A general geometric construction for affine surface area

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Abstract

Let K be a convex body in $\mathbf{R}^n$ and B be the Euclidean unit ball in $\mathbf{R}^n$. We show that

$$\lim_{t\to 0}\frac{|K|-|K_t|}{|B|-|B_t|}=\frac{as(K)}{as(B)},$$

where $as(K)$ respectively $as(B)$ is the affine surface area of K respectively B and $\{K_t\}_{t\geq 0}$, $\{B_t\}_{t\geq 0}$ are general families of convex bodies constructed from K, B satifying certain conditions. As a corollary we get results obtained in [M-W], [Schm],[S-W] and[W].

The affine surface area $as(K)$ was introduced by Blaschke [B] for convex bodies in $\mathbf{R}^3$ with sufficiently smooth boundary and by Leichtweiss [L1] for convex bodies in $\mathbf{R}^n$ with sufficiently smooth boundary as follows

$$as(K)=\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu(x),$$

where $\kappa(x)$ is the Gaussian curvature in $x\in\partial K$ and μ is the surface measure on ∂K. As it occurs naturally in many important questions, so for example in the approximation of convex bodies by polytopes (see the survey article of Gruber [Gr] and the paper by Schütt [S]) or in a priori estimates for PDEs [Lu-O], one wanted to have extensions of the affine surface area to arbitrary convex bodies in $\mathbf{R}^n$ without any smoothness assumptions of the boundary.

Such extensions were given in recent years by Leichtweiss [L2], Lutwak [Lu], Meyer and Werner [M-W], Schmuckenschläger [Schm], Schütt and Werner [S-W] and Werner [W].

The extensions of affine surface area to an arbitrary convex body K in $\mathbf{R}^n$ in [L2], [M-W], [Schm], [S-W] and [W] have a common feature:
first a specific family $\{K_t\}_{t\geq 0}$ of convex bodies is constructed. This family is different in each of the extensions [L2], [M-W], [Schm], [S-W] and [W] but of

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course related to the given convex body K.
Typically the families $\{K_t\}_{t\geq 0}$ are obtained from K through a "geometric" construction. In [L2] respectively [S-W] this geometric construction gives as $\{K_t\}_{t\geq 0}$ the family of the floating bodies respectively the convex floating bodies. In [M-W] the geometric construction gives the family of the Santaló-regions, in [Schm] the convolution bodies and in [W] the family of the illumination bodies.

The affine surface area is then obtained by using expressions involving volume differences $|K| - |K_t|$ respectively $|K_t| - |K|$.

Therefore it seemed natural to ask whether there are completely general conditions on a family $\{K_t\}_{t\geq 0}$ of convex bodies in $\mathbf{R}^n$ that (in connection with volume difference expressions) will give us affine surface area. We give a positive answer to this question which was asked - among others - by A. Pełczyński.

Throughout the paper we shall use the following notations.
$B(a,r) = B^n(a,r)$ is the n-dimensional Euclidean ball with radius r centered at a. We put $B = B(0,1)$. By $||.||$ we denote the standard Euclidean norm on $\mathbf{R}^n$, by $<\cdot,\cdot>$ the standard inner product on $\mathbf{R}^n$. For two points x and y in $\mathbf{R}^n$ $[x,y] = \{\alpha x + (1-\alpha)y : 0 \leq \alpha \leq 1\}$ denotes the line segment from x to y. For a convex set C in $\mathbf{R}^n$ and a point $x \in \mathbf{R}^n \setminus C$, $\text{co}[x,C]$ is the convex hull of x and C.
$\mathcal{K}$ denotes the set of convex bodies in $\mathbf{R}^n$. For $K \in \mathcal{K}$, $\text{int}(K)$ is the interior of K and ∂K is the boundary of K. For $x \in \partial K$, $N(x)$ is the outer unit normal vector to ∂K in x. We denote the n-dimensional volume of K by $\text{vol}_n(K) = |K|$.
Let $K \in \mathcal{K}$ and $x \in \partial K$ with unique outer unit normal vector $N(x)$. We say that ∂K is approximated in x by a ball from the inside (respectively from the outside) if there exists a hyperplane H orthogonal to $N(x)$ such that $H \cap \text{int}(K) \neq \emptyset$ and a Euclidean ball $B(r) = B(x - rN(x), r)$ (respectively $B(R) = B(x - RN(x), R)$) such that
$$B(r) \cap H^+ \subseteq K \cap H^+$$
respectively
$$K \cap H^+ \subseteq B(R) \cap H^+.$$
Here H^+ is one of the two halfspaces determined by H.

Definition 1

For $t \geq 0$, let $\mathcal{F}_t : \mathcal{K} \to \mathcal{K}$, $\quad K \longmapsto \mathcal{F}_t(K) = K_t$, be a map with the following properties

(i) $K_0 = K$ and
either $K_t \subseteq K$ for all $t \geq 0$ and $\mathcal{F}_t$ is decreasing in t (that is $K_{t_1} \subseteq K_{t_2}$ if $t_1 \geq t_2$)
or $K \subseteq K_t$ for all $t \geq 0$ and $\mathcal{F}_t$ is increasing in t.

(ii) For all affine transformations A with $detA \neq 0$, for all t

$$(A(K))_{|detA|t} = A(K_t).$$

(iii) For all $t \geq 0$, B_t is a Euclidean ball with center 0 and radius $f_1(t)$ and

$$lim_{t\to 0}\left|\frac{|B| - |B_t|}{t^{\frac{2}{n+1}}}\right| = c,$$

where c is a constant (depending on n only).

(iv) Let $x \in \partial K$ be approximated from the inside by a ball $B(r)$.
If $H^+ \cap \partial(K_t) \cap \partial(B(r))_s \neq \emptyset$ for some s and t, then $s \leq Ct$ where C is a constant (depending only on n).

(v) Let $\epsilon > 0$ be given and $x \in \partial K$ be such that it is approximated from the inside by a ball $B(\rho - \epsilon)$ and from the outside by a ball $B(\rho + \epsilon)$. There exists a hyperplane H orthogonal to $N(x)$ and t_0 such that whenever

$$H^+ \cap \partial(K_t) \cap \partial(B(\rho - \epsilon))_s \neq \emptyset, \quad for \quad t \leq t_0, \quad s = s(t),$$

respectively

$$H^+ \cap \partial(K_t) \cap \partial(B(\rho + \epsilon))_s \neq \emptyset, \quad for \quad t \leq t_0, \quad s = s(t),$$

then

$$s \leq (1 + \epsilon)t$$

respectively

$$s \geq (1 - \epsilon)t.$$

Remarks 2

(i) Note that the maps $\mathcal{F}_t$ are essentially determined by the invariance property 1 (ii) and by their behaviour with respect to Euclidean balls.

(ii) Let $f_r(t)$ be the radius of $B(0,r)_t$. Then it follows immediately from Definition 1 (ii), (iii) that

$$lim_{t\to 0}\frac{r-f_r(t)}{1-f_1(t)}=r^{\frac{n-1}{n+1}}.$$

(iii) For some examples the following Definition $1'$ is easier to check than Definiton 1.

Definition 1′

(i) - (iii) as in Definition 1.
(iv)′ If $s<t$, then $K_t\subseteq \text{int}(K_s)$.
(v)′ If $K\subset L$ where L is a convex body in $\mathbf{R}^n$, then $K_t\subseteq L_t$ for all $t\geq 0$.

However not all the examples mentioned below satisfy (iv)′ and (v)′. For instance the illumination bodies (defined below) do not satisfy (v)′.

Examples for Definitions 1 and 1′

1. The (convex) floating bodies [S-W]

Let K be a convex body in $\mathbf{R}^n$ and $t\geq 0$. F_t is a (convex) floating body if it is the intersection of all half-spaces whose defining hyperplanes cut off a set of volume t of K. More precisely, for $u\in S^{n-1}$ let a_t^u be defined by

$$t=|\{x\in K:<u,x>\geq a_t^u\}|.$$

Then

$$F_t=\cap_{u\in S^{n-1}}\{x\in K:<u,x>\leq a_t^u\}$$

is a (convex) floating body.
The family $\{F_t\}_{t\geq 0}$ satiesfies Definitions 1 and 1′.

2. The Convolution bodies [K], [Schm]

Let K be a symmetric convex body in $\mathbf{R}^n$ and $t\geq 0$. Let

$$C(t)=\{x\in\mathbf{R}^n:|K\cap(K+x)|\geq 2t\}$$

and

$$C_t=\frac{1}{2}C(t).$$

Then $\{C_t\}_{t\geq 0}$ satisfies Definitons 1 and 1′.

3. The Santaló-regions [M-W]

For $t \in \mathbf{R}$ and a convex body K in $\mathbf{R}^n$ the Santaló-region $S(K,t)$ of K is defined as

$$S(K,t) = \{x \in K : \frac{|K||K^x|}{|B|^2} \leq t\},$$

where $K^x = (K-x)^0 = \{z \in \mathbf{R}^n :< z, y-x > \ \leq 1 \ \text{ for all } y \in K\}$ is the polar of K with respect to x. (We consider only these t for which $S(K,t) \neq \emptyset$).
Put

$$S_t = S(K, \frac{|K|}{t|B|^2}) = \{x \in K : |K^x| \leq \frac{1}{t}\}.$$

Then the family $\{S_t\}_{t\geq 0}$, satisfies Definitions 1 and 1′.

4. The Illumination bodies [W]

Let K be a convex body in $\mathbf{R}^n$ and $t \geq 0$. The illumination body I_t is the convex body defined as

$$I_t = \{x \in \mathbf{R}^n : |\text{co}[x, K] \backslash K| \leq t\}.$$

Then the family $\{I_t\}_{t\geq 0}$ satiesfies Definition 1.

Theorem 3

Let K be a convex body in $\mathbf{R}^n$. For all $t \geq 0$ let K_t respectively B_t be convex bodies obtained from K respectively B by Definition 1 or 1′. Then

$$lim_{t\to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{as(K)}{as(B)}.$$

Remark
Note that

$$as(B) = vol_{n-1}(\partial B) = n|B|.$$

Corollary 4

(i) [S-W]
Let K be a convex body in $\mathbf{R}^n$ *and for* $t \geq 0$ *let* F_t *be a floating body. Then*

$$lim_{t\to 0} c_n \frac{|K| - |F_t|}{t^{\frac{2}{n+1}}} = as(K).$$

where $c_n = 2 \ \ (\frac{|B^{n-1}|}{n+1})^{\frac{2}{n+1}}$.

(ii) [Schm]
Let K be a symmetric convex body in $\mathbf{R}^n$ *and for* $t \geq 0$ *let* C_t *be a convolution body. Then*

$$lim_{t\to 0} c_n \frac{|K| - |C_t|}{t^{\frac{2}{n+1}}} = as(K).$$

where c_n *is as in (i).*

(iii) [M-W]
Let K be a convex body in $\mathbf{R}^n$ *and for* $t \geq 0$ *let* S_t *be a Santaló-region. Then*

$$lim_{t\to 0} e_n \frac{|K| - |S_t|}{t^{\frac{2}{n+1}}} = as(K).$$

where $e_n = \frac{2}{|B|^{\frac{2}{n+1}}}$.

(iii) [W]
Let K be a convex body in $\mathbf{R}^n$ *and for* $t \geq 0$ *let* I_t *be an illumination body. Then*

$$lim_{t\to 0} d_n \frac{|I_t| - |K|}{t^{\frac{2}{n+1}}} = as(K).$$

where $d_n = 2 \ \ (\frac{|B^{n-1}|}{n(n+1)})^{\frac{2}{n+1}}$.

For the proof of Theorem 3 we need several Lemmas. The basic idea of the proof is as in [S-W].

Lemma 5

Let K and L be two convex bodies in $\mathbf{R}^n$ such that $0 \in int(L)$ and $L \subseteq K$. Then
(i)

$$|K| - |L| = \frac{1}{n} \int_{\partial K} < x, N(x) > (1 - (\frac{||x_L||}{||x||})^n) d\mu(x),$$

where $x_L = [0, x] \cap \partial L$ and μ is the usual surface measure on ∂K.
(ii)

$$|K| - |L| = \frac{1}{n} \int_{\partial L} < x, N(x) > ((\frac{||x_K||}{||x||})^n - 1) d\mu(x),$$

where x_K is the intersection of the half-line from 0 through x with ∂K and μ is the usual surface measure on ∂L.

The proof of Lemma 5 is standard.

For $x \in \partial K$ denote by $r(x)$ the radius of the biggest Euclidean ball contained in K that touches ∂K at x. More precisely

$$r(x) = \max\{r : x \in B(y, r) \subseteq K \quad \text{for some y} \in K\}.$$

Remark

It was shown in [S-W] that
(i)If $B \subseteq K$, then

$$\mu\{x \in \partial K : r(x) \geq \beta\} \geq (1 - \beta)^{n-1} \text{vol}_{n-1}(\partial K)$$

(ii)

$$\int_{\partial K} r(x)^{-\alpha} d\mu(x) < \infty \qquad \text{for all} \quad \alpha, \quad 0 \leq \alpha < 1$$

Lemma 6

Suppose 0 is in the interior of K. Then we have for all x with $r(x) > 0$ and for all $t \geq 0$

$$0 \leq \frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \leq g(x),$$

where $\int_{\partial K} g(x) d\mu(x) < \infty$.
$x_t = [0, x] \cap \partial K$ if $K_t \subseteq K$. x_t is the intersection of the half-line from 0 through x with ∂K_t if $K \subseteq K_t$.

Lemma 7 *Let x_t be as in Lemma 6. Then*

$$\lim_{t \to 0} \frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \text{ exists a.e.}$$

and is equal to

(i) $\frac{\rho(x)^{-\frac{n-1}{n+1}}}{n|B|}$ *if the indicatrix of Dupin at* $x \in \partial K$ *is an* $(n-1)$*-dimensional sphere with radius* $\sqrt{\rho(x)}$.

(ii) 0*, if the indicatrix of Dupin at* x *is an elliptic cylinder.*

Remark

(i) $r(x) > 0$ a.e. [S-W] and the indicatrix of Dupin exists a.e. [L2] and is an ellipsoid or an elliptic cylinder.

(ii) If the indicatrix is an ellipsoid, we can reduce this case to the case of a sphere by an affine transformation with determinant 1 (see [S-W]).

Proof of Theorem 3

We may assume that 0 is in the interior of K. By Lemma 5 and with the notations of Lemma 6 we have

$$\frac{|K|-|K_t|}{|B|-|B_t|} = \frac{1}{n}\int_{\partial K} \frac{< x, N(x) > (1-(\frac{||x_t||}{||x||})^n)}{|B|-|B_t|} d\mu(x)$$

By Lemma 6 and the Remark preceding it, the functions under the integral sign are bounded uniformly in t by an L^1-function and by Lemma 7 they are converging pointwise a.e. We apply Lebesgue's convergence theorem.

Proof of Lemma 6

Let $x \in \partial K$ such that $r(x) > 0$. We consider the proof in the case of Definition 1′ and of Definition 1 in the case where $K_t \subseteq K$ for all $t \geq 0$. The case of Definition 1 where $K \subseteq K_t$ for all $t \geq 0$ is treated in a similar way.

As $\|x\| \geq \|x_t\|$, we have for all t

$$\frac{1}{n} < x, N(x) > \left(1-(\frac{\|x_t\|}{\|x\|})^n\right) \leq \; < \frac{x}{\|x\|}, N(x) > \|x - x_t\| \tag{1}$$

Put $r(x) = r$, $x - r(x)N(x) = z$ and $< \frac{x}{\|x\|}, N(x) >= \cos\theta$.
We can assume that there is an $\alpha > 0$ such that

$$B(0,\alpha) \subseteq K \subseteq B(0, \frac{1}{\alpha}), \tag{2}$$

and hence

$$\cos\theta ||x - x_t|| \leq \frac{2}{\alpha}.$$

Let $\epsilon > 0$ be given. By Remark 2 (ii) there exists t_1 such that for all $t \leq t_1$

$$r(1 - \frac{1-f_1(t)}{r^{\frac{2n}{n+1}}}(1+\epsilon)) \leq f_r(t) \leq r(1 - \frac{1-f_1(t)}{r^{\frac{2n}{n+1}}}(1-\epsilon)). \tag{3}$$

Let t_0 be such that $Ct_0 < t_1$. By Definition 1, (i) $f_1(t)$ is decreasing in t, hence we have for all $t \geq t_0$

$$f_1(t) \leq f_1(t_0)$$

and thus for all $t \geq t_0$ with (1) and (2)

$$\frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \leq \frac{2}{\alpha |B|(1 - (f_1(t_0))^n)}.$$

Therefore the expression in question is bounded by a constant in this case and hence integrable. It remains to consider the case when $t < t_0$.

a) Suppose first that

$$\|x - x_t\| \quad < \quad r \quad \cos\theta.$$

For $B(z, r)$ we construct the corresponding inner body $(B(z, r))_s$ such that x_t is a boundary point of $(B(z, r))_s$. By Definition 1 (iii) $(B(z, r))_s$ is a Euclidean ball with center z and radius $f_r(s)$. As x_t is a boundary point of $(B(z, r))_s$,

$$f_r(s) = r(1 - \frac{2||x - x_t||\cos\theta}{r} + \frac{||x - x_t||^2}{r^2})^{1/2} \leq r(1 - \frac{||x - x_t||\cos\theta}{2r}) \qquad (4)$$

The last inequality holds by assumption a).
So far the arguments are the same for Definiton 1 and Definition 1$'$. From now on they differ slightly.
By Definition 1 (iv) $s \leq Ct$, hence by monotonicity $f_r(s) \geq f_r(Ct)$ and thus, as $Ct < t_1$, with (3)

$$f_r(Ct) \geq r(1 - (1 + \epsilon)\frac{1 - f_1(Ct)}{r^{\frac{2n}{n+1}}}),$$

which, using Definition 1 (iii) can be shown to be

$$\geq r(1 - (1 + \epsilon)(C^{\frac{2}{n+1}} + \epsilon)\frac{1 - f_1(t)}{r^{\frac{2n}{n+1}}}). \qquad (5)$$

We get from (4) and (5)

$$1 - f_1(t) \geq \frac{||x - x_t||\cos\theta \quad r^{\frac{n-1}{n+1}}}{2(1 + \epsilon)(C^{\frac{2}{n+1}} + \epsilon)}. \qquad (6)$$

Observe also that

$$|B| - |B_t| = |B|(1 - f_1^n(t)) \geq |B|(1 - f_1(t)).$$

This inequality together with (1) and (6) shows that

$$\frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \leq \frac{2(1 + \epsilon)(C^{\frac{2}{n+1}} + \epsilon)}{|B|} r^{-\frac{n-1}{n+1}}.$$

And the latter is integrable by the Remark preceding Lemma 6.

In the case of Definition $1'$ it follows from $(\mathrm{iv})'$ and $(\mathrm{v})'$ that $s \leq t$. For if not, then $s > t$, therefore by $(\mathrm{iv})'$ $(B(z,r))_s \subset \mathrm{int}(B(z,r))_t$ and by $(\mathrm{v})'$ $\mathrm{int}(B(z,r))_t \subset \mathrm{int}(K_t)$, which contradicts that $x_t \in \partial K_t \cap \partial(B(z,r))_s$. Therefore $f_r(s) \geq f_r(t)$ and thus, as $t < t_1$, with (3)

$$f_r(t) \geq r(1 - (1+\epsilon)\frac{1 - f_1(t)}{r^{\frac{2n}{n+1}}}).$$

We then conclude as above.

b) Now we consider the case when

$$\|x - x_t\| \quad \geq \quad r \quad \cos\theta.$$

We choose α so small that $x_t \notin B(0,\alpha)$. Let H be the hyperplane through 0 orthogonal to x. Then the spherical cone $C = [x, H \cap B(0,\alpha)]$ is contained in K and $x_t \in C$. Let $d = \mathrm{distance}(x_t, C)$. Then

$$d = ||x - x_t|| \frac{\alpha}{(\alpha^2 + ||x||^2)^{\frac{1}{2}}}. \tag{7}$$

Let $w \in [0, x_t]$ such that $||x_t - w|| = \frac{d}{2}$. Let $B(w,R) \subseteq K$ be the biggest Euclidean ball with center w such that $B(w,R) \subseteq K$. Then $\partial B(w,R) \cap \partial K \neq \emptyset$. Moreover $R \geq d$, which implies that $x_t \in B(w,R)$. Let $(B(w,R))_s$ be the corresponding inner ball such that $x_t \in \partial(B(w,R))_s$.
Now we have to distinguish between Definiton 1 and $1'$.
By Definition 1, (iv) $s \leq Ct$. By monotonicity $f_R(s) \geq f_R(Ct)$ which, as above, is

$$\geq R(1 - (1+\epsilon)(C^{\frac{2}{n+1}} + \epsilon)\frac{1 - f_1(t)}{R^{\frac{2n}{n+1}}}).$$

As $R \geq d$, the latter is

$$\geq d(1 - (1+\epsilon)(C^{\frac{2}{n+1}} + \epsilon)\frac{1 - f_1(t)}{d^{\frac{2n}{n+1}}}).$$

On the other hand by construction $f_R(s) = \frac{d}{2}$. Therefore

$$1 - f_1(t) \geq \frac{d^{\frac{2n}{n+1}}}{2(1+\epsilon)(C^{\frac{2}{n+1}} + \epsilon)}.$$

Note also that (2) implies that $\cos\theta \geq \alpha^2$. Hence with (1), (2), (7) and assumption b) we get that

$$\frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \leq \frac{2(1+\alpha^4)^{\frac{n}{n+1}}(1+\epsilon)(C^{\frac{2}{n+1}} + \epsilon)}{|B|\alpha^{\frac{6n-2}{n+1}}} \quad r^{-\frac{n-1}{n+1}}$$

The case of Definition 1′ is treated similarly and the above inequalities hold true with $C = 1$ and $C^{\frac{2}{n+1}} + \epsilon = 1$.

Proof of Lemma 7

We again consider the case when $K_t \subseteq K$ for all $t \geq 0$ for Definition 1. The case $K \subseteq K_t$ for all $t \geq 0$ for Definition 1 and the case of Definition 1′ are done in a similar way (compare the proof of Lemma 6).

As in the proof of Lemma 6 we can choose $\alpha > 0$ such that

$$B(0, \alpha) \subseteq K \subseteq B(0, \frac{1}{\alpha}).$$

Therefore

$$1 \geq \; < \frac{x}{\|x\|}, N(x) > \; \geq \alpha^2. \tag{8}$$

We put again $\cos\theta = < \frac{x}{\|x\|}, N(x) >$. (1) holds, that is

$$\frac{1}{n} < x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right) \leq \; < \frac{x}{\|x\|}, N(x) > \|x \; - x_t\|$$

Since x and x_t are colinear,

$$||x|| = ||x_t|| + ||x - x_t||$$

and hence

$$\frac{1}{n} < x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right) = \frac{1}{n} < x, N(x) > \left((1 - (1 - \frac{\|x - x_t\|}{\|x\|})^n\right)$$

$$\geq$$

$$< \frac{x}{\|x\|}, N(x) > \|x - x_t\| \left(1 - k_1 \cdot \frac{\|x - x_t\|}{\|x\|}\right) \tag{9}$$

for some constant k_1, if we choose t sufficiently large.

(i) Case where the indicatrix is an ellipsoid

We have seen that then we can assume that the indicatrix is a Euclidean sphere. Let $\sqrt{\rho(x)}$ be the radius of this sphere. We put $\rho(x) = \rho$ and we introduce a coordinate system such that $x = 0$ and $N(x) = (0, \dots 0, -1)$. H_0 is the tangent hyperplane to ∂K in $x = 0$ and $\{H_\alpha : \alpha \geq 0\}$ is the family of hyperplanes parallel to H_0 that have non-empty intersection with K and are of distance α from H_0. For $\alpha > 0$, H_α^+ is the half-space generated by H_α that contains $x = 0$. For $a \in \mathbf{R}$, let $z_a = (0, \dots 0, a)$ and $B_a = B(z_a, a)$ be the Euclidean ball with center z_a and radius a. As in [W], for $\varepsilon > 0$ we can choose α_0 so small that for all $\alpha \leq \alpha_0$

$$B_{\rho-\varepsilon} \cap H_\alpha^+ \subseteq K \cap H_\alpha^+ \subseteq B_{\rho+\varepsilon} \cap H_\alpha^+.$$

We choose t so small that $x_t \in \text{int}(B_{\rho-\varepsilon} \cap H_\alpha^+) (\subseteq \text{int}(B_{\rho+\varepsilon} \cap H_\alpha^+))$. For $B_{\rho+\varepsilon}$ we construct the corresponding inner body $(B_{\rho+\varepsilon})_s$ such that x_t is a boundary point of $(B_{\rho+\varepsilon})_s$. $(B_{\rho+\varepsilon})_s$ is a Euclidean ball with center $z_{\rho+\varepsilon}$ and radius $f_{\rho+\varepsilon}(s)$. We have

$$f_{\rho+\varepsilon}(s) = ((\rho+\varepsilon)^2 + \|x - x_t\|^2 - 2(\rho+\varepsilon)\|x - x_t\|\cos\theta)^{\frac{1}{2}},$$

$$\geq (\rho+\varepsilon)(1 - \frac{\|x - x_t\|\cos\theta}{\rho+\varepsilon}).$$

Definition 1, (v) implies that $s \geq (1-\varepsilon)t$, hence by monotonicity $f_{\rho+\varepsilon}(s) \leq f_{\rho+\varepsilon}((1-\varepsilon)t)$, which, for t small enough is (compare with the proof of Lemma 6)

$$\leq (\rho+\varepsilon)(1 - (1-k_2\varepsilon)\frac{1-f_1(t)}{(\rho+\varepsilon)^{\frac{2n}{n+1}}}),$$

where k_2 is a constant. Thus

$$1 - f_1(t) \leq \frac{||x - x_t||\cos\theta(\rho+\varepsilon)^{\frac{n-1}{n+1}}}{1-k_2\varepsilon}.$$

Note that

$$|B| - |B_t| = |B|(1 - f_1^n(t)) \leq n|B|(1 - f_1(t)).$$

Therefore with (9)

$$\frac{< x, N(x) > \left(1 - (\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B| - |B_t|)} \geq (1-k_2\varepsilon)(1 - k_1\frac{||x - x_t||}{||x||})\frac{(\rho+\varepsilon)^{-\frac{n-1}{n+1}}}{n|B|}.$$

This is the lower bound for the expression in question.

To get an upper bound we proceed similarily. For $B_{\rho-\varepsilon}$ we construct the corresponding inner body $(B_{\rho-\varepsilon})_s$ such that x_t is a boundary point of $(B_{\rho-\varepsilon})_s$. $(B_{\rho-\varepsilon})_s$ is a Euclidean ball with center $z_{\rho-\varepsilon}$ and radius $f_{\rho-\varepsilon}(s)$. We have

$$f_{\rho-\varepsilon}(s) = ((\rho-\varepsilon)^2 + \|x-x_t\|^2 - 2(\rho-\varepsilon)\|x-x_t\|\cos\theta)^{\frac{1}{2}},$$

$$\leq (\rho-\varepsilon)(1-\frac{\|x-x_t\|\cos\theta}{\rho-\varepsilon}(1-\frac{\|x-x_t\|}{2(\rho-\varepsilon)\cos\theta})(1+k_3\frac{\|x-x_t\|\cos\theta}{\rho-\varepsilon}(1-\frac{\|x-x_t\|}{2(\rho-\varepsilon)\cos\theta}))),$$

for some constant k_3, if t is small enough. Again by Definiton 1 (v) $s \leq (1+\varepsilon)t$ and therefore $f_{\rho-\varepsilon}(s) \geq f_{\rho-\varepsilon}((1+\varepsilon)t)$ which with arguments similar as before is

$$\geq (\rho-\varepsilon)(1-(1+k_4\varepsilon)\frac{1-f_1(t)}{(\rho-\varepsilon)^{\frac{2n}{n+1}}})$$

with a suitable constant k_4. Thus

$$1-f_1(t) \geq$$

$$\frac{||x-x_t||\cos\theta}{1+k_4\varepsilon}(1-\frac{||x-x_t||}{2(\rho-\varepsilon)\cos\theta})(1+\frac{k_3||x-x_t||\cos\theta}{\rho-\varepsilon}(1-\frac{||x-x_t||}{2(\rho-\varepsilon)\cos\theta}))(\rho-\varepsilon)^{\frac{n-1}{n+1}}. \tag{10}$$

Observe now that

$$|B|-|B_t| = |B|(1-f_1^n(t)) \geq n|B|(1-f_1(t))(1-\frac{n-1}{2}(1-f_1(t))). \tag{11}$$

We choose t so small that $1-f_1(t) < \frac{2\varepsilon}{n-1}$. This together with (1), (10) and (11) implies that

$$\frac{<x, N(x)> \left(1-(\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B|-|B_t|)}$$

$$\leq$$

$$\frac{1+k_4\varepsilon}{(1-\varepsilon)(1-\frac{||x-x_t||}{2(\rho-\varepsilon)\cos\theta})(1+k_3\frac{||x-x_t||\cos\theta}{\rho-\varepsilon}(1-\frac{||x-x_t||}{2(\rho-\varepsilon)\cos\theta}))} \quad \frac{(\rho-\varepsilon)^{-\frac{n-1}{n+1}}}{n|B|}.$$

Note that $\cos\theta \geq \alpha^2$ by (8).
This finishes the proof of Lemma 7 in the case where the indicatrix is an ellipsoid.

(ii) Case where the indicatix of Dupin is an elliptic cylinder
Recall that then we have to show that

$$\lim_{t\to 0}\frac{< x, N(x) > \left(1-(\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B|-|B_t|)}=0.$$

We can again assume (see [S-W]) that the indicatrix is a spherical cylinder i.e. the product of a k-dimensional plane and a $n-k-1$ dimensional Euclidean sphere of radius ρ. We can moreover assume that ρ is arbitrarily large (see also [S-W]).

By Lemma 9 of [S-W] we then have for sufficiently small α and some $\varepsilon > 0$

$$B_{\rho-\varepsilon}\cap H_\alpha^+ \subseteq K \cap H_\alpha^+.$$

Using similar methods, this implies that

$$\lim_{t\to 0}\frac{< x, N(x) > \left(1-(\frac{\|x_t\|}{\|x\|})^n\right)}{n(|B|-|B_t|)}=0.$$

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