NEW HIGHER-ORDER EQUIAFFINE INVARIANTS

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Abstract. We introduce new affine invariants for smooth convex bodies. Some sharp affine isoperimetric inequalities are established for the new invariants.

1. Introduction

Affine invariants play a central role in the general theory of convex bodies. There have been numerous applications of these invariants in asymptotic convex geometry, differential geometry, Banach space theory, ordinary and partial differential equations, and even in seemingly unrelated fields like geometric tomography.

Within the last few years, a substantial amount of research was devoted to investigate in depth the affine invariants occurring in the theory of convex bodies, like volume, Euler characteristic, affine surface area and other valuations [1, 2, 3, 9, 14, 15, 16]. Besides their intrinsic interest, they are essential factors in affine isoperimetric inequalities.

In particular, affine surface area, originally a basic affine invariant from the field of affine differential geometry introduced by Blaschke [6], has been recognized as an important object (see e.g. [19]). The classical affine isoperimetric inequality which gives an upper bound for the affine surface area in terms of volume (see e.g. [22]) proved to be the key ingredient in many problems. To cite just one such result, it was used to show the uniqueness of self-similar solutions of the affine curvature flow, successfully employed in image processing [4, 23, 24]. Simultaneously, affine surface area "measures" the boundary behavior of a convex body and thus comes in naturally in the study of affine PDE's (see e.g. [29] and [30]), in approximation of convex bodies by polytopes (see e.g. [8, 17, 27]) and various deep results in the area of combinatorics (see e.g. [5]).

In this paper, we introduce new higher-order affine invariants related to affine surface area. Their construction resembles the one which allowed to extend the definition of affine surface area (which was originally only defined for sufficiently smooth convex bodies) to all convex bodies (see e.g.

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[10, 18, 19, 21, 26, 31]). As in the "extension problem" we use geometric objects, the convex floating body [26] and the illumination body [31], to obtain these new invariants.

We establish some sharp isoperimetric inequalities relating them and we end by computing these new invariants for the l^p-unit balls in dimension 2.

In a forthcoming paper, we will address the L_p-extensions of higher-order affine invariants.

2. Definitions and main results

Let K be a convex body in $\mathbb{R}^n$ and denote by $h : \mathbb{S}^{n-1} \to \mathbb{R}$ its support function, $h(\mathbf{u}) = \max_{x \in K}(x \cdot \mathbf{u})$.

For each unitary direction $\mathbf{u} \in \mathbb{S}^{n-1}$, there exists a unique hyperplane of normal $\mathbf{u}$ supporting the boundary of K,

$$H_{\mathbf{u}} = \{y \in \mathbb{R}^n \mid \mathbf{u} \cdot y = h(\mathbf{u})\}\,. \tag{1}$$

If $H_{\mathbf{u},\delta}$ denotes the hyperplane parallel to $H_{\mathbf{u}}$ such that the n-dimensional volume of the cap cut from K by $H_{\mathbf{u},\delta}$ is

$$|\,\{y \in K \mid h_\delta(\mathbf{u}) \le \mathbf{u} \cdot y \le h(\mathbf{u})\}\,| = \delta, \tag{2}$$

for some positive $\delta < |K|/2$, then

$$K_\delta = \bigcap_{\mathbf{u} \in \mathbb{S}^{n-1}} \{y \in \mathbb{R}^n \mid \mathbf{u} \cdot y \le h_\delta(\mathbf{u})\} \tag{3}$$

is said to be the *convex floating body* of K of factor δ, [26].

The convex floating body of a convex body always exists. It may be different from the floating body whose definition requires

$$|\,\{y \in K \mid h_\delta(\mathbf{u}) \le \mathbf{u} \cdot y \le h(\mathbf{u})\}\,| \ge \delta,$$

see [7], and may not even exist [12]. Therefore we should emphasize that it is precisely the convex floating body which is used in this paper. For an arbitrary convex body K, it is however true that if its floating body is convex, then it coincides with its convex floating body, see [20].

It is known that the affine surface area of K, denoted here by $\Omega_0(K)$, satisfies

$$\Omega_0(K) = c(n) \lim_{\delta \to 0} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}}, \tag{4}$$

where $c(n) := 2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$ is a constant depending solely on the dimension, making it so that $\Omega_0(B_2^n) = |\partial B_2^n|$, [26]. We denote by B_2^n the Euclidean unit ball in $\mathbb{R}^n$ and by $|\,.\,|$ the top-dimensional volume.

We will identify, in a similar manner to (4), higher-order invariants and we will show that, under sufficient regularity of ∂K, they admit an integral representation.

Definition 2.1. *Let K be a convex body in $\mathbb{R}^n$. We define the higher-order equiaffine invariants Ω_i by the formulas*

$$\Omega_i(K) \;=\; c_i(n) \lim_{\delta\to 0} \frac{\Omega_{i-1}\left(\frac{K_\delta}{|K_\delta|}\right) - \Omega_{i-1}\left(\frac{K}{|K|}\right)}{\delta^{\frac{2}{n+1}}}, \quad i \in \mathbb{N}^\star, \tag{5}$$

provided that the limits exist. The constants c_i are chosen so that $\Omega_i(B_2^n) = |\partial B_2^n|$. They depend only on the dimension n and can be calculated explicitly.

Remark 2.1. *Ω_i are equiaffine invariants of convex bodies.*

Note that Ω_i, as defined, are clearly invariant under any volume preserving linear transformations of $\mathbb{R}^n$, $\Omega_i(\alpha K) = \Omega_i(K)$, $\forall\, \alpha \in SL(n)$ and $\forall\, K$ convex body, respectively invariant under translations $\Omega_i(K+\mathbf{v}) = \Omega_i(K)$, $\forall\, \mathbf{v} \in \mathbb{R}^n$ and $\forall\, K \subset \mathbb{R}^n$ convex body, making Ω_i equiaffine invariants of convex bodies. □

In what follows, we will derive the integral formula of Ω_1 under sufficient regularity assumptions on the boundary of convex bodies. To start with, we assume that K is smooth and strictly locally convex.

We may describe the support function of K_δ, denoted here by h_t via the one-to-one correspondence $t^{(n+1)/2} = \delta$, in terms of the support function of the original body K. To do so, choose $x_1, x_2, ..., x_n$, coordinates in $\mathbb{R}^n$, such that $\{\mathbf{e_1}, ..., \mathbf{e_{n-1}}, \mathbf{u}\}$ is a basis of $\mathbb{R}^n$ and the supporting point, $\{p\} := H_{\mathbf{u}} \cap \partial K$, lies at the origin, so that, after possibly applying a volume preserving linear transformation if necessary, ∂K is locally the graph of

$$x_n = -\frac{1}{2}\mathcal{K}_p^{1/(n-1)} \sum_{i=1}^{n-1} x_i^2 + o(||x||^2), \tag{6}$$

where $||\,.\,||$ is the Euclidean norm in $\mathbb{R}^n$, $\mathcal{K}_p$ is the Gauss-Kronecker curvature of ∂K at p viewed as a function on $\mathbb{S}^{n-1}$, $\mathbf{u} \mapsto \mathcal{K}_p(\mathbf{u})$ and $f = o(s)$ means $f/s \to 0$ as $s \to 0$.

Thus, if $d = h(\mathbf{u}) - h_t(\mathbf{u})$ denotes the distance between the hyperplanes $H_{\mathbf{u}}$ and $H_{\mathbf{u},t}$, one has a description of the cut-off volume, using for example Cavalieri's principle, as

$$\begin{aligned} & |\,\{y \in K \mid h(\mathbf{u}) - d \le \mathbf{u}\cdot y \le h(\mathbf{u})\}\,| \\ =\; & \int_0^d |B_2^{n-1}\left(\sqrt{\frac{2x_n}{\mathcal{K}_p^{\frac{1}{n-1}}(\mathbf{u})}}\right)|\, dx_n + o(d^{(n+1)/2}) \\ =\; & 2^{\frac{n-1}{2}} \cdot |B_2^{n-1}| \cdot \mathcal{K}_p^{-\frac{1}{2}}(\mathbf{u}) \cdot \int_0^d x_n^{\frac{n-1}{2}}\, dx_n + o(d^{(n+1)/2}) \\ =\; & \frac{2^{\frac{n+1}{2}}|B_2^{n-1}|}{n+1}\mathcal{K}_p^{-\frac{1}{2}}(\mathbf{u}) d^{\frac{n+1}{2}} + o(d^{(n+1)/2}), \end{aligned} \tag{7}$$

where the notation $B_2^{n-1}(r)$ stands for the $(n-1)$-dimensional ball $x_1^2 + \ldots + x_{n-1}^2 \leq r^2$.

Consequently

$$h_t(\mathbf{u}) = h(\mathbf{u}) - \frac{\mathcal{K}_p^{\frac{1}{n+1}}(\mathbf{u})}{2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}} t + o(t) = h(\mathbf{u}) - \frac{1}{c(n)} \mathcal{K}_p^{\frac{1}{n+1}}(\mathbf{u}) t + o(t), \tag{8}$$

where $c(n) = 2\left(\frac{|B_2^{n-1}|}{n+1}\right)^{\frac{2}{n+1}}$ is the same constant as in (4).

Definition 2.2. *Let K be a convex body in $\mathbb{R}^n$. We define $D_i\Omega$ recurrently by the formulas $D_0\Omega(K) = \Omega_0(K)$ and*

$$D_i\Omega(K) \;\; = \;\; d_i(n) \lim_{t\to 0} \frac{D_{i-1}\Omega(K) - D_{i-1}\Omega(K_t)}{t}, \quad i \in \mathbb{N}^\star, \tag{9}$$

provided that the limits exist. The constants d_i depend only on the dimension n and can be calculated explicitly.

It should be noted that $D_i\Omega$ are also affine invariants. They will emerge in the definition of $\Omega_i(K)$ and their study will prove essential to the analysis of latter.

We will now recall several facts about mixed curvature functions which can be found as part of an in-depth coverage of this notion in [25]:

Lemma 2.1. *(a) $s(K_1, \ldots, K_{n-1})\, d\mu_{\mathbb{S}^{n-1}}$ is the unique measure from Riesz representation theorem representing the linear functional of mixed volume on convex bodies*

$$V(K, K_1, \ldots, K_{n-1}) := \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \; s(K_1, \ldots, K_{n-1})\, d\mu_{\mathbb{S}^{n-1}}, \tag{10}$$

where h_K is the support function of K. The function $s(K_1, \ldots, K_{n-1})$ is called the mixed curvature function of $K_1, \ldots, K_{n-1}$.

(b) For convex bodies of class $\mathcal{C}^2$, mixed curvature functions can be viewed as second order differential operators on the support functions of convex bodies

$$s(K_1, \ldots, K_{n-1})(\mathbf{u}) = s[h_1(\mathbf{u}), \ldots, h_{n-1}(\mathbf{u})] \tag{11}$$

for any $\mathbf{u} \in \mathbb{S}^{n-1}$. Here h_i denotes the support function of K_i on the unit sphere, $1 \leq i \leq n-1$.

(c) Mixed curvature functions are symmetric and linear in each of their arguments.

(d) If each K_i is a copy of the same convex body K with support function h, then $s[h_1, \ldots, h_{n-1}] = s[h, \ldots, h] = f_K$, the curvature function of K.

Proposition 2.1 (Integral representations for $D_1\Omega(K)$)**.** *If K is a strictly locally convex body with boundary of, at least, class $\mathcal{C}^4$, then*

$$D_1\Omega(K) = \int_{\mathbb{S}^{n-1}} \mathcal{K}^{\frac{1}{n+1}} s[\mathcal{K}^{\frac{1}{n+1}}, h, h, \ldots, h]\, d\mu_{\mathbb{S}^{n-1}}, \tag{12}$$

where s denotes the mixed curvature function of convex bodies extended, if needed, to all positive functions on the unit sphere $\mathbb{S}^{n-1}$.

Remark 2.2. *The extension of the mixed curvature function to positive functions on the unit sphere.*

Recall that, for convex bodies with $\mathcal{C}^2$ support functions, the mixed curvature function can be regarded as a multi-linear, symmetric, second order differential operator on support functions of convex bodies. If K is a convex body of elliptic type, [12], then $\mathcal{K}^{1/(n+1)}$ is itself the support function $h_{\bar{K}}$ of a suitable convex body $\bar{K}$ in $\mathbb{R}^n$. Otherwise, the previous statement considers the extension of the mixed surface area operator s to positive $\mathcal{C}^2$ functions on $\mathbb{S}^{n-1}$. □

Proof. Using (8), we obtain an asymptotic description of the curvature function of K_t, the reciprocal of the Gauss curvature, denoted by f_{K_t}, along all unitary directions $\mathbf{u}$, which for simplicity we will omit to write:

$$f_{K_t} = s[h_t, h_t, \ldots, h_t] = f_K - \frac{(n-1)}{c(n)}\, s[\mathcal{K}_p^{\frac{1}{n+1}}, h, \ldots, h]t + o(t), \tag{13}$$

Then

$$f_{K_t}^{\frac{n}{n+1}} = f_K^{\frac{n}{n+1}} - \frac{n(n-1)}{(n+1)c(n)}\, f_K^{-\frac{1}{n+1}} s[\mathcal{K}_p^{\frac{1}{n+1}}, h, \ldots, h]t + o(t). \tag{14}$$

Thus, applying Lebesgue's convergence theorem,

$$\begin{aligned}
\lim_{t\to 0} \frac{\Omega_0(K) - \Omega_0(K_t)}{t} &= \int_{\mathbb{S}^{n-1}} \lim_{t\to 0} \frac{f_K^{\frac{n}{n+1}} - f_{K_t}^{\frac{n}{n+1}}}{t}\, d\mu_{\mathbb{S}^{n-1}} \\
&= \alpha_1(n) \int_{\mathbb{S}^{n-1}} \mathcal{K}^{\frac{1}{n+1}} s[\mathcal{K}^{\frac{1}{n+1}}, h, \ldots, h]\, d\mu_{\mathbb{S}^{n-1}} \\
&= \alpha_1(n) \int_{\partial K} \mathcal{K}^{\frac{n+2}{n+1}} s[\mathcal{K}^{\frac{1}{n+1}}, h, \ldots, h]\, d\mu_K \\
&= \alpha_1(n) \int_{\partial K} L_1\, dV = d_1(n)^{-1} D_1\Omega(K)
\end{aligned} \tag{15}$$

where $\alpha_1(n) = \dfrac{n(n-1)}{(n+1)c(n)}$, $d_1(n) = \alpha_1(n)^{-1}$, $dV = \mathcal{K}^{1/(n+1)} d\mu_K$ is the Blaschke metric, and L_i will denote the higher affine mean curvatures as in [13]. Since $L_0 = 1$, we may also write $\Omega_0(K) = \int_{\partial K} L_0\, dV$.

Note that (13) implies also

$$f_{K_t}^{-\frac{1}{n+1}} = f_K^{-\frac{1}{n+1}} + \frac{(n-1)}{(n+1)c(n)}\, f_K^{-\frac{n+2}{n+1}} s[\mathcal{K}_p^{\frac{1}{n+1}}, h, \ldots, h]t + o(t) \tag{16}$$

and, furthermore,

$$h_t = h - \frac{1}{c(n)}\,\mathcal{K}_p^{\frac{1}{n+1}} t + \frac{n-1}{(n+1)c(n)^2}\, f_K^{-\frac{n+2}{n+1}} s[\mathcal{K}_p^{\frac{1}{n+1}}, h, \ldots, h]t^2 + o(t^2). \tag{17}$$

□

Similarly integral expressions can be obtained for $D_i\Omega(K)$, $i \geq 2$. However these expressions become increasingly more complicated. Note also that while ∂K of class $\mathcal{C}^2$ suffices for the integral definition of $\Omega(K)$, class $\mathcal{C}^4$ is necessary for $D_1\Omega(K)$, respectively, class $\mathcal{C}^6$ for $D_2\Omega(K)$, etc.

Proposition 2.2. *If K is a strictly locally convex body with boundary of, at least, class $\mathcal{C}^4$, then*

$$(\Omega_0(K))^2 \geq n \cdot |K| \cdot D_1\Omega(K), \tag{18}$$

with equality if and only if K is an ellipsoid.

Proof. Let us assume first that K is an elliptic convex body as in Leichtweiss, [12]. In other words, $\mathcal{K}^{\frac{1}{n+1}}$, viewed as a function on $\mathbb{S}^{n-1}$, is the support function of some convex body in $\mathbb{R}^n$. We denote this convex body by $\bar{K}$.

Recall that $V(K_1, K_2, K_3, ..., K_n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{K_1} s[h_{K_2}, h_{K_3}, \ldots, h_{K_n}]\, d\mu_{\mathbb{S}^{n-1}}$ is the mixed volume of $K_1, K_2, K_3, \ldots, K_n$, [25].

Then $\Omega_0(K) = nV(K, K, ..., K, \bar{K})$, while $D_1\Omega(K) = nV(K, K, ..., K, \bar{K}, \bar{K})$. In this case, the claim follows from one of the Brunn-Minkowski inequalities, Theorem 6.2.1 [25], which states that for n-dimensional convex bodies

$$V^2(K, K, ..., K, \bar{K}) \geq |K| \cdot V(K, ..., K, \bar{K}, \bar{K}), \tag{19}$$

with equality if and only if K and $\bar{K}$ are homothetic. See Leichtweiss [11] for the equality case.

The equality case implies that, for some $\lambda > 0$ and $a \in \mathbb{R}^n$, $\mathcal{K}^{\frac{1}{n+1}}(\mathbf{u}) = \lambda\, h(\mathbf{u}) + < a, \mathbf{u} >$ in all directions $\mathbf{u}$ of the unit sphere. Then K, translated at a, whose support function is $h + < a, . >$ evolves homothetically under the affine curvature flow, so K is an ellipsoid, [4].

If K is not of elliptic type, there exists an $M > 0$ such that for any $c \geq M$, $\mathcal{K}^{\frac{1}{n+1}} + c \cdot h$, viewed as a function on the unit sphere, is the support function of a convex body. To see this, apply the curvature operator s to the above support function. We have

$$s[\mathcal{K}^{\frac{1}{n+1}} + c\cdot h, \mathcal{K}^{\frac{1}{n+1}} + c\cdot h, \dots, \mathcal{K}^{\frac{1}{n+1}} + c\cdot h] = s[\mathcal{K}^{\frac{1}{n+1}}, \mathcal{K}^{\frac{1}{n+1}}, \dots \mathcal{K}^{\frac{1}{n+1}}] + \binom{n-1}{1} cs[h, \mathcal{K}^{\frac{1}{n+1}}, \dots \mathcal{K}^{\frac{1}{n+1}}] + \binom{n-1}{2} c^2 s[h, h, \mathcal{K}^{\frac{1}{n+1}}, \dots \mathcal{K}^{\frac{1}{n+1}}] + \dots$$

$$+\binom{n-1}{n-1} c^{n-1} s[h, h, \dots, h] = c^{n-1}\cdot\left[\frac{s[\mathcal{K}^{\frac{1}{n+1}}, \mathcal{K}^{\frac{1}{n+1}}, \dots \mathcal{K}^{\frac{1}{n+1}}]}{c^{n-1}} + \binom{n-1}{1}\frac{s[h, \mathcal{K}^{\frac{1}{n+1}}, \dots \mathcal{K}^{\frac{1}{n+1}}]}{c^{n-2}} + \dots + \frac{1}{\mathcal{K}}\right] \to \infty \quad \text{as } c \to \infty,$$

due to the fact that K is strictly locally convex, thus its Gauss curvature is strictly positive in all unitary directions.

Hence the Hessian of $\mathcal{K}^{\frac{1}{n+1}} + c\cdot h$ is positive for c large enough and, we will fix such a constant c for which $\mathcal{K}^{\frac{1}{n+1}} + c\cdot h$ is the support function of a convex body, $\bar{K}$.

Furthermore, the inequality (19) for K, $\bar{K}$ implies

$$\begin{aligned}
0 \quad &\le \left[\frac{1}{n}\int_{\partial K}\left(\mathcal{K}^{\frac{1}{n+1}} + c\cdot h\right) d\mu_K\right]^2 \\
&- \frac{|K|}{n}\int_{\mathbb{S}^{n-1}}\left(\mathcal{K}^{\frac{1}{n+1}} + c\cdot h\right) s[\mathcal{K}^{\frac{1}{n+1}} + c\cdot h, h, \dots, h]\, d\mu_{\mathbb{S}^{n-1}} \\
&= \left[\frac{1}{n^2}(\Omega_0(K))^2 - \frac{|K|}{n}\cdot D_1\Omega(K)\right] + \frac{2c}{n}|K|\int_{\partial K}\mathcal{K}^{\frac{1}{n+1}}\, d\mu_K + c^2|K|^2 \\
&- c\frac{|K|}{n}\left[\int_{\mathbb{S}^{n-1}}\mathcal{K}^{\frac{1}{n+1}} s[h, h, \dots, h]\, d\mu_{\mathbb{S}^{n-1}} + \int_{\mathbb{S}^{n-1}} h\, s[\mathcal{K}^{\frac{1}{n+1}}, h, \dots, h]\, d\mu_{\mathbb{S}^{n-1}}\right] \\
&- c^2\frac{|K|}{n}\int_{\mathbb{S}^{n-1}} h\, s[h, h, \dots, h]\, d\mu_{\mathbb{S}^{n-1}} = \left[\frac{1}{n^2}(\Omega_0(K))^2 - \frac{|K|}{n}\cdot D_1\Omega(K)\right],
\end{aligned}$$

which concludes the claim for any K strictly locally convex and sufficiently regular. The equality case implies $\lambda h = \mathcal{K}^{\frac{1}{n+1}} + c\,h + < a, . >$, for some $\lambda > 0$ and $a \in \mathbb{R}^n$, and using the same argument as before, we infer that K is an ellipsoid.

□

Corollary 2.1 (Isoperimetric Inequality for $D_1\Omega(K)$)**.** *If K is a strictly locally convex body with boundary of, at least, class $\mathcal{C}^4$, then*

$$D_1\Omega^{n+1}(K) \le n^{n+1}|B_2^n|^4\cdot|K|^{n-3} \tag{20}$$

with equality if and only if K is an ellipsoid.

Proof. It follows directly from Proposition 2.2 and the classical affine isoperimetric inequality, see for example, [12],

$$\Omega_0^{n+1}(K) \le n^{n+1}|B_2^n|^2\cdot|K|^{n-1},$$

with equality if and only if K is an ellipsoid. □

Unfortunately, the inequality (20) is not extremely powerful as $D_1\Omega$ may be negative if K is not an ovaloid - a convex body whose affine Gauss curvature is positive everywhere, see [12] or [13].

Theorem 2.1 (Integral representations for $\Omega_1(K)$)**.** *If K is a strictly locally convex body with boundary of, at least, class $\mathcal{C}^4$, then*

$$\Omega_1(K) = \frac{c_1(n)}{c(n)} \frac{n(n-1)}{n+1} \frac{1}{|K|^{\frac{n(n-1)}{n+1}}} \left(\frac{(\Omega_0(K))^2}{|K|} - \int_{\partial K} L_1 \, dV \right), \tag{21}$$

where

$$c_1(n) = c(n) \frac{n+1}{n(n-1)^2} |B_2^n|^{\frac{n(n-1)}{n+1}}. \tag{22}$$

Proof. As $\Omega_0(K) = \displaystyle\int_{\mathbb{S}^{n-1}} \mathcal{K}^{-\frac{n}{n+1}} \, d\mu_{\mathbb{S}^{n-1}}$, note that $\Omega_0(\lambda K) = \lambda^{\frac{n(n-1)}{n+1}} \Omega_0(K)$. The proof relies then on the formulas (4) and (12) with its alternate version (15).

Alternatively, this can be proved if one views the definition of Ω_1 as

$$\Omega_1(K) = c_1(n) \frac{d}{dt} \left(\frac{\Omega_0(K_t)}{|K_t|^{\frac{n(n-1)}{n+1}}} \right)_{|t=0}, \tag{23}$$

with (4) and (15) regarded analogously as

$$\frac{d}{dt} \left(|K_t| \right)_{|t=0} = -\frac{1}{c(n)} \Omega_0(K), \tag{24}$$

respectively,

$$\frac{d}{dt} \left(\Omega_0(K_t) \right)_{|t=0} = -\frac{n(n-1)}{c(n)(n+1)} D_1\Omega(K). \tag{25}$$

Recall that $c_1(n)$ is a normalization constant chosen such that $\Omega_1(B_2^n) = |\partial B_2^n|$ hence, via the representation (21),

$$|\partial B_2^n| = \frac{c_1(n)}{c(n)} \frac{n(n-1)}{n+1} \frac{1}{|B_2^n|^{\frac{n(n-1)}{n+1}}} \left(n^2 |B_2^n| - |\partial B_2^n| \right), \tag{26}$$

which leads to (22). □

Corollary 2.2. *If K is a strictly locally convex body with boundary of, at least, class $\mathcal{C}^4$, then*

$$\Omega_1(K) \geq \frac{c_1(n)}{c(n)} \frac{(n-1)^2}{n+1} \frac{(\Omega_0(K))^2}{|K|^{\frac{n^2+1}{n+1}}}, \tag{27}$$

with equality if and only if K is an ellipsoid.

Proof. The statement follows directly from the previous theorem and Proposition 2.2. An essential consequence is that Ω_1 is a positive invariant on all convex bodies whose boundary is sufficiently smooth, as opposed to $D_1\Omega$ whose positivity is guaranteed solely on ovaloids. □

Recall that if K is a convex body in $\mathbb{R}^n$ and if δ is some positive real number, the *illumination body* of K of factor δ is defined by

$$K^\delta = \{x \in \mathbb{R}^n : |\, co[x, K] \setminus K\,| \le \delta\}, \tag{28}$$

where $co[x, K]$ denotes the convex hull of x and K, [31].

Theorem 2.2. *Let K be a convex body in $\mathbb{R}^n$. Then*

$$\Omega_i(K) \;=\; d_i(n) \lim_{\delta \to 0} \frac{\Omega_{i-1}\left(\frac{K}{|K|}\right) - \Omega_{i-1}\left(\frac{K^\delta}{|K^\delta|}\right)}{\delta^{\frac{2}{n+1}}}, \quad i \in \mathbb{N}^\star, \tag{29}$$

provided that the limits exist. The constants d_i are chosen so that $\Omega_i(B_2^n) = |\partial B_2^n|$ and they depend only on the dimension n.

One should note that

$$\Omega_0(K) = d(n) \lim_{\delta \to 0} \frac{|K^\delta| - |K|}{\delta^{\frac{2}{n+1}}}, \tag{30}$$

has been shown in [31].

Proof. In a similar manner with the derivation of formula (8), it was shown in [28] that the support function of the illumination body h^t (with $t = \delta^{2/(n+1)}$) of a convex body with $\mathcal{C}^2$ boundary is described by

$$h^t(\mathbf{u}) = h(\mathbf{u}) + \frac{\mathcal{K}_p^{\frac{1}{n+1}}(\mathbf{u})}{2\left(\frac{|B_2^{n-1}|}{n(n+1)}\right)^{\frac{2}{n+1}}} t + o(t) = h(\mathbf{u}) + \frac{1}{d(n)} \mathcal{K}_p^{\frac{1}{n+1}}(\mathbf{u}) t + o(t). \tag{31}$$

Hence formulas (9) hold, with multiplicative constants depending on the dimension n. Moreover, assuming more regularity for the boundary of K, Ω_1's integral formula can be derived equally using (31).

□

3. $\Omega_1(K)$ for the l^p-unit balls in $\mathbb{R}^2$

For a curve defined by $(x, y(x))$, the affine curvature at a point x is

$$\kappa = -\frac{5}{9}(y'')^{-8/3}(y''')^2 + \frac{1}{3}(y'')^{-5/3}y^{(4)}, \tag{32}$$

where the primes denote here the usual differentiation with respect to x, $\frac{d}{dx}$. References can be found in [11] or [13].

Hence, in $\mathbb{R}^2$,

$$D_1\Omega(K) = \int_{\partial K} k^{1/3}\kappa\, d\mu_K = \int_{\partial K} \kappa(\sigma) d\sigma, \tag{33}$$

where $k(x)$ is the usual Euclidean curvature, while $\kappa(x) = k\, s[k^{1/3}]$ is the affine curvature.

For the l^p-unit balls in $\mathbb{R}^2$, $|x|^p + |y|^p = 1$, one has the Euclidean curvature

$$k(x) = \frac{(1-x^p)^{\frac{1}{p}-2} x^{p-2}(p-1)}{\left(1+(1-x^p)^{\frac{2}{p}-2} x^{2p-2}\right)^{3/2}}, \tag{34}$$

while, using (32), we obtain the analytic expression of the affine curvature

$$\kappa(x) = -\frac{1}{9}\frac{(2p-1)(x^{2p}p + p - p\,x^p + x^{2p} - 2 - x^p)}{x^2\left(\dfrac{(p-1)(1-x^p)^{1/p}x^p}{(1-x^p)^2 x^2}\right)^{2/3} \cdot (1-x^p)^2},$$

or

$$\kappa(x) = -\frac{1}{9}\frac{(2p-1)[(p-2)-(1+p)x^p(1-x^p)]}{\left((p-1)(1-x^p)^{\frac{1}{p}+1}x^{p+1}\right)^{2/3}}. \tag{35}$$

One can note that the affine curvature of the Euclidean unit ball is constantly equal to 1, while it is undefined for the l^1- and l^∞-unit balls. Combining (34) and (35), we get that, for $1 < p < \infty$,

$$D_1\Omega\left(\mathbf{B}_p^2\right) = \frac{4(2p-1)}{9(p-1)^{1/3}}\int_0^1 \frac{[(1+p)x^p(1-x^p)-(p-2)]}{[(1-x^p)^{\frac{1}{p}+4}x^{p+4}]^{1/3}}\, dx. \tag{36}$$

Using the change of variable $u = 1 - x^p$ in (36), one has $du = -px^{p-1}\, dx$ and (36) becomes

$$D_1\Omega\left(\mathbf{B}_p^2\right) = \frac{4(2p-1)}{9p(p-1)^{1/3}}\int_0^1 \frac{[(1+p)u(1-u)-(p-2)]}{[u(1-u)]^{\left(\frac{1}{p}+4\right)\frac{1}{3}}}\, du \tag{37}$$

$$= \frac{4(2p-1)}{9p(p-1)^{\frac{1}{3}}} \cdot \left[(1+p)\cdot 2^{\frac{2}{3p}-\frac{1}{3}} \cdot \sqrt{\pi}\cdot \Gamma\left(\frac{2}{3}-\frac{1}{3p}\right) / \,\Gamma\left(\frac{7}{6}-\frac{1}{3p}\right)\right.$$

$$\left. -(p-2)\cdot 2^{\frac{2}{3p}+\frac{5}{3}} \cdot \sqrt{\pi}\cdot |\,\Gamma\left(-\frac{1}{3}-\frac{1}{3p}\right) / \,\Gamma\left(\frac{1}{6}-\frac{1}{3p}\right)|\right]$$

which, using $\Gamma(x+1) = x\Gamma(x), \quad \forall x \in \mathbb{R}$, we can reduce (37) to

$$= \frac{4(2p-1)}{9p(p-1)^{\frac{1}{3}}} \cdot \frac{2^{\frac{2}{3p}-\frac{1}{3}}}{p+1} \cdot \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{2}{3}-\frac{1}{3p}\right)}{\Gamma\left(\frac{7}{6}-\frac{1}{3p}\right)} \cdot \left[(1+p)^2 - 2(p-2)|p-2|\right].$$

Note that, if $1 < p < \infty$, then $D_1\Omega\left(\mathbf{B}_p^2\right) < \infty$, while $D_1\Omega\left(\mathbf{B}_1^2\right) = +\infty$ and $D_1\Omega\left(\mathbf{B}_\infty^2\right) = -\infty$. Recalling the earlier remark on the non-positivity of $D_1\Omega$, one can additionally note that $D_1\Omega\left(\mathbf{B}_p^2\right) = 0$ for $p_0 = 5+3\sqrt{2}$ and it is strictly negative for p greater than p_0.

On the other hand, the affine surface area of the l^p-unit balls is

$$\Omega_0(\mathbf{B}_p^2) = \int_{\partial K} k^{\frac{1}{3}}\, ds = 4\int_0^1 \frac{\left[(p-1)(1-x^p)^{\frac{1}{p}-2} x^{p-2}\right]^{\frac{1}{3}}}{\left[1+(1-x^p)^{\frac{2}{p}-2} x^{2p-2}\right]^{\frac{1}{2}}}\, dx \tag{38}$$

$$= \frac{4(p-1)^{\frac{1}{3}}}{p} \int_0^1 (1-u)^{\frac{1}{3}\left(\frac{1}{p}-2\right)} u^{\frac{1}{3}\left(\frac{1}{p}-2\right)}\, du$$

$$= \frac{(p-1)^{\frac{1}{3}}}{p} 2^{\frac{1}{3}\left(7-\frac{2}{p}\right)} \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{1}{3p}+\frac{1}{3}\right) / \Gamma\left(\frac{1}{3p}+\frac{5}{6}\right),$$

while

$$|\mathbf{B}_p^2| = 4\int_0^1 (1-x^p)^{\frac{1}{p}}\, dx = 4\left[\Gamma\left(\frac{1}{p}+1\right)\right]^2 / \Gamma\left(\frac{2}{p}+1\right). \tag{39}$$

Hence

$$\Omega_1(\mathbf{B}_p^2) = \left[\frac{\pi}{|\mathbf{B}_p^2|}\right]^{2/3} \cdot \left[\frac{[\Omega_0(\mathbf{B}_p^2)]^2}{|\mathbf{B}_p^2|} - D_1\Omega\left(\mathbf{B}_p^2\right)\right] \tag{40}$$

$$= \left[\frac{\pi}{4\left[\Gamma\left(\frac{1}{p}+1\right)\right]^2 / \Gamma\left(\frac{2}{p}+1\right)}\right]^{2/3} \cdot \left[\frac{\left[\frac{(p-1)^{\frac{1}{3}}}{p} 2^{\frac{1}{3}\left(7-\frac{2}{p}\right)} \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{1}{3p}+\frac{1}{3}\right) / \Gamma\left(\frac{1}{3p}+\frac{5}{6}\right)\right]^2}{4\left[\Gamma\left(\frac{1}{p}+1\right)\right]^2 / \Gamma\left(\frac{2}{p}+1\right)}\right.$$

$$\left. - \frac{4(2p-1)}{9p(p-1)^{\frac{1}{3}}} \cdot \frac{2^{\frac{2}{3p}-\frac{1}{3}}}{p+1} \cdot \sqrt{\pi} \cdot \left[(1+p)^2 - 2(p-2)|p-2|\right] \cdot \frac{\Gamma\left(\frac{2}{3}-\frac{1}{3p}\right)}{\Gamma\left(\frac{7}{6}-\frac{1}{3p}\right)}\right].$$

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