, Yang and Zhang [34] and Lutwak and Zhang [39] proved the following L_p versions of the Blaschke Santaló inequality and the Busemann–Petty inequality; see also Campi and Gronchi [10] for an alternative proof.

THEOREM 2.1 [34, 39]. *Let K be a convex body in $\mathbb{R}^n$ of volume 1. Then, for every $1 \leqslant p \leqslant \infty$,*

$$|Z_p^\circ(K)| \leqslant \left| Z_p^\circ\left(\frac{B_2^n}{|B_2^n|^{1/n}}\right)\right|,$$
$$|Z_p(K)| \geqslant \left| Z_p\left(\frac{B_2^n}{|B_2^n|^{1/n}}\right)\right|$$

with equality if and only if K is an ellipsoid.

A computation shows that $|Z_p(B_2^n/|B_2^n|)|^{1/n} \simeq \sqrt{p/(n+p)}$. Hence, the following inequality, proved in [45] for all $p \geqslant 1$ and a universal constant $c > 0$, can be viewed as an 'Inverse Lutwak–Yang–Zhang inequality':

$$|Z_p(K)|^{1/n} \leqslant c\sqrt{\frac{p}{n+p}} L_K. \tag{2.4}$$

We now want to compare L_p centroid bodies and floating bodies. As K is symmetric and has volume 1, the floating body K_δ, for $\delta \in [0,1]$, may be defined in the following way [49]:

$$K_\delta = \bigcap_{\theta \in S^{n-1}} \{x \in K : |\langle x, \theta\rangle| \leqslant t_\theta\}, \tag{2.5}$$

where $t_\theta = \sup\{t > 0 : |\{x \in K : |\langle x, \theta\rangle| \leqslant t\}| = 1 - \delta\}$. Hence, for every $\theta \in S^{n-1}$, one has that

$$h_{K_\delta}(\theta) = t_\theta. \tag{2.6}$$

THEOREM 2.2. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1. Let $\delta \in (0, \frac{1}{2})$. Then we have, for every $\theta \in S^{n-1}$,*

$$c_1 h_{Z_{\log(e/2\delta)}(K)}(\theta) \leqslant h_{K_\delta}(\theta) \leqslant c_2 h_{Z_{\log(e/2\delta)}(K)}(\theta)$$

or, equivalently,

$$c_1 Z_{\log(e/2\delta)}(K) \subseteq K_\delta \subseteq c_2 Z_{\log(e/2\delta)}(K),$$

where $c_1, c_2 > 0$ are universal constants. Consequently,

$$\frac{1}{c_1} Z^\circ_{\log(e/2\delta)}(K) \supseteq K_\delta^\circ \supseteq \frac{e}{c_2} Z^\circ_{\log(e/2\delta)}(K).$$

Proof. Assume first that $\delta \in (1/e, 1/2)$. Then the fact that K_δ is isomorphic to $Z_2(K)$ has already been proved in [**42**]. Moreover, a result of Latala [**25**] shows that $Z_p(K)$ is isomorphic to $Z_2(K)$ for $p \in (0, 2)$. So we may assume that $\delta \leqslant 1/e$. We apply Markov's inequality in (1.4) and get

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant e h_{Z_p(K)}(\theta)\}| \leqslant e^{-p}.$$

Then (2.6) gives, for all $p \geqslant 1$,

$$e h_{Z_p(K)}(\theta) \geqslant h_{K_{e^{-p}}}(\theta). \tag{2.7}$$

For the other side we use the Paley–Zygmund inequality: If $Z \geqslant 0$ is a random variable with finite variance and $\lambda \in (0, 1)$, then

$$\Pr\{Z \geqslant \lambda E(Z)\} \geqslant (1-\lambda)^2 \frac{E(Z)^2}{E(Z^2)}.$$

Hence, for $Z = |\langle x, \theta\rangle|^p$ we get

$$\left|\left\{x \in K : |\langle x, \theta\rangle|^p \geqslant \lambda \int_K |\langle x, \theta\rangle|^p\, dx\right\}\right| \geqslant (1-\lambda)^2 \frac{(\int_K |\langle x, \theta\rangle|^p\, dx)^2}{\int_K |\langle x, \theta\rangle|^{2p}\, dx}. \tag{2.8}$$

We see that (2.2) implies that $h_{Z_{2p}(K)}(\theta) \leqslant 2c h_{Z_p(K)}(\theta)$ for all $\theta \in S^{n-1}$. So

$$\frac{(\int_K |\langle x, \theta\rangle|^p\, dx)^2}{\int_K |\langle x, \theta\rangle|^{2p}\, dx} \geqslant \left(\frac{1}{2c}\right)^{2p}.$$

Choose $\lambda = \frac{1}{2}$. Then (2.8) becomes

$$|\{x \in K : |\langle x, \theta\rangle| \geqslant \tfrac{1}{2} h_{Z_p(K)}(\theta)\}| \geqslant e^{-c_1 p}.$$

Now we use again (2.6) to get

$$\tfrac{1}{2} h_{Z_p(K)}(\theta) \leqslant h_{K_{e^{-c_1 p}}}(\theta)$$

or

$$h_{K_{e^{-p}}}(\theta) \geqslant \tfrac{1}{2} h_{Z_{p/c_1}(K)}(\theta) \geqslant c_2 h_{Z_p(K)}(\theta), \tag{2.9}$$

where we have used (2.2) again. Equations (2.7) and (2.9) then imply that

$$c_2 h_{Z_p(K)}(\theta) \leqslant h_{K_{e^{-p}}}(\theta) \leqslant e h_{Z_p(K)}(\theta).$$

Now choose $p = \log(e/2\delta)$. This gives the theorem. □

One does not expect that floating bodies and L_q centroid bodies are identical in general. Indeed, observe that, for $p < \infty$, the bodies $Z_p(K)$ are C^∞. However, one can easily check that the floating body of the cube has points of non-differentiability on the boundary.

Theorem 2.2 allows us to 'pass' results about L_p centroid bodies to floating bodies. In particular, (2.1) and (2.3) imply that, for $\delta < e^{-n}$, K_δ is isomorphic to K:

$$K_\delta \subseteq K \subseteq c_1 K_\delta.$$

Moreover, (2.1) and (2.2) imply that

$$K_{\delta_2} \subseteq K_{\delta_1} \subseteq c_2 \frac{\log(e/2\delta_1)}{\log(e/2\delta_2)} K_{\delta_2}, \quad \text{for } \delta_1 \leqslant \delta_2,$$

where $c_1, c_2 > 0$ are universal constants.

As a consequence, we get the following corollary. There, $d(K, L)$ and $d_{\mathrm{BM}}(K, L)$, respectively, mean the geometric Banach–Mazur distance of two convex bodies K and L:

$$d(K, L) = \inf\left\{a \cdot b : \frac{1}{a}K \subset L \subset bK\right\},$$
$$d_{\mathrm{BM}}(K, L) = \inf\{d(K, T(L)) : T \text{ is a linear operator}\}.$$

It is known that one may choose a $T \in \mathrm{SL}(n)$ such that $T(K_{1/2})$ is isomorphic to B_2^n (see [**42**] for details).

Corollary 2.3. *Let K be a symmetric convex body of volume 1. Then, for every $\delta \in (0, 1)$, one has*

$$d_{\mathrm{BM}}(K_\delta, B_2^n) \leqslant c_1 \log\frac{1}{\delta}$$

and

$$d(K_\delta, K) \simeq d(K_\delta, K_{e^{-n}}) \leqslant c_2 \frac{n}{\log(1/\delta)},$$

where $c_1, c_2 > 0$ are universal constants.

Let us note that Theorem 2.1 and (2.4) imply sharp (up to L_K) bounds for the volume of K_δ; namely, letting $c_\delta = \max\{\log(1/\delta), 1\}$,

$$c_1\sqrt{\frac{c_\delta}{n + c_\delta}} \leqslant |K_\delta|^{1/n} \leqslant c_2\sqrt{\frac{c_\delta}{n + c_\delta}} L_K,$$

where $c_1, c_2 > 0$ are universal constants.

Remark. The corollary is also true for non-symmetric K.

In view of a result of Latala and Wojtaszczyk [**26**], Theorem 2.2 has another consequence: The floating body of a symmetric convex body K corresponds to a level set of the Legendre transform of the logarithmic Laplace transform on K.

Let $x \in \mathbb{R}^n$ and K be a symmetric convex body of volume 1. Let

$$\Lambda_K^*(x) := \sup_{u \in \mathbb{R}^n}\left\{\langle x, u\rangle - \log\int_K e^{\langle x, u\rangle}\, dx\right\}$$

be the Legendre transform of the logarithmic Laplace transform on K.

For any $r > 0$ let $B_r(K)$ be the convex body defined as

$$B_r(K) := \{x \in \mathbb{R}^n : \Lambda_K^*(x) \leqslant r\}.$$

It was proved in [**26**] that $B_p(K)$ is isomorphic to $Z_p(K)$,

$$c_1 Z_p(K) \subseteq B_p(K) \subseteq c_2 Z_p(K),$$

where $c_1, c_2 > 0$ are universal constants.

We combine this with Theorem 2.2 and obtain the following proposition.

Proposition 2.4. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$. Then, for every $\delta \in (0, \frac{1}{2})$, one has that*

$$c_1 \left\{ x \in \mathbb{R}^n : \Lambda_K^*(x) \leqslant \log \frac{1}{\delta} \right\} \subseteq K_\delta \subseteq c_2 \left\{ x \in \mathbb{R}^n : \Lambda_K^*(x) \leqslant \log \frac{1}{\delta} \right\},$$

$c_1, c_2 > 0$ are universal constants.

3. *Relative entropy of cone measures and related inequalities*

Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin. For real $p \neq -n$ the L_p-affine surface area $\mathrm{as}_p(K)$ of K was defined in (1.1) and (1.7) in Section 1.

If K is in C^2_+, then (1.1) and (1.7) can be written as integrals over the boundary $\partial B_2^n = S^{n-1}$ of the Euclidean unit ball B_2^n in $\mathbb{R}^n$:

$$\mathrm{as}_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{n/(n+p)}}{h_K(u)^{n(p-1)/(n+p)}}\, d\sigma(u)$$

and

$$\mathrm{as}_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n}\, d\sigma(u) = n|K^\circ|. \tag{3.1}$$

Here $f_K(u)$ is the curvature function, that is, the reciprocal of the Gauss curvature $\kappa(x)$ at that point x in ∂K that has u as the outer normal.

First, we recall results proved in [**56**].

Proposition 3.1 [**56**]. *Let K be a convex body in $\mathbb{R}^n$ such that $\mu\{x \in \partial K : \kappa(x) = 0\} = 0$. Let $p \neq -n$ be a real number. Then the following properties are satisfied.*

(i) *The function $p \to (\mathrm{as}_p(K)/\mathrm{as}_\infty(K))^{n+p}$ is decreasing in $p \in (-n, \infty)$.*
(ii) *The function $p \to (\mathrm{as}_p(K)/n|K^\circ|)^{n+p}$ is decreasing in $p \in (-n, \infty)$.*
(iii) *The function $p \to (\mathrm{as}_p(K)/n|K|)^{(n+p)/p}$ is increasing in $p \in (-n, \infty)$.*
(iv) *We have that $\mathrm{as}_p(K) = \mathrm{as}_{n^2/p}(K^\circ)$.*

Remark. (i) It was shown in [**21**] that, for $p > 0$, (iv) holds without any assumptions on the boundary of K.

(ii) Also, it follows from the proof in [**56**] that (i)–(iii) hold without assumptions on the boundary of K if $p \geqslant 0$.

(iii) Proposition 3.1(ii) is not explicitly stated in [**56**], but follows (without any assumptions on the boundary of K if $p \geqslant 0$) from, for example, inequality [**56**, (4.20)] and the following fact (see [**51**]): Let K be a convex body in $\mathbb{R}^n$. Then

$$\mathrm{as}_\infty(K) \leqslant n|K^\circ| \tag{3.2}$$

with equality if K is in C^2_+.

(iv) Strict monotonicity in Proposition 3.1(i)–(iii).
Proposition 3.1(i)–(iii) was proved in [**56**] using Hölder's inequality. It follows immediately from the characterization of equality in Hölder's inequality, that strict monotonicity holds in Proposition 3.1(i)–(iii) if and only if μ, almost everywhere (a.e) on ∂K

$$\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}} = c,$$

where $c > 0$ is a constant, unless $\kappa(x) = 0$ μ, a.e. on ∂K. If $\kappa(x) = 0$ μ, a.e. on ∂K, then, for all $p > 0$, $(\mathrm{as}_p(K)/\mathrm{as}_\infty(K))^{n+p} = \text{constant} = 0$, $(\mathrm{as}_p(K)/n|K^\circ|)^{n+p} = \text{constant} = 0$ and $(\mathrm{as}_p(K)/n|K|)^{(n+p)/p} = \text{constant} = 0$.

If K is in C^2_+, then the following theorem due to Petty [46] implies that we have strict monotonicity in Proposition 3.1(i)–(iii) unless K is an ellipsoid, in which case the quantities in Proposition 3.1(i)–(iii) are all constant equal to 1.

THEOREM 3.2 [46]. *Let K be a convex body in C^2_+. We have that K is an ellipsoid if and only if, for all x in ∂K,*

$$\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}} = c,$$

where $c > 0$ is a constant.

We now introduce new affine invariants.

DEFINITION 3.3. (i) Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin. We define

$$\Omega_K = \lim_{p\to\infty}\left(\frac{\mathrm{as}_p(K)}{n|K^\circ|)}\right)^{n+p},$$

(ii) Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$, all with their centroids at the origin. We define

$$\Omega_{K_1,\dots,K_n} = \lim_{p\to\infty}\left(\frac{\mathrm{as}_p(K_1,\dots,K_n)}{\mathrm{as}_\infty(K_1,\dots,K_n)}\right)^{n+p}.$$

Here

$$\mathrm{as}_p(K_1,\dots,K_n) = \int_{S^{n-1}} [h_{K_1}(u)^{1-p} f_{K_1}(u)\dots h_{K_n}^{1-p} f_{K_n}(u)]^{1/(n+p)}\, d\sigma(u)$$

is the mixed p-affine surface area introduced for $1 \leqslant p < \infty$ in [33] and for general p in [57]:

$$\begin{aligned}\mathrm{as}_\infty(K_1,\dots,K_n) &= \int_{S^{n-1}} \frac{1}{h_{K_1}(u)}\cdots\frac{1}{h_{K_n}(u)}\, d\sigma(u)\\ &= n\tilde{V}(K_1^\circ,\dots,K_n^\circ)\end{aligned}$$

is the dual mixed volume of $K_1^\circ,\dots,K_n^\circ$, introduced by Lutwak [31].

REMARK. (i) If $\mu\{x\in\partial K : \kappa(x) = 0\} = 0$, then $\Omega_K > 0$. If $\kappa(x) = 0$ μ -a.e. on ∂K, then $\Omega_K = 0$. In particular, $\Omega_P = 0$ for all polytopes P.

(ii) If K is in C^2_+, then, by (3.2), $\mathrm{as}_\infty(K) = n|K^\circ|$ and thus we then also have

$$\Omega_K = \lim_{p\to\infty}\left(\frac{\mathrm{as}_p(K)}{\mathrm{as}_\infty(K)}\right)^{n+p}. \tag{3.3}$$

(iii) As for all $p \neq -n$ and for all linear, invertible transformations T, $\mathrm{as}_p(T(K)) = |\det(T)|^{(n-p)/(n+p)}\mathrm{as}_p(K)$ (see [51]) and $\mathrm{as}_p(T(K_1),\dots,T(K_n)) = |\det(T)|^{(n-p)/(n+p)}\mathrm{as}_p (K_1,\dots,K_n)$ [57], we get that

$$\Omega_{T(K)} = |\det(T)|^{2n}\Omega_K, \tag{3.4}$$

and

$$\Omega_{(T(K_1),\dots,T(K_n))} = |\det(T)|^{2n}\Omega_{K_1,\dots,K_n}.$$

In particular, Ω_K and $\Omega_{K_1,\dots,K_n}$ are invariant under linear transformations T with $|\det(T)| = 1$.

Corollary 3.4. *Let K be a convex body $\mathbb{R}^n$ with its centroid at the origin. Then*

$$\Omega_K = \lim_{p\to 0}\left(\frac{\mathrm{as}_p(K^\circ)}{n|K^\circ|}\right)^{n(n+p)/p}.$$

Proof. By Proposition 3.1(iv) and Remark (i) after it

$$\Omega_K = \lim_{p\to\infty}\left(\frac{\mathrm{as}_p(K)}{n|K^\circ|}\right)^{n+p} = \lim_{p\to\infty}\left(\frac{\mathrm{as}_{n^2/p}(K^\circ)}{n|K^\circ|}\right)^{n+p}$$
$$= \lim_{q\to 0}\left(\frac{\mathrm{as}_q(K^\circ)}{n|K^\circ|}\right)^{n+n^2/q} = \lim_{q\to 0}\left(\frac{\mathrm{as}_q(K^\circ)}{n|K^\circ|}\right)^{n(n+q)/q}. \qquad \square$$

Example. For $1 \leqslant r < \infty$, let $B_r^n = \{x \in \mathbb{R}^n : \left(\sum_{i=1}^n |x_i|^r\right)^{1/r} \leqslant 1\}$ and let $B_\infty^n = \{x \in \mathbb{R}^n : \max_{1\leqslant i\leqslant n} |x_i| \leqslant 1\}$. Then a straightforward, but tedious calculation gives

$$\Omega_{B_r^n} = \frac{\exp(-(n^2(r-2)/r)(\Gamma'((r-1)/r)/\Gamma((r-1)/r) - \Gamma'(n(r-1)/r)/\Gamma(n(r-1)/r)))}{(r-1)^{n(n-1)}}. \tag{3.5}$$

Indeed, it was shown in [51] that

$$\mathrm{as}_p(B_r^n) = \frac{2^n(r-1)^{p(n-1)/(n+p)}}{r^{n-1}}\frac{(\Gamma(n+rp-p)/r(n+p))^n}{\Gamma(n(n+rp-p)/r(n+p))}.$$

Therefore,

$$\frac{\mathrm{as}_p(B_r^n)}{n|(B_r^n)^\circ|} = \frac{1}{(r-1)^{n(n-1)/(n+p)}}\frac{(\Gamma(n+rp-p)/r(n+p))^n}{\Gamma(n(n+rp-p)/r(n+p))}\frac{\Gamma(n(r-1)/r)}{(\Gamma((r-1)/r))^n}$$

and

$$\Omega_{B_r^n} = \lim_{p\to\infty}\left(\frac{\mathrm{as}_p(B_r^n)}{n|(B_r^n)^\circ|}\right)^{n+p}$$
$$= \frac{\exp(-(n^2(r-2)/r)(\Gamma'((r-1)/r)/\Gamma((r-1)/r) - \Gamma'(n(r-1)/r)/\Gamma(n(r-1)/r)))}{(r-1)^{n(n-1)}}.$$

The next propositions describe more properties of Ω_K. Some were already stated in Section 1.

Proposition 3.5. *Let K be a convex body with its centroid at the origin.*

(i) *For all $p > 0$,*

$$\Omega_K \leqslant \left(\frac{\mathrm{as}_p(K^\circ)}{n|K^\circ|}\right)^{n(n+p)/p}.$$

If K is in addition in C_+^2, then equality holds if and only if K is an ellipsoid.

(ii) *For all $p \geqslant 0$,*

$$\Omega_K \leqslant \left(\frac{\mathrm{as}_p(K)}{n|K^\circ|}\right)^{n+p}.$$

If K is in addition in C_+^2, then equality holds if and only if K is an ellipsoid.

(iii) *We have that $\Omega_K \leqslant (|K|/|K^\circ|)^n$. If K is in addition in C_+^2, then equality holds if and only if K is an ellipsoid.*

(iv) *We have that $\Omega_K\Omega_{K^\circ} \leqslant 1$. If K is in addition in C_+^2, then equality holds if and only if K is an ellipsoid.*

Proof. (i) The first part follows from Corollary 3.4, Proposition 3.1(iii) and the Remark (ii) after it. The second part follows from Corollary 3.4, Proposition 3.1(iii) and the Remark (iv) after it.

(ii) The first part follows from the definition of Ω_K, Proposition 3.1(ii) and the Remark (ii) after it. The second part follows from the definition of Ω_K, Proposition 3.1(ii) and the Remark (iv) after it.

(iii) By (ii), $\Omega_K \leqslant (\mathrm{as}_0(K)/n|K^\circ|)^n = (|K|/|K^\circ|)^n$.

(iv) Condition (iv) is immediate from (iii). □

We concentrate on describing the properties of Ω_K. The analog properties for the invariant $\Omega_{K_1,\dots,K_n}$ also hold and are proved similarly using results about the mixed p-affine surface areas proved in [57]. For instance, the analog to Proposition 3.5(ii) holds: For all $p \geqslant 0$

$$\Omega_{K_1,\dots,K_n} \leqslant \left(\frac{\mathrm{as}_p(K_1,\dots,K_n)}{\mathrm{as}_\infty(K_1,\dots,K_n)}\right)^{n+p}.$$

This follows from a monotonicity behavior of $(\mathrm{as}_p(K_1,\dots,K_n)/\mathrm{as}_\infty(K_1,\dots,K_n))^{n+p}$, which was shown in [57]. And the analog to Proposition 3.6(ii) holds:

$$\Omega_{K_1,\dots,K_n} = \exp\left(\frac{1}{\mathrm{as}_\infty(K_1,\dots,K_n)}\int_{S^{n-1}} \frac{\sum_{i=1}^n \log[f_{K_i} h_{K_i}^{n+1}]}{\prod_{i=1}^n h_{K_i}}\, d\sigma\right).$$

Proposition 3.6. *Let K be a convex body $\mathbb{R}^n$ with its centroid at the origin.*

(i)

$$\Omega_K = \exp\left(\frac{1}{|K^\circ|}\int_{\partial K^\circ} \langle x, N_{K^\circ}(x)\rangle \log \frac{\kappa_{K^\circ}(x)}{\langle x, N_{K^\circ}(x)\rangle^{n+1}}\, d\mu_{K^\circ}(x)\right).$$

In addition, if K is in C^2_+, then

(ii)

$$\Omega_K = \exp\left(-\frac{1}{|K^\circ|}\int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} \log \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\, d\mu_K(x)\right).$$

(iii)

$$\begin{aligned}
&\frac{1}{|K|}\int_{\partial K} \langle x, N_K(x)\rangle \log \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\, d\mu_K(x)\\
&\quad\leqslant n \log \frac{|K^\circ|}{|K|}\\
&\quad\leqslant \frac{1}{|K^\circ|}\int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} \log \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\, d\mu_K(x).
\end{aligned}$$

Proof. (i) By Corollary 3.4,

$$\begin{aligned}\log \Omega_K &= \log\left(\lim_{p\to 0}\left(\frac{\mathrm{as}_p(K^\circ)}{n|K^\circ|}\right)^{n(n+p)/p}\right) = \log\left(\lim_{p\to 0}\left(\frac{\mathrm{as}_p(K^\circ)}{n|K^\circ|}\right)^{n^2/p}\right)\\ &= \lim_{p\to 0}\frac{n^2}{p}\log\frac{\mathrm{as}_p(K^\circ)}{n|K^\circ|} = n^2\lim_{p\to 0}\frac{(d/dp)(\mathrm{as}_p(K^\circ))}{\mathrm{as}_p(K^\circ)}\\ &= n^2\lim_{p\to 0}\frac{n(n+p)^{-2}}{\mathrm{as}_p(K^\circ)}\int_{\partial K^\circ}\frac{\kappa_{K^\circ}(x)^{p/(n+p)}}{\langle x, N_{K^\circ}(x)\rangle^{n(p-1)/(n+p)}}\\ &\quad\times\log\frac{\kappa_{K^\circ}(x)}{\langle x, N_{K^\circ}(x)\rangle^{n+1}}\,d\mu_{K^\circ}(x)\\ &= \frac{1}{|K^\circ|}\int_{\partial K^\circ}\langle x, N_{K^\circ}(x)\rangle\log\frac{\kappa_{K^\circ}(x)}{\langle x, N_{K^\circ}(x)\rangle^{n+1}}\,d\mu_{K^\circ}(x).\end{aligned}$$

(ii) If K is in C^2_+, then we have, by (3.3), that

$$\begin{aligned}\log \Omega_K &= \log\left(\lim_{p\to\infty}\left(\frac{\mathrm{as}_p(K)}{\mathrm{as}_\infty(K)}\right)^{n+p}\right) = \lim_{p\to\infty}\frac{\log\left(\mathrm{as}_p(K)/\mathrm{as}_\infty(K)\right)}{(n+p)^{-1}}\\ &= -\lim_{p\to\infty}\frac{(n+p)^2(d/dp)(\mathrm{as}_p(K))}{\mathrm{as}_p(K)}\\ &= -\lim_{p\to\infty}\frac{(n+p)^2}{\mathrm{as}_p(K)}\int_{\partial K}\frac{d}{dp}\left(\exp\left(\log\left(\kappa_K(x)\right)\frac{p}{n+p}\right.\right.\\ &\quad\left.\left.-\ \log\left(\langle x, N_K(x)\rangle\right)\frac{n(p-1)}{n+p}\right)\right)d\mu_K(x)\\ &= -\lim_{p\to\infty}\frac{(n+p)^2}{\mathrm{as}_p(K)}\int_{\partial K}\frac{\kappa_K(x)^{p/(n+p)}}{\langle x, N_K(x)\rangle^{n(p-1)/(n+p)}}\left(\frac{n}{(n+p)^2}\log\left(\kappa_K(x)\right)\right.\\ &\quad\left.-\ \frac{n(n+1)}{(n+p)^2}\log\left(\langle x, N_K(x)\rangle\right)\right)d\mu_K(x)\\ &= -\lim_{p\to\infty}\frac{n}{\mathrm{as}_p(K)}\int_{\partial K}\frac{\kappa_K(x)^{p/(n+p)}}{\langle x, N_K(x)\rangle^{n(p-1)/(n+p)}}\log\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\,d\mu_K(x)\\ &= -\frac{n}{\mathrm{as}_\infty(K)}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n}}\log\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\,d\mu_K(x).\end{aligned}$$

(iii) Combine Proposition 3.5(iii) with (i) and (ii). □

Let (X,μ) be a measure space and let $dP = p\,d\mu$ and $dQ = q\,d\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. The *Kullback–Leibler divergence* or *relative entropy* from P to Q is defined as [**13**]

$$D_{\mathrm{KL}}(P\|Q) = \int_X p\log\frac{p}{q}\,d\mu. \tag{3.6}$$

The information inequality (also called *Gibb's inequality*) [**13**] holds for the Kullback–Leibler divergence: Let P and Q be as above. Then

$$D_{\mathrm{KL}}(P\|Q) \geqslant 0, \tag{3.7}$$

with equality if and only if $P = Q$.

The invariant Ω_K is related to relative entropies on K and a corresponding information inequality holds, which is exactly the inequality of Proposition 3.5(iii).

PROPOSITION 3.7. *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Let*

$$p(x) = \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n n|K^\circ|}, \quad q(x) = \frac{\langle x, N_K(x)\rangle}{n|K|}. \tag{3.8}$$

Then $dP = p\, d\mu_K$ and $dQ = q\, d\mu_K$ are probability measures on ∂K that are absolutely continuous with respect to μ_K and

$$D_{\mathrm{KL}}(P\|Q) = \log\left(\frac{|K|}{|K^\circ|}\Omega_K^{-1/n}\right) \tag{3.9}$$

and

$$D_{\mathrm{KL}}(Q\|P) = \log\left(\frac{|K^\circ|}{|K|}\Omega_{K^\circ}^{-1/n}\right). \tag{3.10}$$

Moreover, the information inequality implies that

$$\Omega_K \leqslant \left(\frac{|K|}{|K^\circ|}\right)^n$$

with equality if and only if K is an ellipsoid.

Proof of Proposition 3.7. As

$$n|K| = \int_{\partial K} \langle x, N_K\rangle\, d\mu_K(x) \quad \text{and} \quad n|K^\circ| = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n}\, d\mu_K(x),$$

$\int_{\partial K} p\, d\mu_K = \int_{\partial K} q\, d\mu_K = 1$ and hence P and Q are probability measures that are absolutely continuous with respect to μ_K on K.

Equation (3.9) or (3.10) follows from the definition of the relative entropy (3.6) and Proposition 3.6(ii) or Proposition 3.6(i), respectively.

By (3.7), equality holds in the inequality of the proposition, if and only if, for all $x \in \partial K$,

$$\frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}} = \frac{|K^\circ|}{|K|} = \text{constant},$$

which holds, by the above-mentioned theorem of Petty [46] if and only if K is an ellipsoid. $\square$

Let K be a convex body in $\mathbb{R}^n$. Recall that the normalized cone measure cm_K on ∂K is defined as follows: For every measurable set $A \subseteq \partial K$,

$$\mathrm{cm}_K(A) = \frac{1}{|K|}|\{ta : a \in A, t \in [0,1]\}|. \tag{3.11}$$

For more information about cone measures we refer to, for example, [6, 16, 43].

The next proposition is well known. It shows that the measures P and Q defined in Proposition 3.7 are the cone measures of K and K°. We include the proof for completeness. We see that $N_K : \partial K \to S^{n-1}$, $x \to N_K(x)$ is the Gauss map.

PROPOSITION 3.8. *Let K be a convex body in $\mathbb{R}^n$ that is C^2_+. Let P and Q be the probability measures on ∂K defined by (3.8). Then*

$$P = N_K^{-1} N_{K^\circ} \mathrm{cm}_{K^\circ} \quad \textit{and} \quad Q = \mathrm{cm}_K,$$

or, equivalently, for every measurable subset A in ∂K

$$P(A) = \mathrm{cm}_{K^\circ}(N_{K^\circ}^{-1}(N_K(A))) \quad \textit{and} \quad Q(A) = \mathrm{cm}_K(A).$$

Proof.

$$Q(A) = \frac{1}{n|K|} \int_A \langle x, N_K(x) \rangle \, d\mu_K(x) = \mathrm{cm}_K(A).$$

Also

$$P(A) = \int_A \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} \frac{d\mu_K(x)}{n|K^\circ|} = \frac{1}{n|K^\circ|} \int_{N_K(A)} \frac{1}{h_K^n(u)} \, d\sigma(u).$$

Let $B \subseteq \partial K^\circ$. Then

$$\mathrm{cm}_{K^\circ}(B) = \frac{1}{|K^\circ|} \left| \left\{ x \in \mathbb{R}^n : \|x\|_{K^\circ} \leqslant 1, \ \frac{x}{\|x\|_2} \in N_{K^\circ}(B) \right\} \right|.$$

Let $\Delta = \{x \in \mathbb{R}^n : \|x\|_{K^\circ} \leqslant 1, \ x/\|x\|_2 \in N_{K^\circ}(B)\}$. We have

$$\begin{aligned} \mathrm{cm}_{K^\circ}(B) = \frac{|\Delta|}{|K^\circ|} &= \frac{1}{|K^\circ|} \int_0^\infty \int_{S^{n-1}} r^{n-1} 1_\Delta(r\theta) \, dr \, d\sigma(\theta) \\ &= \frac{1}{|K^\circ|} \int_{N_{K^\circ}(B)} \int_0^{1/\|\theta\|_{K^\circ}} r^{n-1} \, dr \, d\sigma(\theta) \\ &= \frac{1}{n|K^\circ|} \int_{N_{K^\circ}(B)} \frac{1}{h_K^n(\theta)} \, d\sigma(\theta). \end{aligned}$$

Let $B \in \partial K^\circ$ be such that $N_{K^\circ}(B) = N_K(A)$. This means that $B = N_{K^\circ}^{-1}(N_K(A))$. Then $P(A) = \mathrm{cm}_{K^\circ}(N_{K^\circ}^{-1}(N_K(A)))$, which completes the proof. □

Therefore, with P and Q defined as in (3.8),

$$D_{\mathrm{KL}}(P\|Q) = D_{\mathrm{KL}}(N_K N_{K^\circ}^{-1} \mathrm{cm}_{K^\circ} \| \mathrm{cm}_K), \tag{3.12}$$

and we get as a corollary to Proposition 3.7 that the invariant Ω_K is the exponential of the relative entropy of the cone measures of K and K°.

COROLLARY 3.9. *Let K be a convex body in C_+^2. Then*

$$\Omega_K^{1/n} = \frac{|K|}{|K^\circ|} \exp\left(-D_{\mathrm{KL}}(N_K N_{K^\circ}^{-1} \mathrm{cm}_{K^\circ} \| \mathrm{cm}_K)\right).$$

Finally, an isoperimetric inequality holds for the affine invariant Ω_K.

PROPOSITION 3.10. *Let K be a convex body in C_+^2 of volume 1. Then*

$$\Omega_{K^\circ} \leqslant \Omega_{(B_n^2/|B_n^2|^{1/n})^\circ}$$

with equality if and only if K is a normalized ellipsoid.

Proof. The proof follows from the above information inequality for convex bodies together with the Blaschke Santaló inequality and the fact that $\Omega_{(B_n^2/|B_n^2|^{1/n})^\circ} = |B_n^2|^{2n}$. □

4. $Z_p(K)$ *for* K *in* C_+^2

In this section, we show how Ω_K is related to the L_p centroid bodies. The main theorem of this section is Theorem 4.1. We assume there that K is symmetric, mainly because the bodies $Z_p(K)$ are symmetric by definition. Also, throughout this section we assume that K is of volume 1.

THEOREM 4.1. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 that is in C^2_+. Then:*
(i)

$$\lim_{p\to\infty} \frac{p}{\log p}(|Z_p^\circ(K)| - |K^\circ|) = \frac{n(n+1)}{2}|K^\circ|;$$

(ii)

$$\begin{aligned}
&\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p|Z_p^\circ(K)|\right)\\
&= \lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p|K^\circ|\right)\\
&= -\frac{1}{2}\int_{S^{n-1}} h_K(u)^{-n}\log(2^{n+1}\pi^{n-1}h_K(u)^{n+1}f_K(u))\,d\sigma(u)\\
&= \frac{1}{2}\int_{\partial K}\frac{\kappa(x)}{\langle x, N(x)\rangle^n}\log\left(\frac{\kappa(x)}{2^{n+1}\pi^{n-1}\langle x, N(x)\rangle^{n+1}}\right)d\mu_K(x).
\end{aligned}$$

Thus, Theorem 4.1 shows that if K is a symmetric convex body in C^2_+ of volume 1, then

$$\begin{aligned}
&\lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)\log p}{2p}|Z_p^\circ(K)|\right)\\
&= \lim_{p\to\infty} p\left(|Z_p^\circ(K)| - |K^\circ| - \frac{n(n+1)}{2p}\log p|K^\circ|\right)\\
&= \frac{1}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N(x)\rangle^n}\log\left(\frac{\kappa_K(x)}{2^{n+1}\pi^{n-1}\langle x, N(x)\rangle^{n+1}}\right)d\mu_K(x)\\
&= -\frac{\log(2^{n+1}\pi^{n-1})}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N(x)\rangle^n}\,d\mu_K(x)\\
&\quad + \frac{1}{2}\int_{\partial K}\frac{\kappa_K(x)}{\langle x, N(x)\rangle^n}\log\left(\frac{\kappa_K(x)}{\langle x, N(x)\rangle^{n+1}}\right)d\mu_K(x)\\
&= \log(2^{n+1}\pi^{n-1})\frac{n|K^\circ|}{2} - \frac{|K^\circ|}{2}\log\Omega_K = -\frac{|K^\circ|}{2}\log\frac{\Omega_K}{2^{n(n+1)}\pi^{n(n-1)}}
\end{aligned}$$

or

$$\begin{aligned}
&\lim_{p\to\infty} p\left(\frac{|Z_p^\circ(K)|}{|K^\circ|}\left(1 - \frac{n(n+1)\log p}{2p}\right) - 1\right)\\
&= \lim_{p\to\infty} p\left(\left(1 - \frac{n(n+1)\log p}{2p}\right)\frac{|Z_p^\circ(K)|}{|K^\circ|} - 1\right) = -\frac{1}{2}\log\frac{\Omega_K}{2^{n(n+1)}\pi^{n(n-1)}}.
\end{aligned} \tag{4.1}$$

So we have the following corollary.

COROLLARY 4.2. *Let K and C be symmetric convex bodies of volume 1 in C^2_+. Then:*
(i)

$$\begin{aligned}
&\lim_{p\to\infty}\frac{2p}{n}\left(\frac{(1 - n(n+1)\log p/2p)|Z_p^\circ(K)|}{|K^\circ|} - 1\right)\\
&= \lim_{p\to\infty}\frac{2p}{n}\left(\frac{|Z_p^\circ(K)|}{|K^\circ|} - \left(1 - \frac{n(n+1)\log p}{2p}\right)\right) = -\frac{1}{2}\log\frac{\Omega_K^{1/n}}{2^{n+1}\pi^{n-1}}\\
&= (n+1)\log\left(\frac{2\pi^{(n-1)/(n+1)}}{|K^\circ|}\right) + D_{\mathrm{KL}}(N_K N_{K^\circ}^{-1}\mathrm{cm}_{K^\circ}\|\mathrm{cm}_K);
\end{aligned}$$

(ii)

$$\lim_{p\to\infty} p\left(\left(1-\frac{n(n+1)\log p}{2p}\right)\frac{|Z_p^\circ(K)|}{|K^\circ|}-1\right) \geqslant \frac{1}{2}\log\left(2^{n(n+1)}\pi^{n(n-1)}\frac{|K^\circ|}{|K|}\right).$$

The corresponding statement for $\lim_{p\to\infty} p(|Z_p^\circ(K)|/|K^\circ| - (1-n(n+1)\log p/2p))$ *also holds.*

(iii)

$$\lim_{p\to\infty} p\left(1-\frac{n(n+1)\log p}{2p}\right)\left(\frac{|Z_p^\circ(K)|}{|K^\circ|}-\frac{|Z_p^\circ(C)|}{|C^\circ|}\right) = \frac{1}{2n}\log\frac{\Omega_C}{\Omega_K}.$$

Proof. (i) follows from (4.1) and Corollary 3.9, (ii) follows from Proposition 3.5 and (iii) follows from (4.1).

The remainder of the section is devoted to the proof of Theorem 4.1. We need several lemmas and notation.

Let $x, y > 0$. Let $\Gamma(x) = \int_0^\infty \lambda^{x-1}e^{-\lambda}\,d\lambda$ be the Gamma function and $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\,d\lambda = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ be the Beta function.

We write $f(p) = g(p) \pm o(p)$ if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} ph(p) = 0$, that is, $h(p)$ has terms of order $1/p^2$ and higher. Similarly, $f(p) = g(p) \pm o(p^2)$ if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} p^2h(p) = 0$, that is, $h(p)$ has terms of order $1/p^3$ and higher. We write $f(p) = g(p) \pm O(p)$ if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} h(p) = 0$. □

Lemma 4.3. (i) *Let* $p > 0$. *Then*

$$\begin{aligned}\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p} &= 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\\ &+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\log p\\ &+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2-\frac{n+1}{4}(n(n+1)+2(n+3))\right]\\ &\pm o(p^2).\end{aligned}$$

(ii) *Let* $0 \leqslant a \leqslant 1$. *Then*

$$\begin{aligned}&\left(\int_0^1 u^p(1-u)^{(n-1)/2}(1-a(1-u))^{(n-1)/2}\,du\right)^{n/p}\\ &= 1-\frac{n(n+1)}{2p}\log p\\ &+\frac{n}{p}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\log p\\ &+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2-\frac{(n+1)(n^2+3n+6)}{4}-(n+1)\binom{\frac{n-1}{2}}{1}a\right]\pm o(p^2).\end{aligned}$$

The proof of Lemma 4.3 is in the Appendix.

Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a C^2 log-concave function with $\int_{\mathbb{R}_+} f(t)\,dt<\infty$ and let $p\geqslant 1$. Let $g_p(t) = t^pf(t)$ and let $t_p = t_p(f)$ be the unique point such that $g_p'(t_p) = 0$. We make use of the following lemma due to Klartag [**24**] (Lemmas 4.3 and 4.5).

Lemma 4.4. *Let f be as above. For every $\varepsilon \in (0,1)$,*

$$\int_0^\infty t^p f(t)\,dt \leqslant (1+Ce^{-cp\varepsilon^2})\int_{t_p(1-\varepsilon)}^{t_p(1+\varepsilon)} t^p f(t)\,dt,$$

where $C>0$ and $c>0$ are universal constants.

We think that the next lemma is well known. We give a proof for completeness.

Lemma 4.5. *Let $u \in S^{n-1}$. Let f and t_p be as above and f be also such that it is decreasing and a probability density on $[0,h(u)]$. Then*

$$\lim_{p\to\infty} t_p = h(u).$$

Proof. Since the support of f is $[0,h(u)]$, by the definition of t_p we have that $t_p \leqslant h(u)$ for all p. So we only have to show that $\lim_{p\to\infty} t_p \geqslant h(u)$.

By Hölder's inequality, $(\int_0^{h(u)} t^p f(t)\,dt)^{1/p} \to h(u)$. Thus, for $\varepsilon>0$ given, there exists p_ε such that for all $p \geqslant p_\varepsilon$,

$$\int_0^{h(u)} t^p f(t)\,dt \geqslant (h(u)-\varepsilon)^p.$$

By Lemma 4.4, for all $0<\delta<1$, $\int_0^\infty t^p f(t)\,dt \leqslant (1+Ce^{-cp\delta^2})\int_{t_p(1-\delta)}^{t_p(1+\delta)} t^p f(t)\,dt$. We choose $\delta = 1/p^{1/4}$ with $p>p_\varepsilon$ and get, using the monotonicity behavior of $t^p f$ on the respective intervals, that

$$\begin{aligned}(h(u)-\varepsilon)^p &\leqslant (1+Ce^{-c\sqrt{p}})\left[\int_{t_p(1-\delta)}^{t_p} t^p f(t)\,dt + \int_{t_p}^{t_p(1+\delta)} t^p f(t)\,dt\right]\\ &\leqslant (1+Ce^{-c\sqrt{p}})p^{-1/4} t_p f(t_p) t_p^p.\end{aligned}$$

As f is decreasing, $f(t_p) \leqslant f(0)$. Moreover, $t_p \leqslant h(u)$. Thus, for $p \geqslant p_\varepsilon$ large enough, $((1+Ce^{-c\sqrt{p}})p^{-1/4} t_p f(t_p))^{1/p} \leqslant 1+\varepsilon$ and hence $h(u)-\varepsilon < (1+\varepsilon)t_p$. □

Remark. We will apply Lemma 4.4 to the function $f(t) = |K \cap (u^\perp + tu)|$, $u \in S^{n-1}$. We show below that f is C^2. Thus, t_p is well defined and Lemma 4.4 holds. Also, t_p is an increasing function of p and by Lemma 4.5, $\lim_{p\to\infty} t_p = h_K(u)$.

We also think that the following lemma is well known but we could not find a proof in the literature. Therefore, we include a proof.

Lemma 4.6. *Let K be a convex body C^2_+. Let $u \in S^{n-1}$ and let H_t be the hyperplane orthogonal to u at distance t from the origin. Let $f(t) = |K \cap H_t|$. Then f is C^2. In fact,*

$$f'(t) = -\int_{\partial K \cap H_t} \frac{\langle u, N_K(x)\rangle}{(1-\langle u, N_K(x)\rangle^2)^{1/2}}\,d\mu_{\partial K\cap H_t}(x)$$

and

$$f''(t) = -\int_{\partial K\cap H_t}\left[\frac{\kappa(x_t)^{1/(n-1)}}{(1-\langle N_K(x_t),u\rangle^2)^{3/2}} - \frac{(n-2)\langle N_K(x_t),u\rangle^2}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)}\right]d\mu_{\partial K\cap H_t}(x_t).$$

Proof. We assume that $\mathrm{int}(K)\cap H_t \neq \emptyset$. To show that $f \in C^2$, we compute the derivatives of f. We first show that

$$f'(t) = -\int_{\partial K\cap H_t} \frac{\langle u, N_K(x)\rangle}{(1-\langle u, N_K(x)\rangle^2)^{1/2}}\, d\mu_{\partial K\cap H_t}(x).$$

Indeed, for $x \in \partial K \cap H_t$ let $\alpha(x)$ be the (smaller) angle formed by $N_K(x)$ and u. Then $\cos\alpha(x) = \langle u, N_K(x)\rangle$ and

$$\begin{aligned} f'(t) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(|K\cap H_{t+\varepsilon}| - |K\cap H_t|) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{\partial K\cap H_t}\varepsilon\cot\alpha(x)\, d\mu_{\partial K\cap H_t}(x)\right)\\ &= -\int_{\partial K\cap H_t} \frac{\langle u, N_K(x)\rangle}{(1-\langle u, N_K(x)\rangle^2)^{1/2}}\, d\mu_{\partial K\cap H_t}(x). \end{aligned}$$

We show next that

$$f''(t) = -\int_{\partial K\cap H_t}\left[\frac{\kappa(x_t)^{1/(n-1)}}{(1-\langle N_K(x_t), u\rangle^2)^{3/2}} - \frac{(n-2)\langle N_K(x_t), u\rangle^2}{\langle N_{K\cap H_t}(x_t), x_t\rangle(1-\langle N_K(x_t), u\rangle^2)}\right] d\mu_{\partial K\cap H_t}(x_t).$$

By definition

$$\begin{aligned} f''(t) = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Bigg(&\int_{\partial K\cap H_{t+\varepsilon}} \frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{(1-\langle u, N_K(y_{t+\varepsilon})\rangle^2)^{1/2}}\, d\mu_{\partial K\cap H_{t+\varepsilon}}(y_{t+\varepsilon})\\ &- \int_{\partial K\cap H_t} \frac{\langle u, N_K(x_t)\rangle}{(1-\langle u, N_K(x_t)\rangle^2)^{1/2}}\, d\mu_{\partial K\cap H_t}(x_t)\Bigg). \end{aligned}$$

We project $K\cap H_{t+\varepsilon}$ onto $K\cap H_t$ and we want to integrate both expressions over $\partial K\cap H_t$. To do so, we fix, after the projection, an interior point x_0 in $K\cap H_{t+\varepsilon}$. For $x_t \in \partial K\cap H_t$ let $[x_0, x_t]$ be the line segment from x_0 to x_t and let $x_{t+\varepsilon} = \partial K\cap H_{t+\varepsilon}\cap[x_0, x_t]$. Now observe that

$$d\mu_{\partial K\cap H_{t+\varepsilon}} = \frac{1}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle}\left(\frac{\|x_{t+\varepsilon}\|}{\|x_t\|}\right)^{n-2} d\mu_{\partial K\cap H_t},$$

where $N_{K\cap H_t}(x_t)$ is the outer normal in x_t to the boundary of the $(n-1)$-dimensional convex body $K\cap H_t$ and, similarly, $N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})$ is the outer normal in $x_{t+\varepsilon}$ to the boundary of the $(n-1)$-dimensional convex body $K\cap H_{t+\varepsilon}$.

Note further that

$$\|x_t\| - \|x_{t+\varepsilon}\| = \frac{\varepsilon\langle N_K(x_t), u\rangle\|x_t\|}{\langle N_{K\cap H_t}(x_t), x_t\rangle(1-\langle N_K(x_t), u\rangle^2)^{1/2}} + \text{ higher order terms in } \varepsilon.$$

Therefore,

$$\begin{aligned} \left(\frac{\|x_{t+\varepsilon}\|}{\|x_t\|}\right)^{n-2} &= \left(1 - \frac{\varepsilon\langle N_K(x_t), u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle(1-\langle N_K(x_t), u\rangle^2)^{1/2}}\right)^{n-2}\\ &= 1 - \frac{(n-2)\varepsilon\langle N_K(x_t), u\rangle}{\langle N_{K\cap H_t}(x_t), x_t\rangle(1-\langle N_K(x_t), u\rangle^2)^{1/2}}\\ &\quad + \text{higher order terms in } \varepsilon. \end{aligned}$$

Thus,

$$\begin{aligned}
f''(t) = &- \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\partial K\cap H_t}\left[\frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle(1-\langle u, N_K(y_{t+\varepsilon})\rangle^2)^{1/2}}\right.\\
&\times\left(1-\frac{(n-2)\varepsilon\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)^{1/2}}+\text{ higher order terms in }\varepsilon\right)\\
&\left.-\frac{\langle u, N_K(x_t)\rangle}{(1-\langle u, N_K(x_t)\rangle^2)^{1/2}}\right]d\mu_{\partial K\cap H_t}(x_t)\\
= &-\int_{\partial K\cap H_t}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\langle u, N_K(y_{t+\varepsilon})\rangle}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle(1-\langle u, N_K(y_{t+\varepsilon})\rangle^2)^{1/2}}\right.\\
&\times\left(1-\frac{(n-2)\varepsilon\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)^{1/2}}+\text{higher order terms in }\varepsilon\right)\\
&\left.-\frac{\langle u, N_K(x_t)\rangle}{(1-\langle u, N_K(x_t)\rangle^2)^{1/2}}\right]d\mu_{\partial K\cap H_t}(x_t).
\end{aligned}$$

We can interchange integration and limit using Lebesgue's theorem as the functions under the integral are uniformly (in t) bounded by a constant.

Define $g_x(t)=\langle N_K(x_t),u\rangle/(1-\langle u, N_K(x_t)\rangle^2)^{1/2}$. Then the expression under the integral becomes

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{g_y(t+\varepsilon)}{\langle N_{K\cap H_t}(x_t), N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\rangle}\left(1-\frac{(n-2)\varepsilon\langle N_K(x_t),u\rangle}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)^{1/2}}\right.\right.\\
&\qquad\left.\left.+\text{ higher order terms in }\varepsilon\right)-g_x(t)\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[g_y(t+\varepsilon)-g_x(t)]-\frac{(n-2)\langle N_K(x_t),u\rangle^2}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)}.
\end{aligned}$$

Here we have also used that, as $\varepsilon\to 0$, $x_{t+\varepsilon}\to x_t$, $N_{K\cap H_{t+\varepsilon}}(x_{t+\varepsilon})\to N_{K\cap H_t}(x_t)$ and $g_y(t+\varepsilon)\to g_x(t)$.

To compute $\lim_{\varepsilon\to 0}(1/\varepsilon)[g_y(t+\varepsilon)-g_x(t)]$, we approximate the boundary of ∂K in x_t by an ellipsoid. This can be done as ∂K is C^2_+ by assumption (see Lemma 4.8). To simplify the computations, we assume that the approximating ellipsoid is a Euclidean ball. The case of the ellipsoid is treated similarly; the computations are just slightly more involved. As the expression under the integral depends only on the angles between the vectors involved, we can put the origin so that the approximating Euclidean ball is centered at 0. Let $r=\kappa(x_t)^{-1/(n-1)}$ be its radius. Then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[g_y(t+\varepsilon)-g_x(t)]=\frac{1}{r(1-\langle N_K(x_t),u\rangle^2)^{3/2}}=\frac{\kappa(x_t)^{1/(n-1)}}{(1-\langle N_K(x_t),u\rangle^2)^{3/2}}.$$

Altogether

$$f''(t)=-\int_{\partial K\cap H_t}\left[\frac{\kappa(x_t)^{1/(n-1)}}{(1-\langle N_K(x_t),u\rangle^2)^{3/2}}-\frac{(n-2)\langle N_K(x_t),u\rangle^2}{\langle N_{K\cap H_t}(x_t),x_t\rangle(1-\langle N_K(x_t),u\rangle^2)}\right]d\mu_{\partial K\cap H_t}(x_t).$$

□

Lemma 4.7. *Let K be a symmetric convex body of volume 1 in C^2_+.*

(i) *The functions*

$$\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

are uniformly (in p) bounded by a function that is integrable on S^{n-1}.

(ii) *The functions*

$$\frac{p}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)$$

are uniformly (in p) bounded by a function that is integrable on S^{n-1}.

Proof. (i) Let $u\in S^{n-1}$. Let $x\in\partial K$ be such that $N_K(x)=u$. As K is in C^2_+, by the Blaschke rolling theorem (see [48]), there exists a ball with radius r_0 that rolls freely in K: for all $x\in\partial K$, $B_2^n(x-r_0N(x),r_0)\subset K$. As K is symmetric,

$$\begin{aligned}
h_{Z_p}(u)^n &= \left(2\int_0^{h_K(u)} t^p|\{y\in K:\langle u,y\rangle=t\}|\,dt\right)^{n/p}\\
&\geqslant \left(2\int_{h_K(u)-r}^{h_K(u)} t^p|\{y\in B_2^n(x-r_0u,r_0):\langle u,y\rangle=t\}|\,dt\right)^{n/p}\\
&= 2^{n/p}|B_2^{n-1}|^{n/p}\left(\int_{h_K(u)-r_0}^{h_K(u)} t^p\left(2r_0(h_K(u)-t)\left[1-\frac{h_K(u)-t}{2r_0}\right]\right)^{(n-1)/2}dt\right)^{n/p}.
\end{aligned}$$

The equality holds as the $(n-1)$-dimensional Euclidean ball

$$B_2^n(x-r_0u,r_0)\cap\{y\in\mathbb{R}^n:\langle u,y\rangle=t\}$$

has radius $(2r_0(h_K(u)-t)[1-(h_K(u)-t)/2r_0])^{1/2}$. Now, where, to abbreviate, we write h_K, $h_{Z_p(K)}$, instead of $h_K(u)$, $h_{Z_p(K)}(u)$, and where we use that $\frac{1}{2}\leqslant 1-(h_K(u)-t)/2r_0$,

$$\begin{aligned}
h_{Z_p}(u)^n &\geqslant 2^{n/p}|B_2^{n-1}|^{n/p}(r_0h_K)^{n(n-1)/2p}\left(\int_{h_K-r_0}^{h_K} t^p\left(1-\frac{t}{h_K}\right)^{(n-1)/2}dt\right)^{n/p}\\
&= h_K^n(2|B_2^{n-1}|h_K^{(n+1)/2}r_0^{(n-1)/2})^{n/p}\left(\int_{1-r_0/h_K}^{1} w^p(1-w)^{(n-1)/2}\,dw\right)^{n/p}. \qquad (4.2)
\end{aligned}$$

As K is symmetric, $r_0\leqslant h_K(u)$. If $r_0=h_K(u)$, then

$$\frac{h^n_{Z_p(K)}}{h^n_K}\geqslant(2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|)^{n/p}\left(\int_0^1 w^p(1-w)^{(n-1)/2}\,dw\right)^{n/p}.$$

If $r_0<h_K(u)$, then we apply Lemma 4.4 to the function $f(w)=(1-w)^{(n-1)/2}$. We choose ε so small and p_0 so large that $\varepsilon+(1+\varepsilon)(n-1)/2p_0\leqslant r_0/h_K$. Then Lemma 4.4 holds and we get, for all $p\geqslant p_0$,

$$\begin{aligned}
\frac{h^n_{Z_p(K)}}{h^n_K} &\geqslant (2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|)^{n/p}\left(\int_{1-r_0/h_K}^1 w^p(1-w)^{(n-1)/2}\,dw\right)^{n/p}\\
&\geqslant\left(\frac{2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|}{1+Ce^{-cp\varepsilon^2}}\right)^{n/p}\left(\int_0^1 w^p(1-w)^{(n-1)/2}\,dw\right)^{n/p}\\
&=\left(\frac{2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|}{1+Ce^{-cp\varepsilon^2}}\right)^{n/p}\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p}.
\end{aligned}$$

As

$$(2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|)^{n/p}=1+\frac{n}{p}\log[2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|]\pm o(p),$$

respectively,

$$\left(\frac{2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|}{1+Ce^{-cp\varepsilon^2}}\right)^{n/p}=1+\frac{n}{p}\log\left[\frac{2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|}{1+Ce^{-cp\varepsilon^2}}\right]\pm o(p)$$

we get, together with Lemma 4.3(i),

$$\begin{aligned}\frac{h^n_{Z_p(K)}}{h^n_K}&\geqslant 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left[2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|\Gamma\left(\frac{n+1}{2}\right)\right]\pm o(p)\\ &\geqslant 1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\log[4r_0^{n-1}\pi^{n-1}h_K^{n+1}]\pm o(p),\end{aligned}\tag{4.3}$$

respectively,

$$\frac{h^n_{Z_p(K)}}{h^n_K}\geqslant 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left[\frac{2r_0^{(n-1)/2}h_K^{(n+1)/2}|B_2^{n-1}|\Gamma((n+1)/2)}{1+Ce^{-cp\varepsilon^2}}\right]\pm o(p)\tag{4.4}$$

$$\geqslant 1-\frac{n(n+1)}{2p}\log p+\frac{n}{2p}\log\left[\frac{4r_0^{n-1}\pi^{n-1}h_K^{n+1}}{(1+Ce^{-cp\varepsilon^2})^2}\right]\pm o(p).\tag{4.5}$$

Now note that there is $\alpha>0$ such that

$$B_2^n(0,\alpha)\subset K\subset B_2^n\left(0,\frac{1}{\alpha}\right).$$

This implies that, for all $u\in S^{n-1}$, $\alpha\leqslant h_K\leqslant 1/\alpha$. Moreover, we can choose α so small that we have, for all $p\geqslant p_0>1$,

$$B_2^n(0,\alpha)\subset Z_p(K)\subset K\subset B_2^n\left(0,\frac{1}{\alpha}\right),$$

which implies that, for all $u\in S^{n-1}$, for all $p\geqslant p_0$,

$$\alpha\leqslant h_{Z_p(K)}\leqslant\frac{1}{\alpha}.\tag{4.6}$$

On the one hand, as $Z_p(K)\subset K$,

$$\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)\geqslant 0.$$

On the other hand, we get, by (4.3), (4.4) and (4.6) with a constant c,

$$\begin{aligned}\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)&\leqslant\frac{cn}{\alpha^n}\left(n+1-\frac{1}{\log p}\log\left(4r_0^{n-1}\pi^{n-1}h_K^{n+1}\right)\right)\\ &\leqslant\frac{cn}{\alpha^n}\left(n+1+\frac{1}{\log p_0}\left|\log\left(\frac{4r_0^{n-1}\pi^{n-1}}{\alpha^{n+1}}\right)\right|\right),\end{aligned}$$

respectively,

$$\begin{aligned}\frac{p}{\log(p)}\frac{1}{h_{Z_p(K)}(u)^n}\left(1-\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)&\leqslant\frac{cn}{\alpha^n}\left(n+1-\frac{1}{\log p}\log\left(\frac{4r_0^{n-1}\pi^{n-1}h_K^{n+1}}{(1+Ce^{-cp\varepsilon^2})^2}\right)\right)\\ &\leqslant\frac{cn}{\alpha^n}\left(n+1+\frac{1}{\log p_0}\left|\log\left(\frac{4r_0^{n-1}\pi^{n-1}}{\alpha^{n+1}}\right)\right|\right).\end{aligned}$$

The right-hand side is a constant and hence integrable.

(ii) As K is in C_+^2, there is $R\geqslant r_0>0$ such that, for all $x\in\partial K$, $K\subset B_2^n(x-RN(x),R)$. Then we show similarly to (4.2) that

$$h_{Z_p}(u)^n\leqslant h_K^n(2^{(n-1)/2}|B_2^{n-1}|h_K^{(n+1)/2}R^{(n-1)/2})^{n/p}\left(\int_0^1 w^p(1-w)^{(n-1)/2}\,dw\right)^{n/p},$$

and thus, similar to (4.3),

$$\frac{h^n_{Z_p(K)}}{h^n_K} \leqslant 1 - \frac{n(n+1)}{2p}\log p + \frac{n}{2p}\log[2^{n+1}R^{n-1}\pi^{n-1}h_K^{n-1}] \pm o(p).$$

Hence, together with (4.3), respectively, (4.4)

$$\begin{aligned} &-\frac{n}{2h_{Z_p(K)^n}}\log[2^{n+1}R^{n-1}\pi^{n-1}h_K^{n-1}] \pm O(p) \\ &\leqslant \frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right) \\ &\leqslant -\frac{n}{2h_{Z_p(K)^n}}\log[4r_0^{n-1}\pi^{n-1}h_K^{n+1}] \pm O(p), \end{aligned}$$

respectively,

$$\begin{aligned} &-\frac{n}{2h_{Z_p(K)^n}}\log[2^{n+1}R^{n-1}\pi^{n-1}h_K^{n-1}] \pm O(p) \\ &\leqslant \frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right) \\ &\leqslant -\frac{n}{2h_{Z_p(K)^n}}\log\left[\frac{4r_0^{n-1}\pi^{n-1}h_K^{n+1}}{(1+Ce^{-cp\varepsilon^2})^2}\right] \pm O(p). \end{aligned}$$

Hence, using (4.6), we get, with an absolute constant c for all $p \geqslant p_0$,

$$\begin{aligned} &\left|\frac{p}{h_{Z_p(K)}(u)^n}\left(1 - \frac{h_{Z_p(K)}(u)^n}{h_K(u)^n} - \frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h_{Z_p(K)}(u)^n}{h_K(u)^n}\right)\right| \\ &\quad \leqslant \frac{cn}{\alpha^n}\left|\log\left[\frac{2^{n+1}R^{n-1}\pi^{n-1}}{\alpha^{n-1}}\right]\right|. \end{aligned}$$

Again, the right-hand side is a constant and therefore integrable. □

As $K \in C^2_+$, the indicatrix of Dupin at every $x \in \partial K$ is an ellipsoid. Since the quantities considered in Theorem 4.1 are affine invariant, we can assume that the indicatrix is a Euclidean ball. We have (see [**49**]) the following lemma.

LEMMA 4.8. *Let $K \subset \mathbb{R}^n$ be a convex body in C^2_+. We assume that the indicatrix of Dupin at $x \in \partial K$ is a Euclidean ball. Let $r = r(x) = \kappa(x)^{-1/(n-1)}$ and put $u = N_K(x)$. Let $B(x - ru, r)$ be the Euclidean ball with center at $x - ru$ and radius r. Then, for every $\varepsilon > 0$, there exists $\Delta_\varepsilon > 0$ such that, for all $\Delta \leqslant \Delta_\varepsilon$,*

$$\begin{aligned} &B(x-(1-\varepsilon)ru,(1-\varepsilon)r) \cap H(x-\Delta u,u)^- \\ &\quad \subset K \cap H(x-\Delta u,u)^- \subset B(x-(1+\varepsilon)ru,(1+\varepsilon)r) \cap H(x-\Delta u,u)^-. \end{aligned}$$

Here $H(x - \Delta u, u)$ is the hyperplane with normal u through $x - \Delta u$ and $H(x - \Delta u, u)^-$ is the half space determined by this hyperplane into which u points.

Proof of Theorem 4.1. (i)

$$|Z_p^\circ(K)| - |K^\circ| = \frac{1}{n}\int_{S^{n-1}}\left(\frac{1}{h^n_{Z_p(K)}(u)} - \frac{1}{h^n_K(u)}\right)d\sigma(u).$$

Hence,

$$\lim_{p\to\infty}\frac{p}{\log p}(|(Z_p^\circ(K))|-|K^\circ|)=\frac{1}{n}\lim_{p\to\infty}\frac{p}{\log p}\int_{S^{n-1}}\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u)$$
$$=\frac{1}{n}\int_{S^{n-1}}\lim_{p\to\infty}\frac{p}{\log p}\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u),$$

where we have used Lemma 4.7(i) and Lebesgue's theorem to interchange integration and limit. Let $u\in S^{n-1}$. Let $x\in\partial K$ be such that $N_K(x)=u$. As K is in C^2_+, $\kappa=\kappa_K(x)>0$ and we can assume that the indicatrix of Dupin at x is a Euclidean ball with radius $r=r(x)=\kappa(x)^{-1/(n-1)}$.

$$h^n_{Z_p(K)}(u)=\left(\int_K|\langle y,u\rangle|^p\,dy\right)^{n/p}=\left(2\int_0^{h_K(u)}t^p|\{y\in K:\langle u,y\rangle=t\}|\,dt\right)^{n/p}$$
$$\geqslant\left(2\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)}t^p|\{y\in K:\langle u,y\rangle=t\}|\,dt\right)^{n/p}$$
$$\geqslant\left(2\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)}t^p|\{y\in B(x-(1-\varepsilon)ru,(1-\varepsilon)r):\langle u,y\rangle=t\}|\,dt\right)^{n/p},$$

where we have applied Lemma 4.8. In addition, we also choose Δ_ε of Lemma 4.8 so that $\Delta_\varepsilon\leqslant\min\{\varepsilon,(1-\varepsilon)r\}$.

$B(x-(1-\varepsilon)ru,(1-\varepsilon)r)\cap\{y\in\mathbb{R}^n:\langle u,y\rangle=t\}$ is an $(n-1)$-dimensional Euclidean ball with radius

$$\left(2(1-\varepsilon)r(h_K(u)-t)\left[1-\frac{h_K(u)-t}{2(1-\varepsilon)r}\right]\right)^{1/2},$$

which, by choice of Δ_ε, is larger than or equal to

$$\left(2(1-\varepsilon)r(h_K(u)-t)\left[1-\frac{\varepsilon(h_K(u)+1-\varepsilon)}{2(1-\varepsilon)r}\right]\right)^{1/2}.$$

Hence,

$$h^n_{Z_p(K)}(u)=\left(\int_K|\langle y,u\rangle|^p\,dy\right)^{n/p}$$
$$\geqslant\left(\frac{2|B_2^{n-1}|[2(1-\varepsilon)rh_K(u)]^{(n-1)/2}}{[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)^{n/p}$$
$$\times\left(\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)}t^p\left(1-\frac{t}{h_K(u)}\right)^{(n-1)/2}dt\right)^{n/p}$$
$$=\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)^{n/p}h_K(u)^n$$
$$\times\left(\int_{(1-\varepsilon)(1-\Delta_\varepsilon/h_K(u))}^{1}v^p(1-v)^{(n-1)/2}\,dv\right)^{n/p}.$$

Now we apply Lemma 4.4 to the function $f(v)=(1-v)^{(n-1)/2}$. We see that f is C^2 and $v_p=1/(1+(n-1)/2p)$. Thus, Lemma 4.4 holds. We see that v_p of Lemma 4.4 is an increasing function of p and $\lim_{p\to\infty}v_p=1$. Hence, for $\varepsilon>0$ given, there exists $p_\varepsilon=p_{\varepsilon,\Delta_\varepsilon}$, namely, $p_\varepsilon\geqslant(n-1)(h_K(u)-\Delta_\varepsilon)/2\Delta_\varepsilon$, such that, for all $p\geqslant p_\varepsilon$, $v_p\geqslant(h_K(u)-\Delta_\varepsilon)/h_K(u)$. In addition, we also choose p_ε so large that $p_\varepsilon \geqslant 1/\varepsilon^3$. Thus,

$$\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} \geqslant \left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)^{n/p} \times \left(\int_0^1 v^p(1-v)^{(n-1)/2}\,dv\right)^{n/p}.$$

Now

$$\begin{aligned}&\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)^{n/p}\\ &= 1+\frac{n}{p}\log\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)\\ &+\frac{1}{2}\left(\frac{n}{p}\log\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)\right)^2 \pm o(p^2). \qquad (4.7)\end{aligned}$$

Together with Lemma 4.3(ii) (for $a=0$), we then get the following: For $\varepsilon>0$ given, there exists p_ε such that for all $p \geqslant p_\varepsilon$

$$\begin{aligned}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} &\geqslant 1-\frac{n(n+1)}{2p}\log p\\ &+\frac{n}{2p}\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\\ &+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\\ &\times\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\log p\\ &-\frac{n(n+1)}{2p^2}\left[\frac{(n^2+3n+6)}{4}\right]\\ &+\frac{n^2}{2p^2}\left[\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2\right.\\ &\left.+2\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{\left(1+Ce^{-c/\varepsilon}\right)^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\right]\\ &+\frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)\right)^2\right] \pm o(p^2).\end{aligned} \qquad (4.8)$$

Thus,

$$\begin{aligned}&\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\\ &\leqslant \frac{n(n+1)}{2}-\frac{n}{2\log p}\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{\left(1+Ce^{-c/\varepsilon}\right)^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\pm o(p).\end{aligned} \qquad (4.9)$$

On the other hand, by Lemma 4.6, the function $f(t)=|K\cap(u^\perp+tu)|$ satisfies the assumptions of Lemma 4.4 and t_p is well defined. Also, t_p is an increasing function of p and, by Lemma 4.5, $\lim_{p\to\infty} t_p = h_K(u)$. Hence, for $\varepsilon>0$ given, there exists $p_\varepsilon = p_{\varepsilon,\Delta_\varepsilon}$ such that, for all $p \geqslant p_\varepsilon$, $t_p \geqslant h_K(u) - \Delta_\varepsilon$. In addition, we also choose p_ε so large so that $p_\varepsilon \geqslant 1/\varepsilon^3$. Thus,

$$\begin{aligned} h^n_{Z_p(K)}(u) &= \left(2\int_0^{h_K(u)} t^p|\{y \in K : \langle u, y\rangle = t\}|\,dt\right)^{n/p} \\ &\leqslant \left(2(1+Ce^{-c\varepsilon^2 p})\int_{t_p(1-\varepsilon)}^{h_K(u)} t^p|\{y \in K : \langle u, y\rangle = t\}|\,dt\right)^{n/p} \\ &\leqslant \left(2(1+Ce^{-c/\varepsilon})\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p|\{y \in K : \langle u, y\rangle = t\}|\,dt\right)^{n/p} \\ &\leqslant \left(2(1+Ce^{-c/\varepsilon})\int_{(1-\varepsilon)(h_K(u)-\Delta_\varepsilon)}^{h_K(u)} t^p \right. \\ &\qquad \left. \times\ |\{y \in B(x-(1+\varepsilon)ru, (1+\varepsilon)r) : \langle u, y\rangle = t\}|\,dt\right)^{n/p}. \end{aligned}$$

In the last inequality, we have used Lemma 4.8. The latter is

$$\leqslant \left(2(1+Ce^{-c/\varepsilon})\int_0^{h_K(u)} t^p|\{y \in B(x-(1+\varepsilon)ru, (1+\varepsilon)r) : \langle u, y\rangle = t\}|\,dt\right)^{n/p}.$$

As above, we note that $B(x-(1+\varepsilon)ru, (1+\varepsilon)r) \cap \{y \in \mathbb{R}^n : \langle u, y\rangle = t\}$ is a $(n-1)$-dimensional Euclidean ball with radius

$$\left(2(1+\varepsilon)r(h_K(u)-t)\left[1 - \frac{h_K(u)-t}{2(1+\varepsilon)r}\right]\right)^{1/2},$$

which is smaller than or equal to

$$(2(1+\varepsilon)r(h_K(u)-t))^{1/2}.$$

We continue similarly to above and get that there exists (a new) p_ε (chosen larger than the ones previously chosen and larger than $1/\varepsilon^3$) such that, for all $p \geqslant p_\varepsilon$,

$$\begin{aligned} \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)} &\leqslant 1 - \frac{n(n+1)}{2p}\log p + \frac{n}{2p}\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right) \\ &\quad + \frac{n^2(n+1)^2}{8p^2}(\log p)^2 - \frac{n^2(n+1)}{2p^2}\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right)\log p \\ &\quad - \frac{n(n+1)}{2p^2}\left[\frac{(n^2+3n+6)}{4}\right] \\ &\quad + \frac{n^2}{2p^2}\left[\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2 + 2\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right)\right] \\ &\quad + \frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B_2^{n-1}|((1+\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})^{-1}}\right)\right)\right] \pm o(p^2). \end{aligned} \tag{4.10}$$

Thus,

$$\begin{aligned} &\frac{p}{\log p}\left(1 - \frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right) \\ &\quad \geqslant \frac{n(n+1)}{2} - \frac{n}{2\log p}\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right) \pm o(p). \end{aligned} \tag{4.11}$$

We see that (4.9) and (4.11) give that

$$\lim_{p\to\infty}\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h_K(u)^n}\right)=\frac{n(n+1)}{2}.$$

Hence, also using that, since $|K|=1$, $h_{Z_p(K)}(u)\to h_K(u)$,

$$\begin{aligned}\lim_{p\to\infty}\frac{p}{\log p}(|Z_p^\circ(K)|-|K^\circ|)&=\frac{1}{n}\int_{S^{n-1}}\lim_{p\to\infty}\frac{p}{\log p}\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u)\\&=\frac{1}{n}\int_{S^{n-1}}\lim_{p\to\infty}\frac{1}{h^n_{Z_p(K)}(u)}\lim_{p\to\infty}\frac{p}{\log p}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u)\\&=\frac{n+1}{2}\int_{S^{n-1}}\frac{1}{h^n_K(u)}\,d\sigma(u)\\&=\frac{n(n+1)}{2}|K^\circ|.\end{aligned}$$

This completes (i).
(ii)

$$\begin{aligned}&|Z_p^\circ(K)|-|K^\circ|-\frac{n(n+1)\log p}{2p}|K^\circ|\\&\quad=\frac{1}{n}\int_{S^{n-1}}\left(\frac{1}{h^n_{Z_p(K)}(u)}-\frac{1}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{1}{h^n_K(u)}\right)d\sigma(u)\\&\quad=\frac{1}{n}\int_{S^{n-1}}\frac{1}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u).\end{aligned}$$

Hence,

$$\begin{aligned}&\lim_{p\to\infty}p\left(|Z_p^\circ(K)|-|K^\circ|-\frac{n(n+1)\log p}{2p}|K^\circ|\right)\\&\quad=\frac{1}{n}\int_{S^{n-1}}\lim_{p\to\infty}\frac{p}{h^n_{Z_p(K)}(u)}\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)d\sigma(u),\end{aligned}$$

where we have used Lemma 4.7(ii) and Lebesgue's theorem to interchange integration and limit. By (4.8) we have, for all $p\geqslant p_\varepsilon$,

$$\begin{aligned}&\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\\&\quad\leqslant-\frac{n}{2p}\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\\&\qquad+\frac{n^2(n+1)^2}{8p^2}(\log p)^2+\frac{n(n+1)}{2p^2}\left[\frac{(n^2+3n+6)}{4}\right]-\frac{n^2}{2p^2}\left[\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2\right.\\&\qquad\left.+2\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\right]\\&\qquad-\frac{n^2}{2p^2}\left[\left(\log\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)\right)^2\right]\pm o(p^2).\end{aligned}$$

Thus,

$$\begin{aligned}
&p\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\\
&\leqslant -\frac{n}{2}\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\\
&\quad+\frac{n^2(n+1)^2}{8p}(\log p)^2+\frac{n(n+1)}{2p}\left[\frac{(n^2+3n+6)}{4}\right]-\frac{n^2}{2p}\left[\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2\right.\\
&\quad\left.+\,2\log\left(\frac{\pi^{n-1}((1-\varepsilon)r)^{n-1}\,[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^2[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{n-1}}\right)\right]\\
&\quad-\frac{n^2}{2p}\left[\left(\log\left(\frac{|B_2^{n-1}|((1-\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})[1-\varepsilon(h_K(u)+1-\varepsilon)/2(1-\varepsilon)r]^{(n-1)/2}}\right)\right)^2\right]\pm o(p). \qquad (4.12)
\end{aligned}$$

Similarly, using (4.10), we get, for all $p\geqslant p_\varepsilon$,

$$\begin{aligned}
&p\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)\\
&\geqslant -\frac{n}{2}\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right)+\frac{n^2(n+1)^2}{8p}(\log p)^2\\
&\quad+\frac{n(n+1)}{2p}\left[\frac{(n^2+3n+6)}{4}\right]\\
&\quad-\frac{n^2}{2p}\left[\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2+2\log\left(\frac{\pi^{n-1}((1+\varepsilon)r)^{n-1}\,[2h_K(u)]^{n+1}}{(1+Ce^{-c/\varepsilon})^{-2}}\right)\right]\\
&\quad-\frac{n^2}{2p}\left[\left(\log\left(\frac{|B_2^{n-1}|((1+\varepsilon)r)^{(n-1)/2}[2h_K(u)]^{(n+1)/2}}{(1+Ce^{-c/\varepsilon})^{-1}}\right)\right)^2\right]\pm o(p). \qquad (4.13)
\end{aligned}$$

We see that (4.12) and (4.13) give that

$$\lim_{p\to\infty}p\left(1-\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}-\frac{n(n+1)}{2}\frac{\log(p)}{p}\frac{h^n_{Z_p(K)}(u)}{h^n_K(u)}\right)=-\frac{n}{2}\log(\pi^{n-1}r^{n-1}[2h_K(u)]^{n+1}). \qquad \square$$

The limit $\lim_{p\to\infty}p(|Z_p^\circ(K)|-|K^\circ|-(n(n+1)/2p)\log p|Z_p^\circ(K)|)$ is computed similarly.

5. *Applications*

The fact that Ω_K can be expressed in different ways allows us to compute the integral in the next proposition. This integral is the relative entropy of the (not normalized) cone measures of the l_r^n-unit ball and its polar.

PROPOSITION 5.1. *Let $1<r<\infty$ and let B_r^n be the l_r^n-unit ball and let $(B_r^{n-1})^+$ be the set of all vectors in B_r^{n-1} having non-negative coordinates. Then*

$$\begin{aligned}
&\int_{(B_r^{n-1})^+}\prod_{i=1}^{n-1}|x_i|^{r-2}\log\left[(r-1)^{n-1}\prod_{i=1}^{n}|x_i|^{r-2}\right]x_n^{-1}\,dx_1\ldots dx_{n-1}\\
&\quad=\frac{n}{r^{n-1}}\frac{(\Gamma((r-1)/r))^n}{\Gamma(n(r-1)/r)}\left[\frac{n(r-2)}{r}\left(\frac{\Gamma'((r-1)/r)}{\Gamma((r-1)/r)}-\frac{\Gamma'(n(r-1)/r)}{\Gamma(n(r-1)/r)}\right)+(n-1)\log r\right].
\end{aligned}$$

Proof. In Chapter 3, it was shown that

$$\log \Omega_K = -\frac{n}{\mathrm{as}_\infty(K)} \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n} \log \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}\, d\mu_K(x).$$

We apply this formula to $K = B_r^n$, $1 < r < \infty$. It was also shown in Chapter 3 that

$$\log \Omega_{B_r^n} = -n \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'((r-1)/r)}{\Gamma((r-1)/r)} - \frac{\Gamma'(n(r-1)/r)}{\Gamma(n(r-1)/r)} \right) + (n-1)\log r \right].$$

The curvature at a boundary point of B_r^n is (see [**51**])

$$\kappa(x) = \frac{(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{(n+1)/2}},$$

and the normal is (see [**51**])

$$N_{\partial B_r^n}(x) = \frac{(\mathrm{sgn}(x_1)|x_1|^{r-1}, \dots, \mathrm{sgn}(x_n)|x_n|^{r-1})}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{1/2}}.$$

Thus, we get, where $B_{r'}^n$ is the polar of B_r^n, that is, r' is the conjugate exponent of r,

$$\begin{aligned} &n \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'((r-1)/r)}{\Gamma((r-1)/r)} - \frac{\Gamma'(n(r-1)/r)}{\Gamma(n(r-1)/r)} \right) + (n-1)\log r \right] |B_{r'}^n| \\ &= \int_{\partial B_r^n} \frac{(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{1/2}} \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] d\mu_{\partial B_r^n}(x). \end{aligned}$$

Now we integrate with respect to the variables $x_1, \dots, x_{n-1}$. The volume of a surface element in the plane of the first $n-1$ coordinates equals the volume of the corresponding surface element on ∂B_r^n times

$$|\langle e_n, N_{\partial B_r^n}(x)\rangle| = \frac{|x_n|^{r-1}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{1/2}}.$$

Thus, with $(B_r^{n-1})^+$ being the set of all vectors in B_r^{n-1} having non-negative coordinates,

$$\begin{aligned} &2^n (r-1)^{n-1} \int_{(B_r^{n-1})^+} \prod_{i=1}^n |x_i|^{r-2} \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] x_n^{1-r}\, dx_1 \dots dx_{n-1} \\ &\quad = 2^n (r-1)^{n-1} \int_{(B_r^{n-1})^+} \prod_{i=1}^{n-1} |x_i|^{r-2} \log \left[(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2} \right] x_n^{-1}\, dx_1 \dots dx_{n-1} \\ &\quad = 2^n (r-1)^{n-1} \frac{n}{r^{n-1}} \frac{(\Gamma((r-1)/r))^n}{\Gamma(n(r-1)/r)} \\ &\quad\quad \times \left[\frac{n(r-2)}{r} \left(\frac{\Gamma'((r-1)/r)}{\Gamma((r-1)/r)} - \frac{\Gamma'(n(r-1)/r)}{\Gamma(n(r-1)/r)} \right) + (n-1)\log r \right], \end{aligned}$$

where we have also used that

$$|B_{r'}^n| = \frac{2^n (r-1)^{n-1}}{n r^{n-1}} \frac{(\Gamma((r-1)/r))^n}{\Gamma(n(r-1)/r)}.$$

There are still other ways how Ω_K can be expressed. Similarly to Theorem 4.1, Ω_K appears in the asymptotic behavior of the volume of certain surface bodies and illumination surface bodies [**57**]. We show the result for the surface bodies. For the illumination surface bodies it is done similarly. □

The surface bodies, a variant of floating bodies, were introduced in [**50**, **51**] as follows.

Definition 5.2. Let $s \geqslant 0$ and $f : \partial K \to \mathbb{R}$ be a non-negative, integrable function. The *surface body* $K_{f,s}$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $f\mu_K$-measure less than or equal to s from ∂K. More precisely,

$$K_{f,s} = \bigcap_{\int_{\partial K \cap H^-} f\, d\mu_K \leqslant s} H^+.$$

Proposition 5.3. *Let K be a symmetric convex body in $\mathbb{R}^n$ that is in C^2_+. Then*

$$d_n \lim_{s \to 0} \frac{|K| - |K_{f,s}|}{s^{2/(n-1)}} = \int_{\partial K} \frac{\kappa(x)}{\langle x, N(x)\rangle^n} \log\left(\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}}\right) d\mu(x) = |K^\circ| \log \frac{1}{\Omega_K},$$

where $K_{f,s}$ is the surface body of K for the function

$$f = \frac{\langle x, N_K(x)\rangle^{n(n-1)/2}}{\kappa^{(n-2)/2}} \left(\log\left(\frac{\kappa}{\langle x, N_K(x)\rangle^{n+1}}\right)\right)^{-(n-1)/2},$$

and where $d_n = 2(|B_2^{n-1}|)^{2/(n-1)}$.

Proof. The proof follows immediately from the following formula which was proved in [51, Theorem 14]:

$$d_n \lim_{s \to 0} \frac{|K| - |K_{f,s}|}{s^{2/(n-1)}} = \int_{\partial K} \frac{\kappa^{1/(n-1)}}{f^{2/(n-1)}}\, d\mu_{\partial K}.$$ □

Appendix. *Calculations with Γ-functions*

For $x, y > 0$, $\Gamma(x) := \int_0^\infty \lambda^{x-1} e^{-\lambda}\, d\lambda$ is the Gamma function and $B(x,y) := \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1}\, d\lambda = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function.

Recall that we write $f(p) = g(p) \pm o(p)$, if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} ph(p) = 0$ and, similarly, $f(p) = g(p) \pm o(p^2)$, if there exists a function $h(p)$ such that $f(p) = g(p) + h(p)$ and $\lim_{p\to\infty} p^2 h(p) = 0$.

We shall frequently use the following: For $x \to \infty$,

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x} \left[1 + \frac{1}{12x} + \frac{1}{288x^2} \pm o(x^2)\right]. \tag{A.1}$$

For every $z, w > 0$,

$$z^{1/p} = 1 + \frac{\log z}{p} + \frac{(\log z)^2}{2p^2} \pm o(p^2)$$

and

$$(p+z)^{w/p} = 1 + \frac{w}{p} \log p + \frac{w^2 (\log z)^2}{2p^2} + \frac{wz}{p^2} \pm o(p^2).$$

Note that if $f(p)^2 = o(p)$, then $(1 + f(p))(1 - f(p)) = 1 \pm o(p)$, which means that

$$\frac{1}{1 + f(p)} = 1 - f(p) \pm o(p).$$

Also

$$\frac{a}{p+b} = \frac{a}{p} - \frac{ab}{p^2} \pm o(p^2).$$

Proof of Lemma 4.3. (i) We use (A.1) and get

$$\begin{aligned}
&\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p}\\
&=\left(\frac{\Gamma(p+1)}{\Gamma(p+1+(n+1)/2)}\Gamma\left(\frac{n+1}{2}\right)\right)^{n/p}\\
&=\left(\frac{\begin{array}{c}\Gamma((n+1)/2)e^{(n+1)/2}(p+1)^{p+1/2}\\ \left[1+1/12(p+1)+1/288(p+1)^2\pm o(p^2)\right]\end{array}}{\begin{array}{c}(p+1+(n+1)/2)^{p+1+n/2}\left[1+1/12(p+1+(n+1)/2)\right.\\ \left.+1/288(p+1+(n+1)/2)^2\pm o(p^2)\right]\end{array}}\right)^{n/p}\\
&=\left(\Gamma\left(\frac{n+1}{2}\right)e^{(n+1)/2}\right)^{n/p}\left(\frac{p+1}{p+1+(n+1)/2}\right)^{(n/p)(p+1/2)}\\
&\quad\times\left(\frac{1}{p+1+(n+1)/2}\right)^{n(n+1)/2p}\\
&\quad\times\left(\frac{1+1/12(p+1)+1/288(p+1)^2\pm o(p^2)}{1+1/12(p+1+(n+1)/2)+1/288(p+1+(n+1)/2)^2\pm o(p^2)}\right)^{n/p}.
\end{aligned}$$

Note that

$$\left(\frac{1+1/12(p+1)+1/288(p+1)^2\pm o(p^2)}{1+1/12(p+1+(n+1)/2)+1/288(p+1+(n+1)/2)^2\pm o(p^2)}\right)^{n/p}=1\pm o(p^2).$$

Also

$$\begin{aligned}
\left(\Gamma\left(\frac{n+1}{2}\right)e^{(n+1)/2}\right)^{n/p}&=1+\frac{n}{p}\left[\frac{n+1}{2}+\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right]\\
&\quad+\frac{n^2}{2p^2}\left[\frac{n+1}{2}+\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right]^2\pm o(p^2),\\
\left(\frac{1}{1+(n+1)/2(p+1)}\right)^{n(1+1/2p)}&=\left(\frac{1}{1+(n+1)/2(p+1)}\right)^{n}e^{-(n/2p)\log(1+(n+1)/(2p+2))}\\
&=1-\frac{n(n+1)}{2p}+\frac{n(3+5n+3n^2+n^3)}{8p^2}\pm o(p^2)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{p+1+(n+1)/2}\right)^{n(n+1)/2p}=e^{-(n(n+1)/2p)\log(p+(n+3)/2)}\\
&=1-\frac{n(n+1)}{2p}\log p+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n(n+1)(n+3)}{4p^2}\pm o(p^2).
\end{aligned}$$

Hence,

$$\begin{aligned}\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p} &= \left(1\pm o(p^2)\right)\\ &\quad\times\left(1+\frac{n}{p}\left[\frac{n+1}{2}+\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right]\right.\\ &\quad\left.+\frac{n^2}{2p^2}\left[\frac{n+1}{2}+\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right]^2\pm o(p^2)\right)\\ &\quad\times\left(1-\frac{n(n+1)}{2p}+\frac{n(3+5n+3n^2+n^3)}{8p^2}\pm o(p^2)\right)\\ &\quad\times\left(1-\frac{n(n+1)}{2p}\log p+\frac{n^2(n+1)^2}{8p^2}(\log p)^2\right.\\ &\quad\left.-\frac{n(n+1)(n+3)}{4p^2}\pm o(p^2)\right)\\ &= 1-\frac{n(n+1)}{2p}\log p+\frac{n}{p}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)+\frac{n^2(n+1)^2}{8p^2}(\log p)^2\\ &\quad-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\log p\\ &\quad+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2\right.\\ &\quad\left.-\frac{n+1}{4}\left(n(n+1)+2(n+3)\right)\right]\pm o(p^2).\end{aligned}$$

(ii)

$$\begin{aligned}&\left(\int_0^1 u^p(1-u)^{(n-1)/2}(1-a(1-u))^{(n-1)/2}\,du\right)^{n/p}\\ &=\left(\int_0^1 u^p(1-u)^{(n-1)/2}\left[1-\binom{\frac{n-1}{2}}{1}a(1-u)+\binom{\frac{n-1}{2}}{2}a^2(1-u)^2\pm\dots\right]du\right)^{n/p}\\ &=\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p}\left[1-\binom{\frac{n-1}{2}}{1}aB_3\right.\\ &\quad\left.+\binom{\frac{n-1}{2}}{2}a^2B_5-\binom{\frac{n-1}{2}}{3}a^3B_7\pm\dots\right]^{n/p}\\ &=\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p}\exp\left\{\frac{n}{p}\log\left[1-\binom{\frac{n-1}{2}}{1}aB_3+\binom{\frac{n-1}{2}}{2}a^2B_5\pm\dots\right]\right\}\\ &=\left(B\left(p+1,\frac{n+1}{2}\right)\right)^{n/p}\\ &\quad\times\left[1-\frac{n}{p}\left\{\binom{\frac{n-1}{2}}{1}aB_3-\binom{\frac{n-1}{2}}{2}a^2B_5+\frac{1}{2}\left(\binom{\frac{n-1}{2}}{1}\right)^2a^2B_3^2\pm\dots\right\}\dots\right],\end{aligned}$$

where, for $3\leqslant k\leqslant n-2$ and for a constant c,

$$B_k=\frac{B(p+1,(n+k)/2)}{B(p+1,(n+1)/2)}=\frac{\Gamma((n+k)/2)}{\Gamma((n+1)/2)}\frac{1}{p^{(k-1)/2}}\left(1+\frac{c}{p}\pm o(p)\right).$$

Hence, together with (i),

$$\left(\int_0^1 u^p(1-u)^{(n-1)/2}\,(1-a\,(1-u))^{(n-1)/2}\,du\right)^{n/p} = 1-\frac{n(n+1)}{2p}\log p$$
$$+\frac{n}{p}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)+\frac{n^2(n+1)^2}{8p^2}(\log p)^2-\frac{n^2(n+1)}{2p^2}\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\log p$$
$$+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2-\frac{(n+1)(n^2+3n+6)}{4}-2\binom{\frac{n-1}{2}}{1}a\frac{\Gamma((n+3)/2)}{\Gamma((n+1)/2)}\right]$$
$$+\frac{n}{2p^2}\left[n\left(\log\left(\Gamma\left(\frac{n+1}{2}\right)\right)\right)^2-\frac{(n+1)(n^2+3n+6)}{4}-(n+1)\binom{\frac{n-1}{2}}{1}a\right]$$
$$\pm\, o(p^2).$$
□

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