Confidence Regions for Means of Multivariate Normal Distributions and a Non-Symmetric Correlation Inequality for Gaussian Measure

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Abstract

Let μ be a Gaussian measure (say, on $\mathbf{R}^n$) and let $K, L \subseteq \mathbf{R}^n$ be such that K is convex, L is a "layer" (i.e. $L = \{x : a \leq \langle x, u \rangle \leq b\}$ for some a, $b \in \mathbf{R}$ and $u \in \mathbf{R}^n$) and the centers of mass (with respect to μ) of K and L coincide. Then $\mu(K \cap L) \geq \mu(K) \cdot \mu(L)$. This is motivated by the well-known "positive correlation conjecture" for symmetric sets and a related inequality of Sidak concerning confidence regions for means of multivariate normal distributions. The proof uses an apparently hitherto unknown estimate for the (standard) Gaussian cumulative distribution function: $\Phi(x) > 1 - \frac{(8/\pi)^{\frac{1}{2}}}{3x+(x^2+8)^{\frac{1}{2}}} e^{-x^2/2}$ (valid for $x > -1$).

1 Introduction

Let $\mu = \mu_n$ be the standard Gaussian measure on $\mathbf{R}^n$ with density $(2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}}$ (or any *centered* Gaussian measure on $\mathbf{R}^n$). It is a well known open problem whether any two symmetric (with respect to the origin) convex sets K_1 and K_2 in $\mathbf{R}^n$ are positively correlated with respect to μ, i.e. whether the following inequality holds

$$\mu(K_1 \cap K_2) \geq \mu(K_1)\mu(K_2). \tag{1}$$

Of course once (1) is proved, it follows by induction that the following formally stronger statement is true:

$$\mu(K_1 \cap K_2 \cap \ldots \cap K_N) \geq \mu(K_1)\mu(K_2)\ldots\mu(K_N) \tag{2}$$

for any convex symmetric sets $K_1, K_2, \ldots, K_N$ in $\mathbf{R}^n$ (the same remark applies to any class of sets closed under intersections). In the language of statistics, (1) and (2) can be viewed as statements about confidence regions for means of multivariate normal distributions (cf. Theorem 1A below). In some special cases (1) and (2) are known to be true. Pitt [P] proved in 1977 that (1) (hence (2)) holds in $\mathbf{R}^2$. Also, if $K_1, K_2 \ldots, K_N$ are symmetric layers in $\mathbf{R}^n$, i.e., sets of the form

$$K_i = \{x \in \mathbf{R}^n : |\langle x, u_i \rangle| \leq 1\}, u_i \in \mathbf{R}^n, i = 1, 2, \ldots, N,$$

then (2) holds (note that in that particular case (1) *doesn't* imply (2)). This was proved by Sidak [S] in 1967 and consequently is referred to as Sidak's

Lemma. See also Gluskin [G] for a proof of Sidak's Lemma. The proof gives in fact a version of (1) with K_1 - an arbitrary symmetric convex body and K_2 - a layer; (2) *for layers* follows then by induction. We show in Remark 6 of Section 3 how Sidak's Lemma can be proved easily with the approach of this paper (an argument of this type seems to have recently occurred more or less simultaneously to several people). In 1981 Borell [B] proved that (1) holds for a class of convex symmetric bodies in $\mathbf{R}^n$ with certain additional properties. Recently Hu [H] proved a correlation inequality for Gaussian measure involving convex functions rather than sets. See also [S-S-Z] for a historical survey and other partial results and [K-MS] for related results.

Here we prove the following.

Theorem 1. *Let $K \subseteq \mathbf{R}^n$ be a convex body and $u \in \mathbf{R}^n \setminus \{0\}$ be such that*

$$\int_K (\langle x, u\rangle - c) d\mu_n(x) = 0$$

i.e. the centroid of K with respect to μ_n lies on the hyperplane

$$H_c = \{x \in \mathbf{R}^n : \langle u, x\rangle = c\}.$$

Let $L = L(a,b) = \{x \in \mathbf{R}^n : a \leq \langle x, u\rangle \leq b\}$ where a, $b \in \mathbf{R}$ are such that the centroid of L also lies in H_c. Then

$$\mu_n(K \cap L) \geq \mu_n(K) \cdot \mu_n(L).$$

It is clear that Theorem 1 formally implies an analogous statement with μ_n replaced by *any* gaussian measure on $\mathbf{R}^n$ (centered or not). In the language of "confidence regions", Theorem 1 may be restated as:

Theorem 1A. *Let* $X_1, X_2, \ldots, X_N, Y$ *be jointly Gaussian random variables and* $b_1, b_2, \ldots, b_N, a, b \in \mathbf{R}$ *be such that*

$$\mathbf{E}(Y|X_1 \le b_1, X_2 \le b_2, \ldots, X_N \le b_N, a \le Y \le b) = \mathbf{E}(Y|a \le Y \le b).$$

Then

$$\begin{aligned} & \mathbf{P}(X_1 \le b_1, X_2 \le b_2, \ldots, X_N \le b_N, a \le Y \le b) \\ \ge \quad & \mathbf{P}(X_1 \le b_1, X_2 \le b_2, \ldots, X_N \le b_N) \cdot \mathbf{P}(a \le Y \le b). \end{aligned}$$

We point out that the discrepancy between the degrees of generality of Theorems 1 and 1A (general convex sets vs. "rectangles") is only apparent: passing from rectangular to general parallelepipeds requires only a change of variables; a general convex polytope is a "degenerated" parallelepiped, and any convex set can be approximated by polytopes.

Theorem 1 leads naturally to the following generalization of the "correlation conjecture" (1).

Problem 2. *If* $K_1, K_2 \in \mathbf{R}^n$ *are convex sets (not necessarily symmetric) such that their centroids with respect to* μ_n *coincide, does* (1) *hold?*

It is conceivable that the "equality of the centroids" hypothesis is not the most proper here and that it should be modified. However, we were led to that particular hypothesis while considering some variational arguments related to the original (symmetric) correlation conjecture (those arguments yield, in particular, an alternative proof of the two-dimensional case shown in [P]). Our

Theorem 1 is related to Problem 2 in roughly the same way as Sidak's Lemma is to the original "symmetric" conjecture.

Theorem 1 is proved in Section 3 (with proofs of some technical lemmas relegated to Section 4). In Section 2 we develop some of the tools necessary for the proof. They may also be of independent interest, in particular Proposition 3 which gives an upper estimate on the tail of the Gaussian distribution that is sharper than the corresponding "Komatsu inequality" known from the literature (cf. [I-MK], p. 17; see also [Ba] for another type of estimate).

Proposition 3. *For $x > -1$*

$$\frac{2}{x + (x^2+4)^{\frac{1}{2}}} \le e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \le \frac{4}{3x + (x^2+8)^{\frac{1}{2}}}$$

The lower estimate in Proposition 3 is the other "Komatsu inequality" and is true for any $x \in \mathbf{R}$. The comparison of the upper estimate from Proposition 3 with classical estimates is given in a table in Remark 4 in the next section.

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2 Preliminaries about Gaussian measure.

We start with the

Proof of Proposition 3. We follow the outline given in [I-MK] in the context of Komatsu inequality. Put $g(x) = e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} dt$ and $g_+(x) = \frac{4}{3x+(x^2+8)^{\frac{1}{2}}}$. It is easily checked that $g' = xg - 1$ and somewhat more tediously verified that $g_+' \leq xg_+ - 1$. Moreover, (e.g.) a direct calculation shows that $g(x) \leq \frac{1}{x}$ for $x > 0$. By considering the function $h = g_+ - g$ and its differential inequality $h' \leq xh - 1$ one gets (by the same argument as in [I-MK]) that $h = g_+ - g \geq 0$ on $(0, \infty$, hence on $[0, \infty$. Since $\lim_{x \to -1} g_+(x) = \infty$ whereas $g(-1)$ is finite, it follows that $g_+(x) \geq g(x)$ also for $x \in (-1, 0)$ (otherwise consider $x \in (-1, 0)$ for which h attains its minimum).

The estimate from below is shown in a similar way (and, anyway, it is not new). $\square$

Remark 4. As was mentioned in the introduction, for $x > 0$ the upper estimate of Proposition 3 is sharper than the well known estimate of Komatsu who proved that for $x > 0$

$$\frac{2}{x + (x^2+4)^{\frac{1}{2}}} \leq e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} dt \leq \frac{2}{x + (x^2+2)^{\frac{1}{2}}}$$

(see [I-MK]). We give below the values of relative "errors" (rounded to two significant digits) given by the two upper estimates for some values of x; we also list, for reference, the errors of the lower estimate. All of these were calculated using *Mathematica* and verified with *Maple*. Our estimate is clearly the tightest of the three and vastly superior to the other upper estimate.

x	Our Upper	Komatsu's Upper	Komatsu's Lower
0	.13	.13	$-.20$
2	$.30 \cdot 10^{-2}$	$.67 \cdot 10^{-1}$	$-.17 \cdot 10^{-1}$
4	$.20 \cdot 10^{-3}$	$.25 \cdot 10^{-1}$	$-.25 \cdot 10^{-2}$
6	$.27 \cdot 10^{-4}$	$.13 \cdot 10^{-1}$	$-.61 \cdot 10^{-3}$
8	$.59 \cdot 10^{-5}$	$.74 \cdot 10^{-2}$	$-.21 \cdot 10^{-3}$
10	$.17 \cdot 10^{-5}$	$.48 \cdot 10^{-2}$	$-.92 \cdot 10^{-4}$
20	$.30 \cdot 10^{-7}$	$.12 \cdot 10^{-2}$	$-.61 \cdot 10^{-5}$
30	$.27 \cdot 10^{-8}$	$.55 \cdot 10^{-3}$	$-.12 \cdot 10^{-5}$
40	$.48 \cdot 10^{-9}$	$.31 \cdot 10^{-3}$	$-.39 \cdot 10^{-6}$
50	$.13 \cdot 10^{-9}$	$.20 \cdot 10^{-3}$	$-.16 \cdot 10^{-6}$

Relative errors of estimates for the "Gaussian tail" for selected values of x.

The next result is a fairly easy consequence of Proposition 3.

Proposition 5. *Let, for $x \in \mathbf{R}$,*

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}} dt}.$$

Then

(i) $f(x)$ *is an increasing convex function.*

(ii) $x - f(x)$ *is an increasing (to* 0 *as* $x \to \infty$*) function.*

Proof. (i) We compute

$$f'(x) = \left(\frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt}\right)^2 - x\frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt}$$

Clearly $f' \geq 0$ if and only if

$$\frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt} - x \geq 0.$$

If $x \leq 0$, this inequality holds trivially; if $x > 0$ the inequality holds e.g. by Proposition 3, as $\frac{4}{3x+(x^2+8)^{\frac{1}{2}}} \leq \frac{1}{x}$.

We next have

$$f''(x) = \frac{1}{(e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt)^3}((x^2-1)(e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt)^2 - 3xe^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt + 2).$$

Clearly $f''(x) \geq 0$ if and only if

$$(x^2-1)(e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt)^2 - 3xe^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt + 2 \geq 0.$$

We put $z = e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt$ and consider the expression above as a polynomial in z i.e. $z^2(x^2-1) - 3zx + 2$. As the roots of this polynomial are

$$z_{1/2} = \frac{3x \pm (x^2+8)^{\frac{1}{2}}}{2(x^2-1)} = \frac{4}{3x \mp (x^2+8)^{\frac{1}{2}}},$$

$f'' \geq 0$ holds trivially for $-\infty < x < -1$, and holds for $x > -1$ if

$$e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt \leq \frac{4}{3x + (x^2+8)^{\frac{1}{2}}}$$

which is true by Proposition 3.

(ii) By the calculation from the part (i)

$$(x - f(x))' = 1 - \left(\frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt}\right)^2 + x\frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt}.$$

After putting $z = \frac{e^{-\frac{x^2}{2}}}{\int_x^\infty e^{-\frac{t^2}{2}}\,dt}$, the assertion $(x - f(x))' \geq 0$ becomes

$$1 + xz - z^2 \geq 0.$$

As the roots of this polynomial are

$$z_{1/2} = \frac{x \pm (x^2+4)^{\frac{1}{2}}}{2},$$

the inequality follows, as before, from Proposition 3. $\square$

3 Proof of Theorem 1

The proof of Theorem 1 is achieved in several steps. In the first step we use Ehrhard's inequality [E] to reduce the general case to the 2-dimensional case. In the second step, based on (a rather general) Lemma 7, we reduce the 2-dimensional problem even further to a four-parameter family of "extremal" sets. The final step is based on a careful analysis of dependence of the measures of sets involved on these parameters and uses (computational) Lemmas 8 and 9.

Let K and u be as in Theorem 1 and let H_0 be the hyperplane through 0 orthogonal to u. Without loss of generality we may assume that $\|u\|_2 \leq 1$. For $t \in \mathbf{R}$ put $H_t = H_0 + t \cdot u$ and let $\varphi(t) = \mu_{n-1}(K \cap H_t)$ and $\Phi(x) = \mu_1((-\infty, x])$. By Ehrhard's inequality [E], $\psi(t) = \Phi^{-1}(\varphi(t))$ is a concave function. Therefore it is enough to consider the case $n = 2$ and, in place of K, sets $K_\psi \subseteq \mathbf{R}^2$ of the form

$$K_\psi = \{(x,y) \in \mathbf{R}^2 : y \le \psi(x)\}, \tag{3}$$

with $u = e_1$ and H_0 identical with the y-axis, where ψ is a concave, $\overline{\mathbf{R}}$ - valued function. We will use the convention $\Phi(-\infty) = 0, \Phi(\infty) = 1$. It may also be sometimes convenient to specify the interval $[A, B] = \{x : \psi(x) > -\infty\}$. The assumptions about the centroid become

$$\int_{\mathbf{R}} (x-c)\Phi(\psi(x))d\mu_1(x) = 0 = \int_a^b (x-c)d\mu_1(x) \tag{4}$$

and the assertion becomes

$$\int_a^b \Phi(\psi(x))d\mu_1(x) \ge \int_{\mathbf{R}} \Phi(\psi(x))d\mu_1(x) \int_a^b d\mu_1(x). \tag{5}$$

Remark 6. With this reduction of the general case to the 2-dimensional case we can now give a quick proof of Sidak's Lemma. As was indicated earlier, Sidak's Lemma follows by induction from the "symmetric" variant of Theorem 1, i.e. when L is a 0-symmetric layer ($a = -b$, $b > 0$) and K is a 0-symmetric set (hence $c = 0$). After reduction to the 2-dimensional case, ψ is a concave function that is symmetric about the y-axis (hence decreasing away from the origin) and one has to show that

$$\int_{-b}^b \Phi(\psi(x))d\mu_1(x) \ge \int_{-\infty}^{\infty} \Phi(\psi(x))d\mu_1(x) \int_{-b}^b d\mu_1(x)$$

or equivalently

$$\frac{\int_{-b}^b \Phi(\psi(x))d\mu_1(x)}{\int_{-b}^b d\mu_1(x)} \ge \int_{-\infty}^{\infty} \Phi(\psi(x))d\mu_1(x).$$

The above inequality holds because on the left we are averaging the function $\Phi(\psi(x))$ over the set where it is "biggest", while on the right - over the entire real line.

Actually it is not even necessary to use Ehrhard's inequality for this proof of Sidak's Lemma. What is really used is (a special case of) the Brunn-Minkowski inequality for Gaussian measure (this was pointed out to the authors by A. Giannopoulos) and the fact that the Gaussian measure is a product measure.

Returning to the proof of Theorem 1, we show next that it is enough to prove inequality (5) for "extremal" ψ's which turn out to be linear functions. The reduction to this extremal case holds not only for Gaussian measure on $\mathbf{R}^2$ but for a much more general class of measures on $\mathbf{R}^2$ and is based on Lemma 7 that follows. It will be convenient to introduce the following notation: if $\psi : [a,b] \to \overline{\mathbf{R}}$, let

$$C_\psi = \{(x,y) : a \le x \le b, y \le \psi(x)\}.$$

We then have

Lemma 7. *Let $\psi : [a,b] \to \overline{\mathbf{R}}$ be a concave function not identically equal to $-\infty$ and let ν be a finite measure on $\mathbf{R}^2$ that is absolutely continuous with respect to the Lebesgue measure. Then there exists a linear function $\psi_0(x) = mx + h$ such that*

(i) $\nu(C_\psi) = \nu(C_{\psi_0})$

(ii) $\int_{C_\psi} x d\nu(x,y) = \int_{C_{\psi_0}} x d\nu(x,y)$

(iii) $\psi_((a) \leq \psi_0(a), \quad \psi_((b) \leq \psi_0(b)$

(iv) $\psi_0'(a) \leq \psi'(a), \quad \psi_0'(b) \geq \psi'(b)$

We postpone the rather elementary proof of Lemma 7 until section 4.

For $\alpha < \beta$ let us denote

$$L(\alpha, \beta) = \{(x, y) \in \mathbf{R}^2 : \alpha \leq x \leq \beta\}.$$

In the notation of Lemma 7 the assertion of Theorem 1 (or (5)) then becomes

$$\nu(C_\psi) \geq \nu(K_\psi) \cdot \nu(L(a, b)). \tag{6}$$

(Note that $C_\psi = K_\psi \cap L(a, b)$; the reader is advised to draw a picture at this point to follow the remainder of the argument). Let now $\psi_0(x) = mx + h$ be given by Lemma 7. By symmetry, we may assume that $m \geq 0$. The plan now is to show that, for some (ultimately unbounded) interval $[A, B] \supset [a, b]$ and

$$\psi_1(x) = \begin{cases} mx + h & \text{if } x \in [A, B] \\ -\infty & \text{if } x \notin [A, B] \end{cases} \tag{7}$$

we have

$$\nu(C_{\psi_1}) = \nu(C_\psi) \tag{8}$$

$$\nu(K_{\psi_1}) \geq \nu(K_\psi) \tag{9}$$

while, at the same time, the ν-centroids of K_{ψ_1} and K_ψ lie on the same line $x = c$, i.e.

$$\int_{K_{\psi_1}} (x-c)d\nu(x,y) = \int_{\{A\leq x\leq B, y\leq mx+h\}} (x-c)d\nu(x,y) = \int_{K_\psi} (x-c)d\nu(x,y) = 0. \tag{10}$$

It will then follow immediately that it is enough to prove (6) with ψ replaced by ψ_1, as required for reduction to the "linear" case.

Now (8) is a direct consequence of the assertion (i) of Lemma 7 and (7). On the other hand, it follows from the assertions (iii) and (iv) that $\psi_0(x) = mx+h \geq \psi(x)$ for $x \notin [a,b]$; in other words $K_{\psi_0}\backslash L(a,b) \supset K_\psi\backslash L(a,b)$. In combination with (8) this would imply (9), *if* we were able to set $[A,B] = [-\infty,\infty]$. However, since we also need to ensure the centroid assumption (10), we need to proceed more carefully. Let $A_0 \leq a$ (resp. $B_0 \geq b$) be such that

$$\int_{K_{\psi_0}\cap L(A_0,a)} (x-c)d\nu(x,y) = \int_{K_\psi\cap L(-\infty,a)} (x-c)d\nu(x,y). \tag{11}$$

(resp. $L(b,\infty)$ and $L(b,B_0)$ in place of $L(-\infty,a)$ and $L(A_0,a)$). This is possible since $c \in (a,b)$ and, as we indicated earlier, $\psi_0 \geq \psi$ on $(-\infty,a)$ (resp. on (b,∞)). Since, by (i) and (ii) of Lemma 7,

$$\int_{C_{\psi_0}} (x-c)d\nu(x,y) = \int_{C_\psi} (x-c)d\nu(x,y),$$

it follows that the centroid condition (10) is satisfied if we set $[A,B] = [A_0,B_0]$. Additionally, an elementary argument shows that (11) combined with $\psi_0 \geq \psi$ on $(-\infty,a]$ implies

$$\nu(K_{\psi_0}\cap L(A_0,a)) \geq \nu(K_\psi \cap L(-\infty,a)).$$

This is roughly because the set on the left is "closer" to the axis $x = c$ than the one on the right and so, for the "moment equality" (11) to hold, the former must have a "bigger mass". Similarly, $\nu(K_{\psi_0} \cap L(b, B_0)) \geq \nu(K_\psi \cap L(b, \infty))$, hence the "mass condition" (9) also holds with $[A, B] = [A_0, B_0]$. This reduces the problem to linear functions (more precisely functions of type (7)); to get the full reduction (i.e. to an unbounded interval $[A, B]$) we notice that we may simultaneously (and, for that matter, continuously) move A to the left and B to the right starting from A_0, B_0 respectively so that the centroid condition (10) holds, until A "hits" $-\infty$ or B "hits" $+\infty$; the mass condition (9) will be then *a fortiori* satisfied.

Thus, depending on c, m and h, we end up with one of two possible configurations

$$R_1 = R_1(h, B) = \{(x, y) \in \mathbf{R}^2 : -\infty < x \leq B, y \leq mx + h\}$$

$$R_2 = R_2(h, A) = \{(x, y) \in \mathbf{R}^2 : A \leq x < \infty, y \leq mx + h\},$$

for which we have, for $i = 1$ or $i = 2$ (whichever applicable),

$$\mu_2(R_i \cap L(a, b)) = \mu_2(C_\psi) = \mu_2(K_\psi \cap L(a, b)) \tag{12}$$

$$\mu_2(R_i) \geq \mu_2(K_\psi) \tag{13}$$

$$\int_{R_i} (x - c) d\mu_2 = \int_{K_\psi} (x - c) d\mu_2 = 0 \tag{14}$$

The three conditions above are just a rephrasing of (8)-(10) for $\nu = \mu_2$; in particular it is enough to prove Theorem 1 for the extreme configurations $K =$

$R_i, i = 1, 2$ or, equivalently, to prove (5) for $\psi = \psi_1$ with ψ_1 given by (7) and some unbounded interval $[A, B]$. This will be the last step of the proof of the Theorem.

Let us note here that even though for the configuration $R_1 = R_1(h, B)$ it is possible in principle to have the centroid condition (14) satisfied also for $B < b$, we do not have to consider that case as it would have been "reduced" in the previous step. On the other hand, one always has $A \le a$ for configurations of type R_2 (at least for $m \ge 0$, which we assume all the time). See also the remarks following the statement of Lemma 9.

For $K = R_1$, (5) may be restated as

$$\frac{\int_{-\infty}^{B} \Phi(mx + h) d\mu_1(x)}{\int_a^b \Phi(mx + h) \frac{d\mu_1(x)}{\mu_1((a,b))}} \le 1, \tag{15}$$

while for $K = R_2$

$$\frac{\int_{A}^{\infty} \Phi(mx + h) d\mu_1(x)}{\int_a^b \Phi(mx + h) \frac{d\mu_1(x)}{\mu_1((a,b))}} \le 1. \tag{16}$$

Denote the left hand side of (15) by $F_1(h, w)$; and the left hand side of (16) by $F_2(h, w)$, where $w = \mu_1([a, b])$ is the "Gaussian weight" of the interval $[a, b]$. Note that for fixed c and m, B (resp. A) depends on h as given by (14) with $i = 1$ (resp. $i = 2$). Also note that it perfectly makes sense to consider $h = +\infty$, $w = 0$, $b = +\infty$ or $a = -\infty$ if otherwise allowable.

To study the behavior of F_1 and F_2 we need two more lemmas.

Lemma 8. *With $B = B(h)$ (resp. $A = A(h)$) defined by (14) we have*

$$\frac{dB}{dh} \ge 0, \qquad \frac{dA}{dh} \ge 0\,.$$

Lemma 9. *With $B = B(h)$ defined by (14) we have*

$$\frac{\partial F_1}{\partial h}(h, w) \geq 0\,.$$

Proof of Lemma 8. We give the proof for B (hence $R_1(h, B)$); A and R_2 are treated in a similar way. Showing that $\frac{dB}{dh} \geq 0$ for fixed m and c is equivalent to showing that $\frac{dc}{dh} \leq 0$ for fixed B and m. Note that the centroid of $R_1(h, B)$ is a "weighted average" of the centroids of the half lines $y = mx + \overline{h}$, $-\infty < \overline{h} \leq h$, $-\infty < x \leq B$. Therefore to show that $\frac{dc}{dh} \leq 0$ it is enough to show that the x-coordinates of the centroids of the halflines move further away from the line $x = B$ as h increases. We make a (orthogonal) change of variable such that the line $y = mx + h$ becomes horizontal. Denote the new variables by (u, v). Showing that the x-coordinates of the centroids of the halflines move further away from the line $x = B$ as h increases is equivalent to showing that the u-coordinate of the centroids of the half-lines move further away from the corresponding value $U = U(h)$ on the line corresponding to $x = B$. This means that one has to show that

$$U - \frac{\int_{-\infty}^{U} te^{-\frac{t^2}{2}}\,dt}{(2\pi)^{\frac{1}{2}}\Phi(U)}$$

increases as U increases, which holds by Proposition 5 (ii). $\square$

The computational proof of the Lemma 9 is somewhat involved; we postpone it until the next section.

With Lemmas 8 and 9 we can conclude the proof of the Theorem. Let us start with several observations concerning the qualitative dependence of the regions R_i on c and h (for fixed $m > 0$; m does not *qualitatively* affect that dependence as long as it is positive, the case $m = 0$ being trivial). These observations are only partly used in the proof, but they do clarify the argument nevertheless. First, if $c < 0$ (the special role of 0 follows from the fact that the origin is the centroid of the entire plane), then only configurations of type R_1 appear. As h increases, $B = B(h)$ increases (by Lemma 8) and, as $h \to +\infty$, B approaches some limit value $\tilde{B}$ (of which we may think as $B(\infty)$) defined by the equation

$$\int_{-\infty}^{\tilde{B}} (x - c) d\mu_2 = 0$$

(cf. (14)). It can also be shown that as $h \to -\infty$, $B(h)$ approaches c, but that has no bearing on our argument; we do use only the fact that, for fixed w and c (hence a, b), the condition $B(h) \geq b$ (on which we insist, see the remark following (16)) is, again by Lemma 8, satisfied for h in some interval (of the type $[h^*, +\infty]$ if $c < 0$). If $c = 0$, the picture is similar except that $B(\infty) = \infty$. Finally, if $c > 0$, $B(h)$ is also increasing with h, except that it reaches the limit value $B = +\infty$ for some finite $h = \tilde{h}$, at which point the configuration R_2 "kicks in", the half-plane $R_1(\tilde{h}, +\infty)$ coinciding with $R_2(\tilde{h}, -\infty)$. As h varies from $\tilde{h}$ to $+\infty$, $A(h)$ increases from $-\infty$ to some limit value $\tilde{A} = A(\infty)$ defined by $\int_{\tilde{A}}^{\infty} (x - c) d\mu_2 = 0$, the limit set $R_2(\infty, A(\infty))$ being the half-plane $\{(x, y) \in \mathbf{R}^2 : x \geq A(\infty)\}$.

We first treat $R_1(h, B)$ when $c \leq 0$. By Lemma 9, $\frac{\partial F_1}{\partial h} \geq 0$ for all w. Hence we are done in this case if we show (15) for the extremal configuration when

$h = +\infty$ and $B = \tilde{B}$. But then

$$R_1(\infty, \tilde{B}) = \{(x, y) : x \leq \tilde{B}\}$$

and hence

$$F_1(h, w) \leq F_1(\infty, w) = \mu_1(-\infty, \tilde{B}) \leq 1$$

for all h, w.

Next we consider R_1 when $c \geq 0$. In this case Lemma 9 reduces the deliberation to the extremal configuration with $h = \tilde{h}$ (and $B = +\infty$). Now, as we indicated earlier,

$$R_1(\tilde{h}, +\infty) = \{(x, y) \in \mathbf{R}^2 : y \leq mx + \tilde{h}\} = R_2(\tilde{h}, -\infty)$$

and so the inequality (15) will follow if we show (16) with $A = -\infty$ and the same values of c, m. Thus it remains to handle the case of R_2 i.e. we have to show that

$$F_2(h, w) \leq 1$$

for all h, w or equivalently that

$$\frac{\mu_2(R_2(h, A) \cap L(a, b))}{\mu_2(L(a, b))} \geq \mu_2(R_2(h, A)) \tag{17}$$

for all h, w. Now let us fix h and w (m is fixed throughout the argument) and vary A (hence c). The right hand side of (17) is clearly largest if $A = -\infty$. Similarly the left hand side is smallest if $A = -\infty$; this follows from the fact that, as A is decreasing to $-\infty$, c also decreases and consequently L moves to the left so that $\mu_2(R_2(h, A) \cap L(a, b))$ decreases. So also in the case of R_2 we reduced the argument to the extremal configuration with $A = -\infty$ and $h = \tilde{h}$. It remains to show that

$$\frac{\int_{-\infty}^{\infty} \Phi(mx+h)d\mu_1(x)}{\int_a^b \Phi(mx+h)\frac{d\mu_1(x)}{\mu_1((a,b))}} \leq 1. \tag{18}$$

Throughout the remainder of the proof we will occasionally relax the assumption that c is the Gaussian centroid of (a,b). We first treat the case $h \geq 0$. Observe that in that case $\frac{\int_{-d}^{d} \Phi(mx+h)d\mu_1(x)}{\int_{-d}^{d} d\mu_1(x)}$ decreases as d increases for $d \geq 0$ (this is seen by computing the derivative with respect to d). Therefore

$$\int_{-\infty}^{\infty} \Phi(mx+h)d\mu_1(x) \leq \frac{\int_{-d}^{d} \Phi(mx+h)d\mu_1(x)}{\int_{-d}^{d} d\mu_1(x)}$$

The above is just (18) for $a = -b$. It now formally follows that (18) holds whenever $\frac{a+b}{2} \geq 0$ (or $b \geq -a$): just compare the average of $\Phi(mx+h)$ over [a,b] with that over $[-|a|, |a|]$ and use the fact that $\Phi(mx+h)$ is increasing in x. In particular, if c *is* the Gaussian centroid of (a,b), then, as is easily seen, $\frac{a+b}{2} \geq c \geq 0$, which settles the case $h \geq 0$.

It remains to handle the case $h < 0$.

Let $\Phi_0 = \mu_2(\{(x,y) : y \leq mx+h\})$ and $h_0 = \Phi^{-1}(\Phi_0)$ (i.e. $\Phi_0 = \mu_2(\{(x,y) : y \leq 0 \cdot x + h_0\})$). We need to show that

$$\int_a^b \Phi(mx+h)\frac{d\mu_1(x)}{\mu_1((a,b))} \geq \Phi_0 = \Phi(h_0). \tag{19}$$

Let $x_0 = \frac{h_0-h}{m}$ be the x-coordinate of the point of intersection of the lines $y = h_0$ and $y = mx+h$. If $a \geq x_0$, then (19) holds trivially, hence we only need to consider the case $a < x_0$. We will show that (19) holds provided $\frac{a+b}{2} \geq x_0$.

Figure 1: The case $h < 0$.

In our situation (i.e. when c is the Gaussian centroid of (a, b)) this condition is satisfied since $\frac{a+b}{2} \geq c \geq x_0$. Similarly as in the case of $h \geq 0$, it is enough to consider the case $\frac{a+b}{2} = x_0$ or $b - x_0 = x_0 - a$. To show inequality (19), it is then enough to show

$$\int_a^{x_0} (\Phi_0 - \Phi(mx + h)) d\mu_1(x) \leq \int_{x_0}^b (\Phi(mx + h) - \Phi_0) d\mu_1(x) \tag{20}$$

or equivalently, by rotational invariance of the Gaussian measure, that

$$\begin{aligned} &\textstyle\int_{x_0}^{x_1} (\Phi(mx + h) - \Phi_0) d\mu_1(x) \\ &+ \textstyle\int_{x_1}^{x_2} (\Phi(\frac{x_0 - x}{m} + h_0 + \frac{(1+m^2)^{(1/2)}}{m}(x_0 - a)) - \Phi_0) d\mu_1(x) \\ &\leq \textstyle\int_{x_0}^b (\Phi(mx + h) - \Phi_0) d\mu_1(x), \end{aligned} \tag{21}$$

where $x_1 = x_0 + \frac{x_0 - a}{(1+m^2)^{(1/2)}}$ and $x_2 = x_0 + (1 + m^2)^{(1/2)}(x_0 - a)$ (see Figure 1). Inequality (21) holds, if we can show that

$$\begin{aligned} &\int_b^{x_2} (\Phi(\frac{x_0 - x}{m} + h_0 + \frac{(1 + m^2)^{(1/2)}}{m}(x_0 - a)) - \Phi_0) d\mu_1(x) \\ \leq &\int_{x_1}^b (\Phi(mx + h) - \Phi(\frac{x_0 - x}{m} + h_0 + \frac{(1 + m^2)^{(1/2)}}{m}(x_0 - a)) d\mu_1(x), \end{aligned}$$

which holds as the triangles over which we integrate have the same Lebesgue measure whereas the latter has bigger Gaussian measure as the (restriction of the) reflection which maps the first one into the second is “measure decreasing” with respect to the Gaussian measure.

As shown before, this also completes the proof of $F_1 \leq 1$ and consequently that of the Theorem. $\square$

Remark 10. We wish to reiterate that, at least in the case when K is a half plane $\{(x,y) : y \leq mx + h\}$, the requirement that c is the Gaussian centroid of (a,b) may be relaxed somewhat: to $\frac{a+b}{2} \geq 0$ if $h \geq 0$ and to $\frac{a+b}{2} \geq x_0$ if $h \leq 0$. It follows that the same is true for regions of type R_2. There is also some flexibility in the handling of regions of type R_1, and consequently of an arbitrary K. However, since we do not have any *natural* description of the allowed “relaxation”, we do not pursue this direction.

4 Proofs of the Lemmas.

Proof of Lemma 7. We shall tacitly assume that the density of ν with respect to the Lebesgue measure is strictly positive, which is the case we need in our application; the general case can be easily derived from this one. We shall also assume that ψ doesn't take the value $+\infty$, in particular ψ is continuous (the opposite case is easy to handle directly) and that ψ is not linear (if it is, we are already done). For $m \in \mathbf{R}$ let the line $\psi^{(m)}(x) = mx + h$ be such that

$$\nu(\{(x,y) : a \leq x \leq b,\, y \leq \psi(x)\}) = \nu(\{(x,y) : a \leq x \leq b, y \leq \psi^{(m)}(x)\}), \quad (22)$$

where $h = h(m)$; it follows from our assumptions that $h(\cdot)$ must be a continuous function. The graph of $\psi^{(m)}$ cannot be completely above the graph of ψ on (a, b) nor completely below the graph of ψ on (a, b); otherwise the "mass equality" (22) would not hold. Therefore all the lines satisfying (22) *intersect* the graph of ψ in at least one point $(p, \psi(p))$ with $a < p < b$.

Now suppose there is a line $\psi_0(x) = m_0 x + h$ for which (22) holds, for which the "moment equality"

$$\int_{\{(x,y):a\leq x\leq b, y\leq\psi(x)\}} x d\nu = \int_{\{(x,y):a\leq x\leq b, y\leq\psi_0(x)\}} x d\nu \qquad (23)$$

holds and which has *exactly* one point of intersection $(p, \psi_0(p))$ with the graph of ψ. Then $\psi \leq \psi_0$ on one of the intervals $[a, p], [p, b]$ and $\psi \geq \psi_0$ on the other. On the other hand, it follows from (22) and (23) that

$$\int_{\{(x,y):a\leq x\leq b, y\leq\psi(x)\}} (x-p) d\nu = \int_{\{(x,y):a\leq x\leq b, y\leq\psi_0(x)\}} (x-p) d\nu,$$

which is inconsistent with the preceding remark if ψ and ψ_0 are not identical. Consequently, the line $y = \psi_0(x)$ with the required properties (22) and (23) has to intersect the graph of ψ in at least two points $(p_1, \psi(p_1))$, $(p_2, \psi(p_2))$ with $a < p_1 < p_2 < b$ and, by concavity of ψ, in *exactly* two such points. Again by concavity of ψ this is only possible if the assertions (iii) and (iv) of Lemma 7 hold.

It thus remains to show that among the linear functions $\psi^{(m)}$ for which the "mass equality" (22) (hence (i)) holds there is one for which also the "moment equality" (23) (hence (ii)) holds. To this end, observe that as $m \to +\infty$, the lines $y = \psi^{(m)}(x)$ "converge" to a vertical line $x = a_1$, where a_1 is defined by $\nu(L(a_1, b) = \nu(C_\psi)$. One clearly has

$$\int_{C_\psi} x d\nu < \int_{L(a_1,b)} x d\nu.$$

Similarly, as $m \to -\infty$, the sets $C_{\psi^{(m)}}$ "converge" to a strip $L(a, b_1)$ satisfying $\int_{L(a,b_1)} x d\nu < \int_{C_\psi} x d\nu$. By continuity, there must be $m_0 \in \mathbf{R}$ such that $\psi_0 = \psi^{(m_0)}$ verifies (ii). This finishes the proof of the Lemma. □

For the proof of Lemma 9 we shall need an elementary auxiliary result.

Lemma 11. *Let g be a convex function on an interval $[\alpha, \beta]$ and let ρ be a positive measure on $[\alpha, \beta]$. Let α', β' be such that $\alpha \le \alpha' < \beta' \le \beta$ and suppose that*

$$\left(\frac{\int_\alpha^\beta x d\rho(x)}{\rho([\alpha, \beta])} - \frac{\int_{\alpha'}^{\beta'} x d\rho(x)}{\rho([\alpha', \beta'])}\right)(g(\beta') - g(\alpha')) \ge 0. \tag{24}$$

Then

$$\frac{\int_\alpha^\beta g(x) d\rho(x)}{\rho([\alpha, \beta])} \ge \frac{\int_{\alpha'}^{\beta'} g(x) d\rho(x)}{\rho([\alpha', \beta'])}.$$

Note that if, in particular, $\frac{(\int_\alpha^\beta x d\rho(x)}{\rho([\alpha,\beta])} = \frac{\int_{\alpha'}^{\beta'} x d\rho(x)}{\rho([\alpha',\beta'])}$ or if $g(\beta') = g(\alpha'))$, then the assertion holds. We skip the proof (the reader is advised to draw a picture).

Proof of Lemma 9. We recall that by the comments following the statement of Lemma 9 (see also the remark preceding (15)), for fixed w and c (hence fixed a, b), we do need to consider $h^* \le h \le \tilde{h}$, where $h = h^*$ corresponds to $B = b$ while $\tilde{h} = +\infty$ if $c \le 0$ and $\tilde{h}$ $(< +\infty)$ is defined by $B(\tilde{h}) = +\infty$ (or $A(\tilde{h}) = -\infty$)

if $c > 0$. We have

$$\frac{\partial F_1}{\partial h} = \frac{\int_a^b \Phi(mx+h)\frac{d\mu_1(x)}{\mu_1(a,b)}\left[B'e^{-\frac{1}{2}B^2}\Phi(mB+h) + \int_{-\infty}^{B} e^{-\frac{1}{2}(mx+h)^2}d\mu_1(x)\right]}{(2\pi)^{\frac{1}{2}}\left(\int_a^b \Phi(mx+h)\frac{d\mu_1(x)}{\mu_1((a,b))}\right)^2}$$

$$-\frac{\int_{-\infty}^{B} \Phi(mx+h)d\mu_1(x)\int_a^b e^{-\frac{1}{2}(mx+h)^2}\frac{d\mu_1(x)}{\mu_1(a,b)}}{(2\pi)^{\frac{1}{2}}\left(\int_a^b \Phi(mx+h)\frac{d\mu_1(x)}{\mu_1((a,b))}\right)^2}.$$

As $B' \geq 0$ by Lemma 8, $\frac{\partial F_1}{\partial h} \geq 0$ will follow if

$$\frac{\int_{-\infty}^{B} \frac{e^{-\frac{1}{2}y^2}}{\Phi(y)}\Phi(y)d\mu_1(x)}{\int_{-\infty}^{B} \Phi(y)d\mu_1(x)} \geq \frac{\int_a^b \frac{e^{-\frac{1}{2}y^2}}{\Phi(y)}\Phi(y)d\mu_1(x)}{\int_a^b \Phi(y)d\mu_1(x)} \tag{25}$$

where $y = mx + h$.

By Proposition 5, $g(y) = \frac{e^{-\frac{1}{2}y^2}}{\Phi(y)}$ is a convex decreasing function (note that $g(y) = f(-y)$, where f is as in Proposition 5. Moreover

$$\frac{\int_{-\infty}^{B} x\Phi(y)d\mu_1(x)}{\int_{-\infty}^{B} \Phi(y)d\mu_1(x)} = c = \frac{\int_a^b xd\mu_1(x)}{\int_a^b d\mu_1(x)} \leq \frac{\int_a^b x\Phi(y)d\mu_1(x)}{\int_a^b \Phi(y)d\mu_1(x)},$$

as $\Phi(y)$ is increasing, and so the condition (24) is satisfied with $d\rho(x) = \Phi(y)d\mu_1(x)$. Consequently Lemma 11 yields (25), completing the proof of Lemma 9. □

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