The illumination body of almost polygonal bodies

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The affine surface area was introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficently smooth boundary and by Leichtweiss [L1] for convex bodies in $\mathbb{R}^n$ with sufficently smooth boundary. As it occurs naturally in many important questions so for example in the approximation of convex bodies by polytopes (see the survey article of Gruber [G]) or in a priori estimates for PDEs [Lu2] one wanted to have extensions of the affine surface area to arbitrary convex bodies in $\mathbb{R}^n$ without any smoothness assumptions of the boundary.

Such extensions were given in recent years by Leichtweiss [L2], Lutwak[Lu1], Schuett and Werner [S-W1] and Werner [W1]. In many of these extensions the affine surface area is defined using the volume differences $\text{vol}_n(K)$ - $\text{vol}_n(K_\delta)$ respectively $\text{vol}_n(K^\delta)$ - $\text{vol}_n(K)$ where K_δ is a floating body (see [L1] and [S-W1] for the definition) and $K^\delta$is an illumination body.

Therefore one is interested in the precise behaviour of such volume differences.

We give here the analogue for illumination bodies to results obtained in [S-W2] for floating bodies; namely we investigate what kind of functions can occur as the difference volume $\text{vol}_2(K^\delta)$ - $\text{vol}_2(K)$ for convex bodies in $\mathbb{R}^2$. The result and the methods of proof are similar to [S-W2].

Let K be a convex body in $\mathbb{R}^n$ and $\delta > 0$ be given. For $x \in \mathbb{R}^n \setminus K$ co$[x, K]$ denotes the convex hull of x and K. Then the convex body

$$K^\delta = \{x \in \mathbb{R}^n : \text{vol}_n(co[x, K]) \leq \delta\}$$

is called an illumination body.

If K has smooth boundary then we know that $\lim_{\delta \to 0} \text{vol}_n(K^\delta)$ - $\text{vol}_n(K)$ is a positive real number [W1] and if K is a polytope then $\text{vol}_n(K^\delta)$ - $\text{vol}_n(K)$ is a polynomial in δ of degree n (see [W2]). Here we show

THEOREM 1

Let h be a positive, 2-times continuously (on (0,1]) differentiable function with h(0)=0. Suppose $\frac{h(t^3)}{t^2}$ is strictly decreasing and strictly convex on (0,1] and $\lim_{t \to 0} \frac{\frac{h(t^3)}{t^2}}{lnt} = -\infty$. Then there are a constant c, a $\delta_0 > 0$ and a convex

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body K such that we have for all $\delta, 0 < \delta < \delta_0$

$$\frac{1}{c}h(\delta) \le \mathrm{vol}_2(K^\delta) - \mathrm{vol}_2(K) \le ch(\delta)$$

As in [S-W2] a "polygonal" body K(a) associated to a sequence $a = (a_n)_{n=0}^{\infty}$ of positive real numbers satisfies the statement of Theorem 1. The strategy of the proof is as follows.

For such a body K(a) we estimate $\mathrm{vol}_2(K(a)^\delta)$ - $\mathrm{vol}_2(K(a))$ in Proposition 2 below and find that this expression is proportional to

$$h(\delta) = \delta^{2/3} a_{n(\delta)} + n(\delta)\delta + \delta^2 \sum_{n=0}^{n(\delta)} \frac{1}{(a_n - a_{n+1})^3}$$

where

$$\delta^{1/3} \sim a_{n(\delta)} - a_{n(\delta)+1}$$

Actually we will show in Corollary 3 that for our purposes the above expression can be simplified to

$$\delta^{2/3} a_{n(\delta)} + n(\delta)\delta$$

as we are only interested in the order of magnitude.

It is because of this simplification that the conditions in Theorem 1 are somewhat nicer than the corresponding one's in [S W]. We then solve a 'continuous' version of the problem:

we interpret the sequence $a = (a_n)_{n=0}^{\infty}$ as a convex, differentiable function a(n) on the positive the real line and the differences $a_n - a_{n-1}$ as the derivatives $a'(n)$.Thus we have to solve

$$h((-a')^3) = a'^2 a + n(-a')^3$$

We then go back to the 'discrete'version.

Let us start by giving a precise description of K(a).

Let $a = (a_n)_{n=0}^{\infty}$ be a sequence of positive real numbers with the following properties

$$(1) \quad 1 = a_0 \ge ... \ge a_n \ge a_{n+1} \ge ... \ge 0 \quad \text{for all} \quad n = 1, 2, ...$$

$$(2) \quad a_{n-1} - a_n \ge a_n - a_{n+1} \quad \text{for all} \quad n = 1, 2, ...$$

$$(3) \quad lim_{n \to \infty} a_n = 0.$$

Then K(a) is obtained as the convex hull of the points

$$(4) \quad (\pm a_n, \pm(a_n^2 - a_0^2)), \quad n = 1, 2, ...$$

By $n(\delta)$ we denote the smallest integer so that

$$(5) \quad a_{n(\delta)} - a_{n(\delta)+1} \leq \delta^{1/3}.$$

Then we have

PROPOSITION 2

Let a be a sequence of positive real numbers satisfying (1),(2) and (3) and let K(a) be the associated convex body (4). Then there are a constant c and δ_0 so that we have for all $\delta, 0 < \delta < \delta_0$

$$\frac{1}{c}(a_{n(\delta)}\delta^{2/3} + n(\delta)\delta + \delta^2 \sum_{n=0}^{n(\delta)-1} \frac{1}{(a_n - a_{n+1})(a_n - a_{n+2})(a_{n+1} - a_{n+2})})$$

$$\leq \mathrm{vol}_2(K(a)^\delta) - \mathrm{vol}_2(K(a)) \leq$$

$$c(a_{n(\delta)}\delta^{2/3} + n(\delta)\delta + \delta^2 \sum_{n=0}^{n(\delta)-1} \frac{1}{(a_n - a_{n+1})(a_n - a_{n+2})(a_{n+1} - a_{n+2})})$$

Proof of Proposition 2

To simplify the computations we take the shifted copy of K(a) that has center of symmetry at (0, a_0^2). By symmetry it is enough to consider the points $P_n = (a_n, a_n^2)$, n=0,1,2,...

Let $\delta < \delta_0$ be given with δ_0 small enough.

Let $n(\delta)$ be the smallest natural number such that

$$\frac{1}{2}\frac{(a_{n(\delta)} - a_{n(\delta)+1})^2(a_{n(\delta)-1} - a_{n(\delta)+1})(a_{n(\delta)} - a_{n(\delta)+2})}{(a_{n(\delta)} - a_{n(\delta)+1}) + (a_{n(\delta)-1} - a_{n(\delta)+2})} \geq \delta.$$

If $n \leq n(\delta)$ $\quad K(a)^\delta$ behaves as if K(a) is a polygone and if $n \geq n(\delta)$ $\quad K(a)^\delta$ behaves as if the boundary of K(a) is smooth. Consequently the boundary of the shifted $K(a)^\delta$ can be described as follows (below the line $y = a_0^2$ -which by symmetry is all we need):

within the range where K(a) behaves like a polygone it is the convex hull of the points

$$(\overline{x_0}, a_0^2) \quad \text{with} \quad \overline{x_0} = \frac{x_0(a_0 - a_2) + 2a_0^2 + a_0a_1 + a_1a_2}{2(a_0 + a_1)}$$

$$\overline{Q_0} = (\overline{x_0}, \overline{y_0}) \quad \text{with} \quad \overline{y_0} = -\frac{1}{2}(x_0 - a_1)(a_0 - a_2) + a_0^2$$

and for $n = 0, 1, ..., n(\delta) - 1$

$$Q_n = (x_n, y_n) \quad \text{with} \quad \begin{array}{c} x_n = a_{n+1} + \frac{2\delta}{(a_n - a_{n+2})(a_n - a_{n+1})} \\ y_n = (a_{n+1} + a_{n+2})x_n - a_{n+1}a_{n+2}. \end{array}$$

and for $n = 1, 2, ..., n(\delta)$

$$\overline{Q_n} = (\overline{x_n}, \overline{y_n}) \quad \text{with} \quad \begin{array}{c} \overline{x_n} = \frac{a_{n-1}a_n - (a_n - a_{n+2})x_n - a_{n+1}a_{n+2}}{(a_{n-1} - a_{n+1})} \\ \overline{y_n} = (a_n + a_{n+1})(\overline{x_n} - x_n) + y_n \end{array} .$$

This holds as

$$vol_2(co[Q_n, P_n, P_{n+1}]) = \delta, \quad n = 0, 1, ..., n(\delta) - 1$$

and

$$vol_2(co[\overline{Q_n}, P_n, P_{n+1}]) = \delta, \quad n = 1, 2, ..., n(\delta)$$

Notice that the point Q_n is on the line determined by the points P_{n+1} and P_{n+2} and $\overline{Q_n}$ is on the line through P_{n-1} and P_n. Moreover the line through Q_n and $\overline{Q_n}$ is parallel to the line through P_n and P_{n+1}.

Therefore in the "polygonal range" we get as a contribution to the volume difference $\mathrm{vol}_2(K(a)^\delta) - \mathrm{vol}_2(K(a))$ the area of the triangles spanned by the points $Q_n, \overline{Q_{n+1}}$ and P_{n+1},$n = 0, 1, ..., n(\delta) - 1$ which is

$$\frac{2\delta^2}{(a_n - a_{n+1})(a_n - a_{n+2})(a_{n+1} - a_{n+2})}$$

and the area of the trapezoid spanned by the points $Q_n, \overline{Q_n}, P_n$ and P_{n+1}, $n = 1, ..., n(\delta) - 1$ which is

$$2\delta - \frac{2\delta^2}{(a_n - a_{n+1})^2} \quad (\frac{1}{(a_{n-1} - a_{n+1})} + \frac{1}{(a_n - a_{n+2})}).$$

In addition we get

$$\mathrm{vol}_2(co[P_0, P_1, Q_0, \overline{Q_0}, (\overline{x_0}, a_0^2)]) =$$

$$2\delta - \frac{2 \quad \delta^2(a_1 + a_2)}{(a_0 - a_1)^2(a_0 - a_2)^2} - \frac{1}{2}\frac{\delta^2}{(a_0 + a_1)(a_0 - a_1)^2}.$$

Therefore the "polygonal" contribution is

$$8(n(\delta) - 1)\delta + 8\delta^2 \sum_{n=0}^{n(\delta)-1} \frac{1}{(a_n - a_{n+1})(a_n - a_{n+2})(a_{n+1} - a_{n+2})}$$

$$-8\delta^2 \sum_{n=1}^{n(\delta)-1} \frac{1}{(a_n - a_{n+1})^2}(\frac{1}{(a_{n-1} - a_{n+1})} + \frac{1}{(a_n - a_{n+2})})$$

$$-\frac{4 \quad \delta^2(a_1 + a_2)}{(a_0 - a_1)^2(a_0 - a_2)^2} - \frac{\delta^2}{(a_0 + a_1)(a_0 - a_1)^2}.$$

Within the range where K(a) behaves as if it has smooth boundary, the boundary of $K(a)^\delta$ is descibed by the curve $k^\delta(x)$ where

$$k^\delta(x) \geq x^2 - \frac{1}{4}(12\delta)^{2/3}, \quad 0 \leq x \leq \overline{x_{n(\delta)}}$$

and

$$k^{\delta}(x) \leq \begin{matrix} x^2 - \frac{1}{4}(12\delta)^{2/3}, \quad 0 \leq x \leq \ a_{n(\delta)+1} \\ \frac{\overline{y_{n(\delta)}} - a_{n(\delta)}^2 + \frac{1}{4}(12\delta)^{2/3}}{\overline{x_{n(\delta)}} - a_{n(\delta)}} x + b, \quad a_{n(\delta)+1} \leq x \leq \overline{x_{n(\delta)}} \end{matrix}$$

where $b = \overline{y_{n(\delta)}} - \overline{x_{n(\delta)}} \frac{\overline{y_{n(\delta)}} - a_{n(\delta)}^2 + \frac{1}{4}(12\delta)^{2/3}}{\overline{x_{n(\delta)}} - a_{n(\delta)}}$.

Hence in the "smooth range" the contribution to the volume difference $\text{vol}_2(K(a)^{\delta}) - \text{vol}_2(K(a))$ is

$$\leq \frac{a_{n(\delta)}}{4}(12\delta)^{2/3}$$

and

$$\geq \frac{a_{n(\delta)+1}}{4}(12\delta)^{2/3} + \frac{1}{4}(12\delta)^{2/3} \frac{\delta}{(a_{n(\delta)} - a_{n(\delta)+1})(a_{n(\delta)} - a_{n(\delta)+2})}.$$

Taking into account the additional $\delta \sim \text{vol}_2(co[P_{n(\delta)}, P_{n(\delta)+1}, Q_{n(\delta)}])$ we get with a suitably chosen constant c the statement of the proposition.

COROLLARY 3

There is a constant $c\ >$ such that

$$\frac{1}{c}(a_{n(\delta)}\delta^{2/3} + n(\delta)\delta) \leq \text{vol}_2(K(a)^{\delta}) - \text{vol}_2(K(a)) \leq c(a_{n(\delta)}\delta^{2/3} + n(\delta)\delta)$$

Proof

Next we give three examples that show the range of functions that can occur within this construction. The examples and the computations are the same as the ones given in [S-W2]. Therefore we give here only the results and refer to [S-W2] for the details.

If $a_n = (n+1)^{-\beta}$, $0 < \beta < \infty$, then

$$\text{vol}_2(K(a)^{\delta}) - \text{vol}_2(K(a)) \sim \delta^{\frac{3\beta+2}{3\beta+3}}.$$

If $a_n = e^{-n}$, $n = 0, 1, 2, ...$, then

$$\text{vol}_2(K(a)^{\delta}) - \text{vol}_2(K(a)) \sim \delta + \delta ln\frac{1}{\delta}.$$

If $a_n = \frac{1}{lnln(n+30)}$, $n = 1, 2, ...$, then

$$\text{vol}_2(K(a)^{\delta}) - \text{vol}_2(K(a)) \sim \frac{\delta^{2/3}}{ln\frac{1}{\delta}(lnln\frac{1}{\delta})^2}.$$

Now we come to the proof of Theorem 1. Hence assume that the function h satiesfies the conditions of Theorem 1 and let

$$g(t) = 3h'(t^3) - 6\frac{h(t^3)}{t^3}.$$

Suppose that

$$(6) \quad \text{tg(t) is decreasing and strictly convex on (0,1]}$$

and that

$$(7) \quad lim_{t\to 0}\frac{g(t)}{lnt-1}=\infty.$$

First we observe

LEMMA 3
(i) If $lim_{t\to 0}\frac{h(t^3)}{t^2}=0$, then $lim_{t\to 0}tg(t)=0$ and $lim_{t\to 0}t^2g'(t)=0$

(ii) If $\lim_{t\to 0}\frac{g(t)}{lnt-1}=\infty$, then $\lim_{t\to 0}(tg(t))'=-\infty$.
Proof
By L'Hospital's rule we have

$$0=lim_{t\to 0}\frac{h(t^3)}{t^2}=lim_{t\to 0}\frac{3t^2h'(t^3)}{2t}=lim_{t\to 0}\frac{3th'(t^3)}{2}$$

and

$$0=lim_{t\to 0}\frac{h'(t^3)}{t^{-1}}=lim_{t\to 0}t^4h''(t^3).$$

This proves (i) as $g'(t)=9t^2h''(t^3)-18\frac{h'(t^3)}{t}+18\frac{h(t^3)}{t^4}$.
To prove (ii), notice that again by L'Hospital and (i)

$$\infty=\lim_{t\to 0}\frac{g(t)}{lnt-1}=\lim_{t\to 0}\frac{tg(t)}{tlnt-t}=$$

$$=\lim_{t\to 0}\frac{(tg(t))'}{lnt}.$$

Consequently we have $\lim_{t\to 0}(tg(t))'=-\infty$.

To prove Theorem 1 we distinguish two cases. In the first case we have

$$lim_{t\to 0}\frac{h(t^3)}{t^2}=c>0.$$

This means that $h(t)\sim ct^{2/3}$ for small t. In this case we may take the Euclidean ball as the convex body K and we have

$$\text{vol}_2(K^\delta)-\text{vol}_2(K)\sim t^{2/3}.$$

In the second case we have

$$lim_{t\to 0}\frac{h(t^3)}{t^2}=0.$$

We will show that in this case a body K(a) satisfies Theorem 1. The sequence a will be obtained as follows. We define

$$(8) \quad A(t) = \frac{1}{t^2}\int_0^t \frac{1}{s}(\int_0^s g'(u)u^3 du)ds$$

A(t) is a solution of the differential equation

$$(9) \quad \frac{d^2A}{dt^2} + \frac{5}{t}\frac{dA}{dt} + 4\frac{A}{t^2} = g'(t)$$

We put

$$(10) \quad n(t) = \frac{1}{9}(\frac{dA}{dt} - 4\frac{A}{t} - g(t))$$

and then define

$$(11) \quad a(n(t)) = A(t) \quad \text{for t} \in (0,1]$$

We also need the following lemmas.

LEMMA 4 Suppose that assumption (6) holds. Then
(i) $-t\frac{dn}{dt} = \frac{dA}{dt}$ for t $\in (0,1]$

(ii) $A(t)$ is well defined, $A(t) > 0$ on (0,1] and $lim_{t\to 0}A(t) = 0$.

(iii) $A'(t) > 0$ on (0,1]

(iv) $n(t) > 0$ on (0,1]

Proof
(i) By (10),

$$\frac{dn}{dt} = \frac{1}{9}(\frac{d^2A}{dt^2} - \frac{4}{t}\frac{dA}{dt} + 4\frac{A}{t^2} - g')$$

which implies

$$-t\frac{dn}{dt} = \frac{dA}{dt}$$

by (9).

(ii) Note first that $(tg)'' > 0$ is equivalent to $(t^2g')' > 0$.
Therefore we have for all s, $0 < s \leq 1$

$$0 < \int_0^s (u^2g')'du = s^2g' - lim_{u\to 0}u^2g' = s^2g',$$

by Lemma 3, (i). Consequently $A(t) = \frac{1}{t^2}\int_0^t \frac{1}{s}(\int_0^s u^3g'du)ds > 0$, provided it exists which we show next.

As s^2g' is strictly increasing on (0,1], we have

$$A(t) = \frac{1}{t^2}\int_0^t \frac{1}{s}(\int_0^s u^2g'udu)ds$$

$$\leq \frac{1}{t^2}\int_0^t \frac{1}{s}s^2 g' \frac{s^3}{2} ds$$

$$\leq \frac{1}{4}t^2 g',$$

again as $s^2 g'$ is strictly increasing. Therefore A(t) is well defined and $\lim_{t\to 0} A(t) = 0$ by Lemma 3,(i).

(iii)

$$A'(t) = -\frac{2}{t}A(t) + \frac{1}{t^3}\int_0^t s^3 g' ds.$$

Consequently $A'(t) >\ 0$ if and only if

$$\int_0^t (s^3 g' - \frac{2}{s}(\int_0^s u^3 g' du))ds \ >\ 0.$$

This holds if and only if

$$\frac{2}{s}(\int_0^s u^3 g' du)\ <\ s^3 g'$$

which holds as $g'u^2$ is stricly increasing.

(iv)

$$(12)\quad 9n(t) = \frac{dA}{dt} - 4\frac{A}{t} - g = 3A' - \frac{2}{t^3}\int_0^t g'u^3 du - g.$$

Hence, as by (iii) $A'(t) > 0$, to show that $n(t) > 0$ it is enough to show that

$$0 \geq \frac{2}{t^3}\int_0^t g'u^3 du + g = 3g - \frac{6}{t^3}\int_0^t gu^2 du,$$

which holds as tg is decreasing.

LEMMA 5
(i)$\lim_{t\to 0} n(t) = \infty$

(ii) The function a defined by a(n(t)) = A(t) is well defined on $[n(1),\infty)$ such that $a > 0$, $a' < 0$, $a'' > 0$ and $lim_{n\to\infty} a(n) = 0$. Moreover there is a constant $c \in$ such that

$$(*) h((-a')^3) = (-a')^2 a + 3n(-a')^3 - 3(-a')^6 \int_0^n \frac{dk}{(-a'(k))^3} + \quad c(-a')^6$$

(iii) There is a constant $c > 0$ such that we have for all n

$$c(-a'(n-1)) \leq -a'(n)$$

Proof

(i) As $A' \geq 0$, it is by (12) enough to show that

$$-\infty = lim_{t\to 0}\frac{2\int_0^t g's^3ds + gt^3}{t^3}.$$

Using de L'Hospital this is equivalent to showing that $-\infty = lim_{t\to 0}(tg)'$, which holds by Lemma 3, (ii).

(ii) $\frac{dn}{dt} = -\frac{1}{7t}\frac{dA}{dt}$ and by Lemma 4 (iii) $\frac{dA}{dt} > 0$. Therefore $\frac{dn}{dt} < 0$, hence n(t) is injective and hence a is well defined on n((0,1]). $a > 0$ as $A > 0$. Since $lim_{t\to\infty} n(t) = \infty$, a is defined on $n((0,1]) = [n(1),\infty)$ and $lim_{n\to\infty} a(n) = lim_{t\to 0} a(n(t)) = lim_{t\to 0} A(t) = 0$. As $\frac{1}{7}A(t) = a(n(t))$, we get $\frac{d^2a}{dn^2} = -(\frac{dn}{dt})^{-1} > 0$.

Now we check equation (*). First we divide by $(-a')^6$ to obtain

$$c + \frac{h((-a')^3)}{(-a')^6} = \frac{3a}{(-a')^4} + \frac{n}{(-a')^3} + \int_0^n \frac{dk}{(-a'(k))^3}.$$

Then we differentiate with respect to n and get after multiplying through with $\frac{(-a')^4}{(-a'')}$

$$3h'((-a')^3) - 6\frac{h((-a')^3)}{(-a')^3} = \frac{-a'}{a''} - \frac{12a}{(-a')} - 3n.$$

As $\frac{da}{dn} = -t$, we get

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