Geometry of sets of quantum maps: a generic positive map acting on a high-dimensional system is not completely positive

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Abstract

We investigate the set a) of positive, trace preserving maps acting on density matrices of size N, and a sequence of its nested subsets: the sets of maps which are b) decomposable, c) completely positive, d) extended by identity impose positive partial transpose and e) are superpositive. Working with the Hilbert-Schmidt (Euclidean) measure we derive tight explicit two-sided bounds for the volumes of all five sets. A sample consequence is the fact that, as N increases, a generic positive map becomes not decomposable and, *a fortiori*, not completely positive. Due to the Jamiołkowski isomorphism, the results obtained for quantum maps are closely connected to similar relations between the volume of the set of quantum states and the volumes of its subsets (such as states with positive partial transpose or separable states) or supersets. Our approach depends on systematic use of duality to derive *quantitative* estimates, and on various tools of classical convexity, high-dimensional probability and geometry of Banach spaces, some of which are not standard.

1 Introduction

Processing of quantum information takes place in physical laboratories, but it may be conveniently described in a finite dimensional Hilbert space. The standard set of tools of a quantum mechanician includes density operators which represent physical states. A density operator ρ is Hermitian, positive semi-definite and normalized. The set of density operators of "size" 2 is equivalent, with respect to the Hilbert-Schmidt (Euclidean) geometry, to a three ball, usually called the Bloch ball. The set of density operators of "size" N forms an N^2-1-dimensional convex body which naturally embeds into $\mathcal{M}_N$, the space of $N \times N$ (complex) matrices.

The interesting geometry of these non-trivial, high–dimensional sets attracts a lot of recent attention [1, 2, 3, 4, 5]. In particular one computed their Euclidean volume and hyper-area of their surface [6], and investigated properties of its boundary [7].

If the dimension N of the Hilbert space $\mathcal{H}_N$ is a composite number, the density operator can describe a state of a bipartite system. If such a state has the tensor product structure, $\rho = \rho_A \otimes \rho_B$, then it represents uncorrelated subsystems. In general, following [8], a state is called *separable* if it can be written as a convex combination of product states. In the opposite case the state is called *entangled* and it is valuable for quantum information processing [9], since it may display non–classical correlations.

The set $\mathcal{M}_N^{\mathrm{sep}}$ of separable states forms a convex subset of positive volume of the entire set of states, which we will denote by $\mathcal{M}_N^{\mathrm{tot}}$ [10]. Some estimations of the relative size of the set of separable states were obtained in [11, 12, 13, 14, 15, 16, 17], while its geometry was analyzed in [18, 19, 20, 21]. Similar issues for infinite-dimensional systems were studied in [22].

Quantum information processing is inevitably related with dynamical changes of the physical system. Transformations that are discrete in time can be described by linear *quantum maps*, or *super-operators*, $\Phi : \mathcal{M}_N \to \mathcal{M}_N$ (or, more generally, $\Phi : \mathcal{M}_K \to \mathcal{M}_N$). A map is called *positive* (or *positivity-preserving*) if any positive (semi-definite) operator is mapped into a positive operator. A map Φ called *completely positive* (CP) if the extended map $\Phi \otimes \mathcal{I}_k$ is positive for any size k of the extension. Here $\mathcal{I}_k$ is the identity map on $\mathcal{M}_k$. We will denote the *cones* of positive and completely positive maps (on $\mathcal{M}_N$) by $\mathcal{P}_N$ and $\mathcal{CP}_N$ respectively, or simply by $\mathcal{P}$ and $\mathcal{CP}$ if the size of the system is fixed or clear from the context.

Conservation of probability in physical processes imposes the *trace preserving* (TP) property: $\operatorname{Tr} \Phi(\rho) = \operatorname{Tr} \rho$. It is a widely accepted paradigm that any physical process may be described by a *quantum operation*: a completely positive, trace preserving map. (In the context of quantum communication, quantum operations are usually called *quantum channels*.)

The set $\mathcal{CP}_N^{\mathrm{TP}}$ of quantum operations, which act on density operators of size N, forms a convex set of dimension $N^4 - N^2$. Due to Jamiołkowski isomorphism [23, 24] the set

$N^{-1}\mathcal{CP}_N^{\mathrm{TP}}$ can be considered as a subset of the (N^4-1)–dimensional set $\mathcal{M}_{N^2}^{\mathrm{tot}}$ of density operators acting on an extended Hilbert space, $\mathcal{H}_N \otimes \mathcal{H}_N$. This useful fact contributes to our understanding of properties the set of quantum operations, but its geometry is nontrivial even in the simplest case of $N=2$ [25, 26].

The main aim of the present work is to derive tight two-sided bounds for the Hilbert–Schmidt (Euclidean) volume of the set $\mathcal{CP}_N^{\mathrm{TP}}$ of quantum operations acting on density operators of size N and analogous estimates for the volume of the sets $\mathcal{P}_N^{\mathrm{TP}}$ of positive trace preserving maps, and of similar subsets of the *superpositive* cone $\mathcal{SP}_N$ (see (14) and/or [27]) or the cone $\mathcal{D}_N$ of *decomposable* maps (see (18)) etc. We show that, for large N, some subsets cover only a very small fraction of its immediate superset, while in some other cases the gap between volumes is relatively small. These bounds are related to (and indeed derived from, making use of the Jamiołkowski isomorphism) analogous relations between the volumes of various subsets of the set of quantum states such as those consisting of separable states or of states with *positive partial transpose* (PPT) (see the paragraph following (16)) and their dual objects. Our methods are quite general and allow to produce tight two-sided estimates for many other sets of quantum states or of quantum maps.

The paper is organized as follows. In the next section we introduce some necessary definitions involving the set of trace preserving positive maps and its relevant subsets or supersets, which will allow us to present an overview of the results obtained in this paper (summarized in Tables 2-4). Section 3 contains more definitions and various preliminary results. Most of those results are not new, but many of them are not well-known in the quantum information theory community. In section 4 we state precise versions of our results and outline their proofs. Some details of the proofs and technical results (from all sections) are relegated to Appendices.

2 Positive and trace preserving maps: notation and overview of results

2.1 Cones of maps and matrices

Let $\Phi : \mathcal{M}_N \to \mathcal{M}_N$ be a linear quantum map, or a super-operator. More general maps $\Phi : \mathcal{M}_K \to \mathcal{M}_N$ may also be considered and analyzed by essentially the same methods, but we choose to focus on the case $K = N$ to limit proliferation of parameters.

Let $\rho \in \mathcal{M}_N$; the transformation $\rho' = \Phi(\rho)$ can be described by

$$\rho'_{n\nu} \;=\; \Phi_{\substack{n\nu \\ m\mu}}\, \rho_{m\mu}\,, \tag{1}$$

where we use the usual Einstein summation convention. The pair of upper indices ${}^{n\nu}$ defines its "row," while the lower indices ${}_{m\mu}$ determine the "column." This agrees with the usual linear algebra convention of representing linear maps as matrices. The relevant basis of $\mathcal{M}_N$ is here $E_{ij} := |e_i\rangle\langle e_j|$, $i, j = 1, \ldots, N$, where $(e_i)_{i=1}^N$ is an orthonormal basis of $\mathcal{H}_N$ (which can be identified with $\mathbb{C}^N$), and the ${}_{m\mu}$-th "column" of $\Phi_{\substack{n\nu \\ m\mu}}$, i.e., the $N \times N$ matrix $\big(\Phi_{\substack{n\nu \\ m\mu}}\big)_{n,\nu=1}^N$, is indeed $\Phi(E_{m\mu}) = \Phi_{\substack{n\nu \\ m\mu}} E_{n\nu}$.

By appropriately reshuffling elements of $\Phi_{\substack{n\nu \\ m\mu}}$ we obtain another matricial representation of a quantum map, the dynamical matrix D_Φ [28], sometimes also called in the literature "the Choi matrix" of Φ. The dynamical matrix is obtained as follows

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{n\nu \\ m\mu}}\,. \tag{2}$$

An alternative (and useful) description of the dynamical matrix is as follows

$$D_\Phi := \big(\mathcal{I}_N \otimes \Phi\big)\rho_{\max} = \sum_{m,\mu=1}^{N} E_{m\mu} \otimes \Phi(E_{m\mu}),$$

where $\rho_{\max} = |\xi\rangle\langle\xi|$, with $|\xi\rangle = \sum_{m=1}^N e_m \otimes e_m$, is a maximally entangled pure state on $\mathcal{H}_N \otimes \mathcal{H}_N$.

We point out that the order of indices of the matrix D in (2) is different than in the previous work [24, 26]. (The reason for this change will be elucidated in the next paragraph.) Note that in the present notation the operation of "reshuffling," which converts matrix Φ into D, corresponds to a "cyclic shift" of the four indices.

It is sometimes convenient to arrange the row and column indices of D_Φ (^{mn} and ${}_{\mu\nu}$ respectively) in the lexicographic order, thus obtaining a standard "flat" $N^2 \times N^2$ matrix with a natural block structure: the leading indices ${}^{m}_{\mu}$ indicate the position of the block and the second pair of indices ${}_{\nu}^{n}$ refers to the position of the entry within a block. In other words, the ${}_{m\mu}$-th block of D_Φ is $\Phi(E_{m\mu})$ or

$$D_\Phi = (\Phi(E_{m\mu}))_{m,\mu=1}^{N}, \tag{3}$$

an $N \times N$ block matrix with each block belonging to $\mathcal{M}_N$.

If a super-operator Φ belongs to the positive cone $\mathcal{P}$ (i.e., Φ is positivity-preserving), then it also maps Hermitian matrices to Hermitian matrices. This in turn is equivalent to Φ commuting with complex conjugation †; in what follows we will generally consider only maps with this property. It is easy to check that Hermiticity-preserving is equivalent to the following relation (which has no obvious interpretation)

$$\Phi_{\substack{n\nu \\ m\mu}} = \overline{\Phi_{\substack{\nu n \\ \mu m}}}. \tag{4}$$

However, expressing condition (4) in terms of the dynamical matrix we obtain

$$D_{\substack{mn \\ \mu\nu}} = \overline{D_{\substack{\mu\nu \\ mn}}} \ ,$$

which just means that D_Φ is Hermitian. Thus one may describe linear Hermiticity-preserving maps on $\mathcal{M}_N$ via Hermitian dynamical $N^2 \times N^2$ matrices. The property of being positive can be characterized just as elegantly. A theorem of Jamiołkowski [23] states that a map Φ is positive, $\Phi \in \mathcal{P}$, if and only if the corresponding dynamical matrix D_Φ is *block positive*. [A (square) block matrix (M_{ij}) (say, with $M_{ij} \in \mathcal{M}_N$ for all i, j) is said to be block positive iff, for every sequence of complex scalars $\xi = (\xi_j)$, the $N \times N$ matrix $\sum_{i,j} M_{ij}\bar{\xi}_i\xi_j$ is positive semi-definite.]

Arguably the most useful upshot of the dynamical matrix point of view arises in the study of CP maps. A theorem of Choi [29] states that a map Φ is completely positive, $\Phi \in \mathcal{CP}$, iff D_Φ is positive semi-definite. Therefore, to each CP map on $\mathcal{M}_N$ corresponds an $N^2 \times N^2$ (positive semi-definite) matrix, and *vice versa*. In particular, the rescaled dynamical matrix D associated with a (non-zero) CP map represents a state of a bi–partite system, $\sigma := D/\mathrm{Tr}\, D \in \mathcal{M}_{N^2}^{\mathrm{tot}}$ – see e.g. [23, 24], an element of the *base* of the positive semi-definite cone obtained by intersecting that cone with the hyperplane of trace one matrices.

If the dimension of the cones or other sets under consideration is relevant, we will explicitly use a lower index, writing, e.g., $\mathcal{CP}_2$ for the set of one–qubit completely positive maps.

2.2 Trace preserving maps

The trace preserving property, $\mathrm{Tr}\,\Phi(\rho) = \mathrm{Tr}\rho$, is equivalent to a condition for the *partial trace* of the dynamical matrix

$$\sum_n D_{\substack{mn \\ \mu n}} = \delta_{m\mu}\,, \quad \text{or} \quad \mathrm{Tr}_B D = I_A\,. \tag{5}$$

Therefore the compact set $\mathcal{CP}_N^{\mathrm{TP}}$ of quantum operations may be defined as a common part of the affine plane representing the condition (5) and the cone of positive semi-definite dynamical matrices - see Figure 1 in section 3.

In (5) and (occasionally) in what follows we use the labels A, B to distinguish between the space on which the original state ρ acts, namely $\mathcal{H}_A$, and the space of $\Phi(\rho)$, denoted $\mathcal{H}_B$. In particular, I_A stands for the identity operator on $\mathcal{H}_A$. Since such conventions are somewhat arbitrary (as was the ordering of indices of D), some care needs to be exercised when comparing (5) and similar formulae with other texts (such as, e.g., [26]).

2.3 Bases of cones

Let $H^0 = \{M \in \mathcal{M}_d : \operatorname{Tr} M = 0\}$. Next, let $H^{\mathrm{b}} = \{M \in \mathcal{M}_d : \operatorname{Tr} M = d^{1/2}\}$ and let $H^+ = \{M \in \mathcal{M}_d : \operatorname{Tr} M \geq 0\}$. If $\mathcal{C} \subset \mathcal{M}_d$ is a cone, we will denote by $\mathcal{C}^{\mathrm{b}} := \mathcal{C} \cap H^{\mathrm{b}}$ the corresponding *base* of $\mathcal{C}$. (This definition makes good sense if $\mathcal{C} \subset H^+$ or, equivalently, if the $d \times d$ identity matrix I_d belongs to the dual cone $\mathcal{C}^*$ (see (12)). In this case the cones generated by $\mathcal{C}^{\mathrm{b}}$ and $\mathcal{C}$ coincide, perhaps after passing to closures.) We will use the same notation for the sets of quantum maps corresponding to matrices via the Choi-Jamiołkowski isomorphism. Thus, for example, $\Phi : \mathcal{M}_N \to \mathcal{M}_N$ belongs to H^{b} iff $\operatorname{Tr} D_\Phi = \operatorname{Tr} \Phi(I_N) = N$. (Here the identity matrix I_N and its image $\Phi(I_N)$ are $N \times N$ matrices, while D_Φ is a $d \times d$ matrix, with $d = N^2$; in particular the two trace operations take place in different dimensions.) Then $\mathcal{P} \cap H^{\mathrm{b}} = \mathcal{P}^{\mathrm{b}}$ is a base of the cone $\mathcal{P}$, $\mathcal{CP} \cap H^{\mathrm{b}} = \mathcal{CP}^{\mathrm{b}}$ is a base of the cone $\mathcal{CP}$, and similarly for other cones that will be introduced later. The (real) dimension of the bases is $N^4 - 1$.

The reason behind our somewhat non-standard normalization $\operatorname{Tr} M = d^{1/2}$ is twofold. First, the condition can be rewritten as $\langle M, e\rangle_{HS} = 1$, where $\cdot, \cdot\rangle_{HS}$ is the Hilbert-Schmidt inner product (see (11)) and $e = I_d/d^{1/2}$ is a matrix whose Hilbert-Schmidt norm is equal to one; this allows to treat e as a distinguished element of cones and – at the same time – of their duals. Next, the primary objects of our analysis are quantum *maps*, and the chosen normalization assures that TP (and, dually, unital; see Appendix 6.5) maps are in H^{b}. When we are primarily interested in *states*, the normalization $\operatorname{Tr} M = 1$ can be thought of as more natural (the distinguished element I_d/d is the then the maximally mixed state, usually denoted by ρ_*).

While all the matrix spaces or spaces of maps are *a priori* complex, all cones of interest will live in fact in the real space $\mathcal{M}_d^{\mathrm{sa}}$ of Hermitian matrices or in the space of Hermicity-preserving maps. We will use the same symbols H^0, H^{b} etc. to denote the smaller *real* (vector or affine) subspaces; this should not lead to misunderstanding.

2.4 Other cones, all sets of interest compiled in one table

Analogous point of view will be employed when studying other cones of quantum maps such as

• the cone $\mathcal{S}P$ of *superpositive* maps (also called *entanglement breaking*, see (14) and the paragraphs that follow)

• the cone $\mathcal{D}$ of *decomposable* maps (see (18))

• the cone $\mathcal{T}$ of maps which extended by identity impose positive partial transpose (see (16) and the paragraphs that follow).

In all cases we will identify the corresponding cone of $N^2 \times N^2$ matrices and will relate in various ways bases of the cones and their sections corresponding to the trace preserving restriction. For easy reference, we list all objects of interest in the table below; see also Figures 1 and 3 in section 3. The missing definitions and unexplained relations (generally appealing to duality) will also be clarified there.

Table 1: Sets of quantum maps and the sets of quantum states associated to them via the Jamiołkowski–Choi isomorphism, cf. (6)-(9) below. The inclusion relation holds in each collumn, e.g. $\mathcal{P}_N \supset \mathcal{D}_N \supset \mathcal{CP}_N \supset \mathcal{T}_N \supset \mathcal{SP}_N$. The symbols $\circ$ and $\star$ in the rightmost column denote sets consisting of also non-positive semi-definite matrices which technically are not states (and are not readily identifiable with objects appearing in the literature).

Maps	$\Phi : \mathcal{M}_N \to \mathcal{M}_N$					States $\sigma \in \mathcal{M}_{N^2}$
	cones		$\mathrm{Tr} D_\Phi = N$		$\mathrm{Tr}_B D_\Phi = I_N$	$\mathrm{Tr}\sigma = 1$
positive	$\mathcal{P}_N$	$\supset$	$\mathcal{P}_N^{\mathrm{b}}$	$\supset$	$\mathcal{P}_N^{\mathrm{TP}}$	$\circ$
decomposable	$\mathcal{D}_N$	$\supset$	$\mathcal{D}_N^{\mathrm{b}}$	$\supset$	$\mathcal{D}_N^{\mathrm{TP}}$	$\star$
completely positive	$\mathcal{CP}_N$	$\supset$	$\mathcal{CP}_N^{\mathrm{b}}$	$\supset$	$\mathcal{CP}_N^{\mathrm{TP}}$	$\mathcal{M}_{N^2}^{\mathrm{tot}}$
PPT inducing	$\mathcal{T}_N$	$\supset$	$\mathcal{T}_N^{\mathrm{b}}$	$\supset$	$\mathcal{T}_N^{\mathrm{TP}}$	$\mathcal{M}_{N^2}^{\mathrm{PPT}}$
super positive	$\mathcal{SP}_N$	$\supset$	$\mathcal{SP}_N^{\mathrm{b}}$	$\supset$	$\mathcal{SP}_N^{\mathrm{TP}}$	$\mathcal{M}_{N^2}^{\mathrm{sep}}$

The action of the Jamiołkowski–Choi isomorphism, associating cones of maps to cones of matrices and their respective bases, can be summarized as

$$\Phi \text{ is positive } (\Phi \in \mathcal{P}_N) \quad \Leftrightarrow \quad \sigma \text{ is block-positive} \tag{6}$$

$$\begin{aligned} \Phi \text{ is completely positive } (\Phi \in \mathcal{CP}_N) \quad &\Leftrightarrow \quad \sigma \text{ is positive semi-definite} \\ \Phi \in \mathcal{CP}_N^{\mathrm{b}} \quad &\Leftrightarrow \quad \sigma = \frac{1}{N} D_\Phi \in \mathcal{M}_{N^2}^{\mathrm{tot}} \end{aligned} \tag{7}$$

$$\begin{aligned} \Phi \text{ is PPT inducing } (\Phi \in \mathcal{T}_N) \quad &\Leftrightarrow \quad \sigma \in \mathcal{PPT} \\ \Phi \in \mathcal{T}_N^{\mathrm{b}} \quad &\Leftrightarrow \quad \sigma = \frac{1}{N} D_\Phi \in \mathcal{M}_{N^2}^{\mathrm{PPT}} \end{aligned} \tag{8}$$

$$\begin{aligned} \Phi \text{ is superpositive } (\Phi \in \mathcal{SP}_N) \quad &\Leftrightarrow \quad \sigma \text{ is separable} \\ \Phi \in \mathcal{SP}_N^{\mathrm{b}} \quad &\Leftrightarrow \quad \sigma = \frac{1}{N} D_\Phi \in \mathcal{M}_{N^2}^{\mathrm{sep}} \end{aligned} \tag{9}$$

The description of the matricial cone associated to the cone $\mathcal{D}_N$ of decomposable maps is largely tautological: the sum of the positive semi-definite cone and its image via the partial transpose. We likewise have $\frac{1}{N} D_\Phi \in \circ$ (resp., $\in \star$) iff $\Phi \in \mathcal{P}_N^{\mathrm{b}}$ (resp., $\in \mathcal{D}_N^{\mathrm{b}}$).

2.5 Comparing sets via volume radii, overview of results

Explicit formulae for volumes of high dimensional sets are often not very transparent (when they can be figured out at all, that is). This may be exemplified by the closed expression for the volume of the d^2-1–dimensional set $\mathcal{M}_d^{\mathrm{tot}}$, the set of density operators of size d that has been computed in [6]

$$\mathrm{vol}\big(\mathcal{M}_d^{\mathrm{tot}}\big) = \sqrt{d}\,(2\pi)^{d(d-1)/2} \frac{\Gamma(1)\ldots\Gamma(d)}{\Gamma(d^2)}. \tag{10}$$

Given the complexity of formulae such as (10), the following concept is sometimes convenient. Given an m-dimensional set K, we define $\mathrm{vrad}(K)$, the *volume radius* of K, as the radius of an Euclidean ball of the same volume (and dimension) as K. Equivalently, $\mathrm{vrad}(K) = \big(\mathrm{vol}(K)/\mathrm{vol}(B_2^m)\big)^{1/m}$, where B_2^m is the unit Euclidean ball. It is fairly easy (if tedious) to verify that (10) implies a much more transparent relation $\mathrm{vrad}\big(\mathcal{M}_d^{\mathrm{tot}}\big) = e^{-1/4} d^{-1/2}(1 \pm O(d^{-1}))$ as $d \to \infty$, and similar two-sided estimates valid for all d.

This point of view allows to present in a compact way the gist of our results. We start by listing, in Table 2, bounds and asymptotics for volume radii of bases of various cones of maps acting on N–level density matrices. Observe that the bounds for volume radii of the three middle sets ($\mathcal{D}$, $\mathcal{CP}$ and $\mathcal{T}$) do not depend on dimensionality. On the other hand, the volume radii of the base for the largest set $\mathcal{P}$ of positive maps grow as $\sqrt{N}$, while the volume radii of the smallest set $\mathcal{SP}$ of superpositive maps decrease as $1/\sqrt{N}$.

The base of the set of completely positive maps acting on density matrices of size N is up to a rescaling by the factor $1/N$ equivalent to the set of mixed states $\mathcal{M}_d^{\mathrm{tot}}$ of

Table 2: Volume radii for the bases of mutually nested cones of positive maps which act on N–level density matrices. Here r_{CP} denotes the volume radius of the base $\mathcal{CP}_N^{\mathrm{b}}$ of the set of completely positive maps. The last column characterizes the asymptotical properties, where $r_{\mathcal{X}}^{\lim} := \lim_{N\to\infty} \mathrm{vrad}(\mathcal{X}_N^b)$ with $\mathcal{X}$ standing for $\mathcal{D}$, $\mathcal{CP}$ or $\mathcal{T}$. It is tacitly assumed that the limits exist, which we do not know for $\mathcal{X} \neq \mathcal{CP}$ (the rigorous statements would involve then lim inf or lim sup, cf. Theorem 5). The question marks "?" indicate that we do not have asymptotic information that is more precise than the one implied by the bounds in the middle column. It is an interesting open problem whether $r_{\mathcal{T}}^{\lim}$ admits a nontrivial (i.e., < 1) *upper* bound; cf. remark (c) following Theorem 5.

Sets of maps	Bounds for volume radii					Asymptotics
positive $\mathcal{P}$	$\frac{1}{4}\sqrt{N}$	$\leq$	$\mathrm{vrad}(\mathcal{P}_N^{\mathrm{b}})$	$\leq$	$6\sqrt{N}$	?
decomposable $\mathcal{D}$	$r_{\mathcal{CP}}$	$\leq$	$\mathrm{vrad}(\mathcal{D}_N^{\mathrm{b}})$	$\leq$	$8r_{\mathcal{CP}}$	$r_{\mathcal{D}}^{\lim} \leq 2$
completely positive $\mathcal{CP}$	$\frac{1}{2}$	$\leq$	$r_{\mathcal{CP}} := \mathrm{vrad}(\mathcal{CP}_N^{\mathrm{b}})$	$\leq$	1	$r_{\mathcal{CP}}^{\lim} = e^{-1/4}$
PPT inducing $\mathcal{T}$	$\frac{1}{4}r_{\mathcal{CP}}$	$\leq$	$\mathrm{vrad}(\mathcal{T}_N^{\mathrm{b}})$	$\leq$	$r_{\mathcal{CP}}$	$r_{\mathcal{T}}^{\lim} \geq \frac{1}{2}$
super positive $\mathcal{SP}$	$\frac{1}{6}\frac{1}{\sqrt{N}}$	$\leq$	$\mathrm{vrad}(\mathcal{SP}_N^{\mathrm{b}})$	$\leq$	$4\frac{1}{\sqrt{N}}$	?

dimensionality $d = N^2$, and similarly for other cones of maps – see eq. (7)-(9). Therefore, the results implicit in the last three rows of Table 2 are equivalent to the following bounds, presented in Table 3, for the volume radii of the set of quantum states and its subsets, some of which were known.

Finally, we list in Table 4 the volume radii of the main objects of study in this paper: the set $\mathcal{CP}_N^{\mathrm{TP}}$ of quantum operations and of other "ensembles" of trace preserving maps. Each of these sets forms a $N^4 - N^2$ cross-section of the corresponding $N^4 - 1$-dimensional base (i.e., of $\mathcal{CP}_N^{\mathrm{b}}$ etc.).

Although the volume of the larger set is sometimes known (10), the cross-sections appear much harder to analyze. Our approach does not aim at producing exact values (even though here and in the previous tables we made an effort to obtain "reasonable" values for the numerical constants appearing in the formulae). Instead, we produce two-sided estimates for the volume radius of $\mathcal{CP}_N^{\mathrm{TP}}$, which are quite tight in the asymptotic sense (as the dimension increases) and analogous bounds for the sets of positive, decomposable, PPT–inducing and super–positive trace preserving maps. Note that these bounds are similar to the results for the bases of all five sets presented in Table 2, but are not their formal consequences.

Table 3: Volume radii for the set of states $\mathcal{M}_d^{\text{tot}}$ of size d and its subsets $\mathcal{M}_d^{\text{PPT}}$ and $\mathcal{M}_d^{\text{sep}}$. The latter two sets are well defined if the dimensionality d is a square of an integer. Here $a \sim b$ means that $\lim_{d\to\infty} a/b = 1$, while $a \gtrsim b$ stands for $\liminf_{d\to\infty} a/b \geq 1$.

Sets of states	Bounds for volume radii					Asymptotics
all states	$\frac{1}{2}\frac{1}{\sqrt{d}}$	$\leq$	$r_{\text{tot}} := \text{vrad}(\mathcal{M}_d^{\text{tot}})$	$\leq$	$\frac{1}{\sqrt{d}}$	$r_{\text{tot}} \sim e^{-1/4}\frac{1}{\sqrt{d}}$
PPT states	$\frac{1}{4}\frac{1}{\sqrt{d}}r_{\text{tot}}$	$\leq$	$r_{\text{PPT}} := \text{vrad}(\mathcal{M}_d^{\text{PPT}})$	$\leq$	$\frac{1}{\sqrt{d}}r_{\text{tot}}$	$r_{\text{PPT}} \gtrsim \frac{1}{2}\frac{1}{\sqrt{d}}$
separable states	$\frac{1}{6}\frac{1}{d}$	$\leq$	$\text{vrad}(\mathcal{M}_d^{\text{sep}})$	$\leq$	$4\frac{1}{d}$	?

While we concentrate in this work on the study of various classes of trace preserving maps, our approach allows deriving estimates of comparable degree of precision for other sets of quantum maps. As an illustration, we sketch in Appendix 6.5 an argument giving tight bounds for the volume of *trace non-increasing* (TNI) maps. An exact formula for that volume was recently found by a different method [30] independently from the present work.

Finally, let us point out that formula (10) is valid only in the case when the underlying Hilbert space is complex, and that our analysis focuses on the complex setting, as it is the one that is of immediate physical interest. However, all the discussion preceding (10) can be carried out also for real Hilbert spaces, and virtually all results that follow do have real analogues. This is because even when closed formulae are not available, the methods of geometric functional analysis allow to derive two-sided dimension free bounds on volume radii and similar parameters. Accordingly, while in the real case one may be unable to precisely calculate coefficients such as $e^{-1/4}$ above, it will be generally possible to determine the relevant quantities up to universal multiplicative constants.

Table 4: Asymptotic properties of volume radii for five nested sets of trace preserving maps. Same caveat as in Table 2 applies to the limits in the second column. Upper and lower bounds valid for all N (as in the middle columns of Tables 2 and 3) can be likewise obtained.

Sets of trace preserving maps	asymptotics of their volume radii				
positive $\mathcal{P}$	$\frac{1}{4}$	$\leq$	$\lim_{N\to\infty} \frac{\mathrm{vrad}(\mathcal{P}_N^{\mathrm{TP}})}{\sqrt{N}}$	$\leq$	6
decomposable $\mathcal{D}$	$e^{-1/4}$	$\leq$	$\lim_{N\to\infty} \mathrm{vrad}(\mathcal{D}_N^{\mathrm{TP}})$	$\leq$	2
completely positive $\mathcal{CP}$			$\lim_{N\to\infty} \mathrm{vrad}(\mathcal{CP}_N^{\mathrm{TP}})$	$=$	$e^{-1/4}$
PPT inducing $\mathcal{T}$	$\frac{1}{2}$	$\leq$	$\lim_{N\to\infty} \mathrm{vrad}(\mathcal{T}_N^{\mathrm{TP}})$	$\leq$	$e^{-1/4}$
super positive $\mathcal{SP}$	$\frac{1}{6}$	$\leq$	$\lim_{N\to\infty} \frac{\mathrm{vrad}(\mathcal{SP}_N^{\mathrm{TP}})}{1/\sqrt{N}}$	$\leq$	4

2.6 A generic positive map acting on a high dimensional system is not decomposable

This is immediate from Table 4: the volume radius of the set of positive trace preserving maps acting on an N dimensional system is of order $\sqrt{N}$, while the volume radius of the corresponding set of decomposable trace preserving maps is $O(1)$. Thus, for large N, the latter set constitutes a very small part of the former one. Note that in order to compare volumes we need to raise the ratio of the volume radii to the power $N^4 - N^2$, which yields roughly $N^{-N^4/2}$, a fraction that is (strictly) subexponential in the dimension of the set.

3 Known and preliminary results

3.1 Duality of cones

Spaces of operators or matrices are endowed with the canonical Hilbert-Schmidt inner product structure. The Choi-Jamiołkowski isomorphisms transfers this structure to the space of quantum maps. We define

$$(\Phi, \Psi) := \langle D_\Phi, D_\Psi \rangle_{HS} := \mathrm{Tr}\, D_\Phi^\dagger D_\Psi. \tag{11}$$

The spaces in question and the corresponding inner products are *a priori* complex. However, if we restrict our attention to the *real* vector spaces of Hermicity-preserving maps Φ and Hermitian matrices D_Φ, which we will do in what follows, the scalar product becomes real and we may simply write

$$(\Phi, \Psi) = \mathrm{Tr}\, D_\Phi D_\Psi.$$

We next define a duality * for cones of maps via their representation (or dynamical) matrix by

$$\mathcal{C}^* := \{\Psi : \mathcal{M}_N \to \mathcal{M}_N : (\Phi, \Psi) \geq 0 \text{ for all } \Phi \in \mathcal{C}\}. \tag{12}$$

This is a very special case of associating to a cone in a vector space the dual cone in the dual space (here $\mathcal{M}_d$ is identified with its dual via the inner product $\langle \cdot, \cdot \rangle_{HS}$). Duality for cones of matrices and cones of maps is the same by definition.

We point out that all the cones $\mathcal{C}$ we consider are non-degenerate, i.e., they are of full dimension in the real vector space $\mathcal{M}_{N^2}^{\mathrm{sa}}$ of Hermitian matrices, or in the space of linear maps commuting with † (equivalently, every map/matrix – Hermicity-preserving or Hermitian, as appropriate – can be written as the difference of two elements of $\mathcal{C}$) and further $-\mathcal{C} \cap \mathcal{C} = \{0\}$. Consequently, their duals are also non-degenerate.

Since the cone of positive semi-definite matrices is self-dual, it follows that

$$\mathcal{CP}^* = \mathcal{CP}. \tag{13}$$

The superpositive cone $\mathcal{SP}$ may be defined via duality

$$\mathcal{SP} := \mathcal{P}^*. \tag{14}$$

By the bipolar theorem for cones ($(\mathcal{C}^*)^* = \mathcal{C}$), we then have

$$\mathcal{SP}^* = \mathcal{P}. \tag{15}$$

[Note that the bipolar theorem for *closed* cones follows, for example, from the easily verifiable identity $\mathcal{C}^* = -\mathcal{C}^\circ$, where $^\circ$ is the standard polar defined by $K^\circ = \{x : \langle x, y\rangle \le 1 \text{ for all } y \in K\}$, and from the bipolar theorem for the standard polar, i.e., from the equality $(K^\circ)^\circ = K$ valid whenever K is a closed convex set containing 0.] Clearly

$$\mathcal{SP} \subset \mathcal{CP} \subset \mathcal{P},$$

see Figure 1.

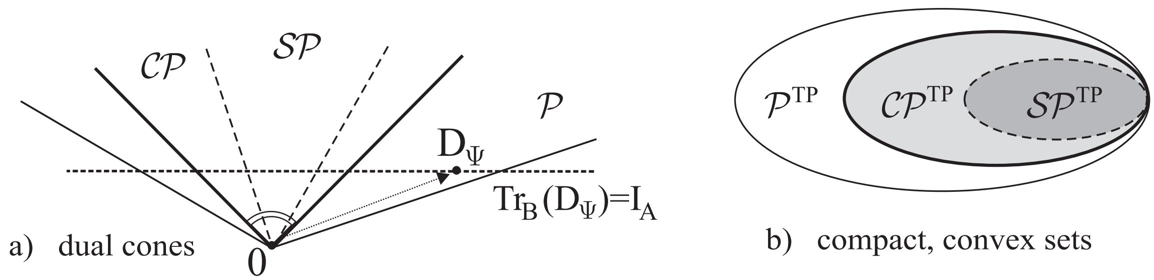

Figure 1: Sketch of sets of maps. a) The cone $\mathcal{P}$ of positive maps includes the cone $\mathcal{CP}$ of completely positive maps and its subcone $\mathcal{CP}$ containing the superpositive maps, dual to $\mathcal{P}$. Trace preserving maps belong to the cross–section of the cones with an affine plane of dimension $N^4 - N^2$ (and of codimension N^2), representing the condition $\mathrm{Tr}_B D = I_A$. b) The sets of trace preserving maps in another perspective. This is a complete picture for $N = 2$ since some of the cones coincide, namely $\mathcal{P} = \mathcal{D}$ and $\mathcal{T} = \mathcal{S}P$. For $N \ge 3$, the complete picture is more complicated, see Figure 3.

To clarify the duality relations (14), (15) and the structure of the cone $\mathcal{SP}$, we recall that Φ is positive iff D_Φ is block positive, which – by definition – is equivalent to $\Phi(\rho_\xi) \ge 0$ for every matrix of the form $\rho_\xi := |\xi\rangle\langle\xi|$, that is, for every rank one positive semi-definite matrix. In other words, for any $\xi \in \mathcal{H}_A$ and for any $\eta \in \mathcal{H}_B$,

$$0 \le \langle \Phi(|\xi\rangle\langle\xi|)\eta, \eta\rangle_{HS} = \mathrm{Tr}\,\Phi(|\xi\rangle\langle\xi|)\,|\eta\rangle\langle\eta| = \mathrm{Tr}\,D_\Phi(\rho_\xi \otimes \rho_\eta) = \langle D_\Phi, \rho_\xi \otimes \rho_\eta\rangle_{HS}$$

where the first tracing takes place in $\mathcal{H}_B$ (or $\mathcal{M}_N$) and the other in $\mathcal{H}_A \otimes \mathcal{H}_B$, or in $\mathcal{M}_{N^2}$ (and similarly for the two Hilbert-Schmidt scalar products). This is the same as saying that D_Φ belongs to the cone of matrices that is dual to the separable cone (the cone generated by all $\rho_\xi \otimes \rho_\eta = \rho_{\xi\otimes\eta}$ or, equivalently, by all products $\rho_A \otimes \rho_B$ of positive semi-definite matrices). By the bipolar theorem for cones, this is equivalent to the cone $\{D_\Phi : \Phi \in \mathcal{SP}\}$ being *exactly* the separable cone.

An alternative description of $\mathcal{SP}$, which justifies the "entanglement breaking" terminology, is as follows: Φ is superpositive iff for every k the extended quantum map $\Phi \otimes \mathcal{I}_k$ maps positive semi-definitive matrices to (positive semi-definite) separable matrices, or states to separable states if Φ is trace preserving.

Sometimes (see, e.g., Appendix 6.2) it is useful to work with extended sets of maps such as the convex hulls of $\mathcal{P}_N^{\mathrm{TP}} \cup \{0\}$ or $\mathcal{P}_N^{\mathrm{b}} \cup \{0\}$. For technical reasons, we find the latter one more useful; we will denote it by $\mathcal{P}^{\mathrm{E}} = \mathcal{P}_N^{\mathrm{E}}$, and similarly for other cones. Here 0 denotes the "zero" map, which may be chosen as a reference point. Further, one may consider symmetrized sets such as $\mathcal{CP}^{\mathrm{sym}} = \mathcal{CP}_N^{\mathrm{sym}}$, the convex hull of $-\mathcal{CP}^{\mathrm{b}} \cup \mathcal{CP}^{\mathrm{b}}$, where $-\mathcal{CP}^{\mathrm{b}}$ is the symmetric image of $\mathcal{CP}^{\mathrm{b}}$ with respect to 0. (Note that $\mathcal{CP}^{\mathrm{sym}}$ is also the convex hull of $-\mathcal{C}P^{\mathrm{E}} \cup \mathcal{C}P^{\mathrm{E}}$, see Figure 2 below.) The advantage in using 0-symmetric sets is that, first, they often admit an interpretation as unit balls with respect to natural norms and, second, that symmetric convex bodies have been studied more extensively than general ones convex bodies.

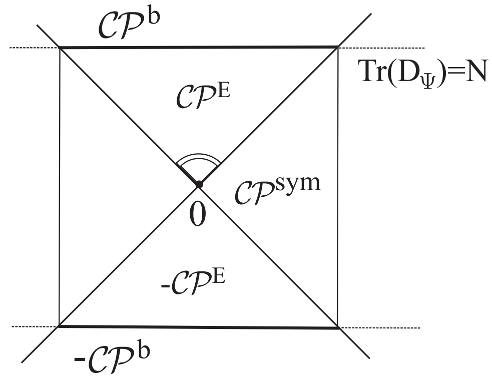

Figure 2: The set $\mathcal{CP}^{\mathrm{b}}$ of normalized quantum maps arises as a cross-section of the unbounded cone of CP maps with the hyperplane representing the condition $\operatorname{Tr} D = N$. The set $\mathcal{CP}^{E}$ of maps extended by the zero map is the convex hull of $\mathcal{C}P^{\mathrm{b}} \cup \{0\}$, while $\mathcal{CP}^{\mathrm{sym}}$ is the symmetrized set, the convex hull of $-\mathcal{CP}^{\mathrm{b}} \cup \mathcal{CP}^{\mathrm{b}}$. $\mathcal{CP}^{\mathrm{sym}}$ may be identified with a ball in trace class norm, whose radius equals N. Analogous notation (and similar identifications) may be employed for other sets of maps including $\mathcal{P}$, $\mathcal{SP}$ etc., or for abstract cones.

We next introduce the auxiliary cone of completely co-positive (CcP) maps

$$\mathcal{C}c\mathcal{P} = \{\Phi : T \circ \Phi \in \mathcal{CP}\},$$

where $T : \mathcal{M}_N \to \mathcal{M}_N$ is the transposition map (which is positive, but not completely positive for $N > 1$), and the cone

$$\mathcal{T} := \mathcal{CP} \cap \mathcal{C}c\mathcal{P}. \tag{16}$$

In terms of dynamical (Choi) matrices, $D_{T\circ\Phi}$ is obtained from D_Φ by transposing each block, i.e., by the partial transpose in the second system. This means that $\{D_\Phi \ : \ \Phi \in \mathcal{T}\}$ is exactly $\mathcal{PPT}$, the *positive partial transpose cone* (positive semi-definite matrices whose partial transpose is also positive semi-definite). Since, as is easy to check, separable matrices are in $\mathcal{PPT}$, it follows that

$$\mathcal{SP} \subset \mathcal{T} \subset \mathcal{CP} \tag{17}$$

For $N = 2$ the sets $\mathcal{T}$ and $\mathcal{SP}$ coincide, while for larger dimensions the inclusion $\mathcal{SP} \subset \mathcal{T}$ is proper as shown in Figure 3.

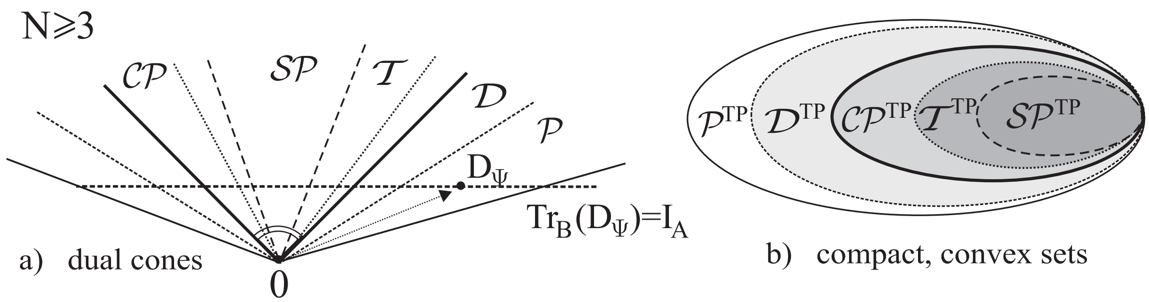

Figure 3: Sketch of sets of maps for $N \geq 3$. a) The cone $\mathcal{P}$ of positive maps includes a sequence of nested subcones: the cone $\mathcal{D}$ of decomposable maps, the cone $\mathcal{C}P$ of completely positive maps, the cone $\mathcal{T}$ of maps which extended by identity impose positive partial transpose, and the cone $\mathcal{S}P$ of superpositive maps. b) The sequence of nested subsets of the compact set of positive trace preserving maps.

Similarly to superpositive maps, there is an alternative description of $\mathcal{T}$ in the language of extended quantum maps: $\Phi \in \mathcal{T}$ iff $\Phi \otimes \mathcal{I}_k$ is *PPT inducing* for any size k of the extension, i.e., for any state ρ acting on the bipartite system its image, $\rho' = \Phi \otimes \mathcal{I}(\rho) \in \mathcal{PPT}$. [The necessity of the latter condition follows by noticing that the partial transpose of ρ' equals $(T \otimes \mathcal{I})\rho' = (T \circ \Phi \otimes \mathcal{I})\rho$, which is positive semidefinite due to $T \circ \Phi$ being CP.]

A quantum map Φ is called *decomposable*, if it may be expressed as a sum of a CP map Ψ_1 and a another CP map Ψ_2 composed with the transposition T,

$$\Phi = \Psi_1 + T \circ \Psi_2 \tag{18}$$

or, equivalently, as a sum of a CP map and a CcP map. In other words, the cone $\mathcal{D}$ of decomposable maps is defined by

$$\mathcal{D} := \mathcal{CP} + \mathcal{C}c\mathcal{P}$$

(the Minkowski sum). Since the transposition preserves positivity, $\mathcal{D} \subset \mathcal{P}$. It is known [31, 32] that every one-qubit positive map is decomposable, so the sets $\mathcal{P}_2$ and $\mathcal{D}_2$ coincide.

However, already for $N = 3$ there exist positive, non–decomposable maps [33], so $\mathcal{D}_3$ forms a proper subset of $\mathcal{P}_3$ – see Figure 3.

It follows from the identity $(\Phi, T\circ\Psi) = (T\circ\Phi, \Psi)$ valid for all Φ, Ψ that

$$\mathcal{C}c\mathcal{P}^* = \mathcal{C}c\mathcal{P}.$$

Accordingly, the dual cone $\mathcal{D}^*$ verifies

$$\mathcal{D}^* = (\mathcal{CP} + \mathcal{C}c\mathcal{P})^* = \mathcal{CP}^* \cap \mathcal{C}c\mathcal{P}^* = \mathcal{CP} \cap \mathcal{C}c\mathcal{P} = \mathcal{T} \tag{19}$$

This is a special case of the identity $(\mathcal{C}_1 + \mathcal{C}_2)^* = \mathcal{C}_1^* \cap \mathcal{C}_2^*$ (the Minkowski sum) valid for any two convex cones $\mathcal{C}_1, \mathcal{C}_2$. It now follows by the bipolar theorem that

$$\mathcal{D} = \mathcal{T}^*. \tag{20}$$

As $\mathcal{SP} \subset \mathcal{T} \subset \mathcal{CP}$ by (17), it follows by duality that

$$\mathcal{CP} \subset \mathcal{D} \subset \mathcal{P}.$$

3.2 Bases of cones and duality; the inradii and the outradii. The symmetrized sets

We now return to the analysis of *bases* of cones of matrices, as defined in section 2.3. As was to be expected, natural set-theoretic and algebraic operations on cones induce analogous operations on bases of cones. Sometimes this is trivial as in $(\mathcal{C}_1 \cap \mathcal{C}_2)^{\mathrm{b}} = \mathcal{C}_1^{\mathrm{b}} \cap \mathcal{C}_2^{\mathrm{b}}$, in other cases simple: $(\mathcal{C}_1 + \mathcal{C}_2)^{\mathrm{b}} = \mathrm{conv}(\mathcal{C}_1^{\mathrm{b}} \cup \mathcal{C}_2^{\mathrm{b}})$, where conv stands for the convex hull. What is more interesting and somewhat surprising is that also duality of cones carries over to precise duality of bases in the following sense.

Lemma 1 *Let $\mathcal{V}$ be a real Hilbert space, $\mathcal{C} \subset \mathcal{V}$ a closed convex cone and let $e \in \mathcal{V}$ be a unit vector such that $e \in \mathcal{C} \cap \mathcal{C}^*$. Set $V^{\mathrm{b}} := \{x \in \mathcal{V} \,:\, \langle x, e\rangle = 1\}$ and let and $\mathcal{C}^{\mathrm{b}} = \mathcal{C} \cap V^{\mathrm{b}}$ and $(\mathcal{C}^*)^{\mathrm{b}} = \mathcal{C}^* \cap V^{\mathrm{b}}$ be the corresponding bases of $\mathcal{C}$ and $\mathcal{C}^*$. Then*

$$(\mathcal{C}^*)^{\mathrm{b}} := \mathcal{C}^* \cap V^{\mathrm{b}} = \{y \in V^{\mathrm{b}} \,:\, \forall x \in \mathcal{C}^{\mathrm{b}} \;\; \langle -(y-e), x-e\rangle \le 1\}. \tag{21}$$

In other words, if we think of V^{b} as a vector space with the origin at e, and of $\mathcal{C}^{\mathrm{b}}$ and $(\mathcal{C}^*)^{\mathrm{b}}$ as subsets of that vector space, then $(\mathcal{C}^*)^{\mathrm{b}} = -(\mathcal{C}^{\mathrm{b}})^{\circ}$.

Recall that for abstract cones $\mathcal{C} \subset \mathcal{V}$, the dual cone $\mathcal{C}^*$ is defined (cf. (12)) via

$$\mathcal{C}^* := \{x \in \mathcal{V} \,:\, \forall\, y \in \mathcal{C}\; \langle x, y\rangle \ge 0\}.$$

This elementary Lemma seems to be a folklore result, but does not appear in standard references for convexity (the best source we were pointed to after consulting specialists was Exercise 6, §3.4 of [34]). However, once stated, the Lemma is straightforward to prove. If $\langle x, e\rangle = \langle y, e\rangle = 1$, then $\langle -(y-e), x-e\rangle = -\langle y, x\rangle + 1$ and so the condition from (21) can be restated as

$$\forall x \in \mathcal{C}^{\mathrm{b}}\ -\langle y, x\rangle + 1 \leq 1 \quad \Leftrightarrow \quad \forall x \in \mathcal{C}^{\mathrm{b}}\ \langle y, x\rangle \geq 0.$$

Since under our hypotheses $\mathcal{C}^{\mathrm{b}}$ generates $\mathcal{C}$, the latter condition is equivalent to $\langle y, x\rangle \geq 0$ for all $x \in \mathcal{C}$, i.e., to $y \in \mathcal{C}^*$, as required.

Let us now return to our more concrete setting of $\mathcal{V} = \mathcal{M}_d^{\mathrm{sa}}$ (endowed with the Hilbert-Schmidt scalar product) and $e = I_d/d^{1/2}$. Even more specifically, we will consider $\mathcal{V} = \mathcal{M}_{N^2}^{\mathrm{sa}}$, identified via the Choi-Jamiołkowski isomorphism with the space of Hermicity preserving quantum maps on $\mathcal{M}_N$, and the cones that we defined in prior section. Note that the quantum map associated to $e = I_{N^2}/N$ is the so-called "completely depolarising map," which is usually denoted by Φ_* and whose action is described by $\Phi_*(M) = (N^{-1}\mathrm{Tr}\, M)\, I_N$. The duality relations for cones (13), (14), (15) and (19), (20) combined with Lemma 1 imply now

Corollary 2 *We have the following duality relations for the bases of cones*

$$(\mathcal{CP}^{\mathrm{b}})^\circ = -\mathcal{CP}^{\mathrm{b}},\ (\mathcal{SP}^{\mathrm{b}})^\circ = -\mathcal{P}^{\mathrm{b}},\ (\mathcal{P}^{\mathrm{b}})^\circ = -\mathcal{SP}^{\mathrm{b}}$$

$$(\mathcal{D}^{\mathrm{b}})^\circ = -\mathcal{T}^{\mathrm{b}},\ (\mathcal{T}^{\mathrm{b}})^\circ = -\mathcal{D}^{\mathrm{b}}, \tag{22}$$

where both the polarity and the negative signs refer to the vector structure in $H^{\mathrm{b}} = \{\Phi : \mathrm{Tr}\, D_\Phi = \mathrm{Tr}\,\Phi(I_N) = N\}$ *with* Φ_* *as the origin.*

In other words, we have for example

$$\mathcal{D}^{\mathrm{b}} = \{\Phi \in H^{\mathrm{b}}\ :\ \forall \Psi \in \mathcal{T}^{\mathrm{b}}\ (-(\Phi - \Phi_*), (\Psi - \Phi_*)) \leq 1\}.$$

While the duality relations *for cones* described in the preceding subsection are rather well known, the duality *for bases* in the present generality appears to be a new observation. When combined with standard results from convex geometry, most notably Santaló and inverse Santaló inequalities [35, 36] (see below), and other tools of geometric functional analysis, it allows for relating volumes of bases of cones to those of the dual cones, and ultimately for asymptotically precise estimates of these volumes and of volumes of the corresponding sets of trace preserving maps.

Let us also note here one immediate but interesting (and presumably known) consequence of the duality relations.

Corollary 3 *For each of the sets $\mathcal{CP}_N^{\rm b}, \mathcal{SP}_N^{\rm b}, \mathcal{P}_N^{\rm b}, \mathcal{D}_N^{\rm b}$ and $\mathcal{T}_N^{\rm b}$, the Euclidean (i.e., Hilbert-Schmidt) in-radius is $(N^2-1)^{-1/2}$ and the Euclidean out-radius is $(N^2-1)^{1/2}$.*

We observe first that, for each of the above sets, Φ_* is the only element that is invariant under isometries of the set. Accordingly, it is enough to restrict attention to Hilbert-Schmidt balls centered at Φ_*. For $\mathcal{CP}_N^{\rm b}$, the assertion is just a reflection of the elementary fact that $\mathcal{M}_d^{\rm tot}$ contains a Hilbert-Schmidt ball of radius $1/\sqrt{d(d-1)}$ centered at the maximally mixed state ρ_*, and that the distance from ρ_* to pure states is $\sqrt{1-1/d}$. For $\mathcal{SP}_N^{\rm b}$, it is a consequence of equality of in-radii of $\mathcal{M}_{N^2}^{\rm tot}$ and $\mathcal{M}_{N^2}^{\rm sep}$ (in the bivariate case) established in [13] (the out-radius of the latter is of course attained on pure separable states). It then follows that the in- and out-radii must be the same for the intermediate set $\mathcal{T}_N^{\rm b}$. Finally, since the out-radius of K° is the reciprocal of the in-radius of K (and *vice versa*), we deduce the assertion for $\mathcal{P}_N^{\rm b}$ and $\mathcal{D}_N^{\rm b}$ via (22).

It is curious to note that the statement about the out-radius of $\mathcal{P}_N^{\rm b}$ is equivalent – via simple geometric arguments – to the following fact (which *a posteriori* is true)

If $M=(M_{jk})_{j,k=1}^N$ is a block-positive matrix, then $\mathrm{Tr}\,(M^2) \le (\mathrm{Tr}\,M)^2$.

It would be nice to have a simple direct proof of the above inequality, as it would yield (via Lemma 1 and (22)) an alternate derivation of the result from [13] concerning the in-radius of the set of separable states in the bivariate case.

Similarly, the best (i.e., the smallest) constant R in the inclusion

$$\mathcal{CP}^{\rm b} - \Phi_* \subset R\,(\mathcal{SP}^{\rm b} - \Phi_*)$$

is the same as the best constant in

$$\mathcal{P}^{\rm b} - \Phi_* \subset R\,(\mathcal{CP}^{\rm b} - \Phi_*).$$

It has been shown in [13] that the optimal R satisfies $N^2/2+1 \le R \le N^2-1$. [The upper bound follows just from the formulae for the inradius of $\mathcal{SP}^{\rm b}$ and the outradius of $\mathcal{CP}^{\rm b}$ (or, equivalently, $\mathcal{M}^{\rm sep}$, $\mathcal{M}^{\rm tot}$).] Again, there could be a more direct elementary argument.

Remark 4 *The Euclidean inradii and outradii of $\mathcal{CP}_N^{\rm TP}, \mathcal{SP}_N^{\rm TP}, \mathcal{P}_N^{\rm TP}, \mathcal{D}_N^{\rm TP}$ and $\mathcal{T}_N^{\rm TP}$ are the same as for the larger $\mathcal{C}^{\rm b}$-type sets, i.e., $(N^2-1)^{-1/2}$ and $(N^2-1)^{1/2}$.*

As pointed out in the arguments following the statement of Corollary 3, while the fact that the inradii and outradii of all sets in that Corollary are identical is nontrivial, there is no mystery about at least some of the maps (or directions) that witness them. In the language of the sets of states (i.e., matrices with trace one normalization) such witnesses for outradii are pure states, and universal witnesses that work for the outradii of all five sets are pure separable states. By duality (i.e., Lemma 1), direction that witness inradii (for all sets) are obtained by reflecting a pure separable state with respect to the maximally mixed state ρ_*. In the language of quantum maps purity (i.e., the Choi matrix being of rank one) corresponds to the map being of the form $\rho \to v^\dagger \rho v$ (Kraus rank one), and the trace preserving condition is then equivalent to v being unitary. If that unitary is separable (i.e., a tensor product of two unitaries acting on the first and second system), the corresponding pure state will be separable. This means that universal witnesses of outradii of $\mathcal{C}^{\mathrm{b}}$-type sets exist also in the smaller set by the trace preserving condition (5), i.e., inside the $\mathcal{C}^{\mathrm{TP}}$-type sets. Since condition (5) defines an affine subspace, the "opposite" directions giving witnesses to the inradii also belong there.

An alternative use of duality considerations involves symmetrized sets (cf. Figure 2). If $\mathcal{C} \subset \mathcal{V}$ is a cone and $\mathcal{C}^{\mathrm{b}}$ its base, we define $\mathcal{C}^{\mathrm{sym}} := \mathrm{conv}(-\mathcal{C}^{\mathrm{b}} \cup \mathcal{C}^{\mathrm{b}})$; the minus sign referring now to the symmetric image with respect to 0. If, as earlier, e is the distinguished point of $\mathcal{C} \cap \mathcal{C}^*$ defining $\mathcal{C}^{\mathrm{b}}$ and $(\mathcal{C}^*)^{\mathrm{b}}$, then

$$(\mathcal{C}^{\mathrm{sym}})^\circ = (e - \mathcal{C}^*) \cap (-e + \mathcal{C}^*), \tag{23}$$

where the polarity has now the standard meaning (i.e., inside the entire space $\mathcal{V}$ *and* with respect to the origin). In other words, the polar of $\mathcal{C}^{\mathrm{sym}}$ is the order interval $[-e, e]$, in the sense of the order induced by the cone $\mathcal{C}^*$. The advantages of this approach is that we find ourselves in the category of centrally symmetric convex sets, which is better understood than that of general convex sets, and that frequently the object in question ($\mathcal{C}^{\mathrm{sym}}$ and its polar) have natural functional analysis interpretation as balls in natural normed spaces. One disadvantage is that in place of one very simple operation (symmetric image with respect to e) we have two elementary and manageable, but somewhat non-trivial operations (symmetrization and passing to order intervals). We postpone the discussion of (23) and related issues to the Appendix.

3.3 Volume radii and duality: Santaló and inverse Santaló inequalities

The classical Santaló inequality [35] asserts that if $K \subset \mathbb{R}^m$ is a 0-symmetric convex body and K° its polar body, then $\mathrm{vol}(K)\,\mathrm{vol}(K^\circ) \;\leq\; \big(\mathrm{vol}\big(B_2^m\big)\big)^2$ or, in other words

$$\mathrm{vrad(K)\ vrad(K^\circ)} \leq 1. \tag{24}$$

Moreover, the inequality holds also for not-necessarily-symmetric convex sets after an appropriate translation, in particular if the origin is the centroid of K or of K°, a condition that will be satisfied for all sets we will consider in what follow. Even more interestingly, there is a converse inequality [36], usually called "the inverse Santaló inequality,"

$$\mathrm{vrad(K)\ vrad(K^\circ)} \geq c \tag{25}$$

for some universal numerical constant $c > 0$, independent of the convex body K (symmetric or not) and, most notably, of its dimension m.

The inequalities (24), (25) together imply that, under some natural hypotheses (which are verified in most of cases of interest), the volume radii of a convex body and of its polar are approximately (i.e., up to a multiplicative universal numerical constant) reciprocal. By Lemma 1, the same is true for the base of a cone and that of the dual cone. This observation reduces, roughly by a factor of 2, the amount of work needed to determine the asymptotic behavior of volume radii of, say, sets from the third column of Table 1. We note, however, that since, at present, there are no good estimates for the constant c from (25) if K is not symmetric, it is often more efficient to revisit arguments from [14, 16] which allow to estimate volume radii of polar bodies without resorting to the inverse Santaló inequality. (An argument yielding reasonable value of c for *symmetric* bodies was recently given in [37].)

4 Volume estimates: precise statements and approximate arguments

The results stated in section 3, in combination with known facts, allow us to determine the asymptotic orders (as $N \to \infty$) for the volume radii (and hence reasonable estimates for the volumes) of bases of all cones of quantum maps discussed up to this point. Our goal is slightly more ambitious; we want to find not just the asymptotic order of each quantity, but also establish inequalities valid in every fixed dimension and involving explicit fairly sharp numerical constants. Specifically, we will show the following

Theorem 5 *We have the following inequalities, valid for all N, and the following asymptotic relations*

(i) $\frac{1}{2} \leq \mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{b}}\big) \leq 1, \quad \lim_{N\to\infty} \mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{b}}\big) = e^{-1/4} \approx 0.779$
(ii) $\frac{1}{4} N^{1/2} \leq \mathrm{vrad}\big(\mathcal{P}_N^{\mathrm{b}}\big) \leq 6N^{1/2}$
(iii) $\frac{1}{6} N^{-1/2} \leq \mathrm{vrad}\big(\mathcal{SP}_N^{\mathrm{b}}\big) \leq 4N^{-1/2}$
(iv) $\frac{1}{4} \leq \frac{\mathrm{vrad}\left(\mathcal{T}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 1, \quad \frac{e^{1/4}}{2} \leq \liminf_N \frac{\mathrm{vrad}\left(\mathcal{T}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)}$
(v) $1 \leq \frac{\mathrm{vrad}\left(\mathcal{D}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 8, \quad \limsup_N \frac{\mathrm{vrad}\left(\mathcal{D}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 2e^{1/4}$

Remarks : (a) Estimates on volume radii listed in Table 2 are either identical to the corresponding inequalities stated above, or follow by the same argument.
(b) Since the asymptotic orders of the volume radii of the families $\mathcal{CP}_N^{\mathrm{b}}$, $\mathcal{T}_N^{\mathrm{b}}$ and $\mathcal{D}_N^{\mathrm{b}}$ are the same, we chose – for greater transparence – to compare the volume radii of the two latter sets to that of $\mathcal{CP}_N^{\mathrm{b}}$ in (iv) and (v), rather than give separate estimates for each of these quantities.
(c) It is an interesting open problem whether there exists a universal constant $\alpha < 1$ such that $\mathrm{vrad}\big(\mathcal{T}_N^{\mathrm{b}}\big) \leq \alpha\, \mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{b}}\big)$ for all $N > 2$ or, equivalently, "is $\mathrm{vol}\big(\mathcal{M}_{N^2}^{\mathrm{PPT}}\big) \leq \alpha^{N^4} \mathrm{vol}\big(\mathcal{M}_{N^2}^{\mathrm{tot}}\big)$ for some $\alpha < 1$ and *all* $N > 2$?" Analogous question may be asked about comparing $\mathrm{vrad}\big(\mathcal{D}_N^{\mathrm{b}}\big)$ and $\mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{b}}\big)$. Inquiries to similar effect can be found in the literature [10, 38].
(d) It is likely that the *asymptotic* bound $2e^{1/4}$ in (v) holds actually for all N. Indeed, there is a strong numerical evidence that the estimate $\mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{b}}\big) \geq e^{-1/4}$ from (i) is valid for all N and not just in the limit. (In view of the explicit character of the formula (10) this issue shouldn't be too difficult to resolve.) Should that be the case, the next step would be to carefully analyze the dependence of $\mathrm{vrad}\big(\mathcal{D}_N^{\mathrm{b}}\big)$ on N given by the arguments presented in this paper.

Since the bases of cones, whose volume radii are described by Theorem 5, are effectively homothetic images, with ratio N, of the corresponding sets of trace one matrices (see Table 1 and the formulae that follow it), some of the inequalities/relations of Theorem 5 follow from known estimates for the volumes of various sets of states, particularly if we do not insist on obtaining "good" numerical constants that are included in the statements. For example, the estimates in statement (iii) are contained in Theorem 1 from [16]; one obtains the constants $\frac{1}{6}$ and 4 by going over the proof of that Theorem specified to bilateral systems. Similarly, the statement (iv) is (essentially) a version of Theorem 4 from [16] which asserts that, in the present language, $\mathrm{vrad}\big(\mathcal{M}_{N^2}^{\mathrm{PPT}}\big)/\mathrm{vrad}\big(\mathcal{M}_{N^2}^{\mathrm{tot}}\big) \geq c_0$ for some constant $c_0 > 0$ independent of the dimension N (the upper estimate with constant 1 is trivial). However, the argument from [16] yields only $c_0 = \frac{1}{8}$ and $\frac{e^{-1/4}}{4}$ for the asymptotic lower bound.

Next, the asymptotic relation in (i) follows from the explicit formula (10); see the comments following (10). Presumably, the estimates in (i) can also be derived from (10), but there are more elementary arguments. For a simple derivation of the lower bound from the classical Rogers-Shephard inequality [39] see [14], section II. And here is an apparently new proof of the upper bound: combine the duality results of the preceding section, specifically the identification $(\mathcal{CP}^{\mathrm{b}})^\circ = -\mathcal{CP}^{\mathrm{b}}$ from (22), with the Santaló inequality (24) to obtain

$$1 \geq \mathrm{vrad}\big(\mathcal{CP}^{\mathrm{b}}\big)\mathrm{vrad}\big((\mathcal{CP}^{\mathrm{b}})^\circ\big) = \mathrm{vrad}\big(\mathcal{CP}^{\mathrm{b}}\big)\mathrm{vrad}\big(-\mathcal{CP}^{\mathrm{b}}\big) = \mathrm{vrad}\big(\mathcal{CP}^{\mathrm{b}}\big)^2,$$

as required. We recall that, in the context of (22), the operations $^\circ$ and $-$ take place in the space H^{b} of quantum maps verifying $\mathrm{Tr}\, D_\Phi = N$, with Φ_* thought of as the origin; note that Φ_* is the centroid of $\mathcal{CP}^{\mathrm{b}}$ and so (24) with $K = \mathcal{CP}^{\mathrm{b}}$ indeed does apply in that setting.

Arguments parallel to the last one lead to versions of the remaining statements with *some* universal constants. For example, the identification $(\mathcal{SP}^{\mathrm{b}})^\circ = -\mathcal{P}^{\mathrm{b}}$ combined with the Santaló inequality (24) and its inverse (25) leads to

$$1 \geq \mathrm{vrad}\big(\mathcal{SP}^{\mathrm{b}}\big)\mathrm{vrad}\big(\mathcal{P}^{\mathrm{b}}\big) \geq c,$$

where c is the (universal) constant from (25). Combining the above inequality with (iii) we obtain $\frac{c}{4}N^{1/2} \leq \mathrm{vrad}\big(\mathcal{P}_N^{\mathrm{b}}\big) \leq 6N^{1/2}$. Similarly, $1 \geq \mathrm{vrad}\big(\mathcal{T}^{\mathrm{b}}\big)\mathrm{vrad}\big(\mathcal{D}^{\mathrm{b}}\big) \geq c$ combined with (the already shown version of) (iv) and with (i) implies $\frac{\mathrm{vrad}\left(\mathcal{D}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 32$ and $\limsup_N \frac{\mathrm{vrad}\left(\mathcal{D}_N^{\mathrm{b}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 4e^{3/4}$. As the constants in Theorem 5 are not meant to be optimal, we relegate the somewhat more involved (but still based on classical facts) arguments yielding them to Appendix 6.1.

The inequalities of Theorem 5 compare volumes of bases of cones, that is, sets of maps Φ normalized by the condition that the trace of D_Φ, the corresponding Choi (or dynamical) matrix, is N (or $\mathrm{Tr}\,\Phi(I_N) = N$). [Of course, any other normalization – most notably $\mathrm{Tr}\,D_\Phi = 1$ leading to sets of states – would work just as well for comparing volumes provided we were consistent.] However, if we want to study quantum operations, i.e., trace-preserving quantum maps (or, similarly, unital maps), then – as explained in the previous sections – the corresponding constraints are stronger than just normalization by trace: in each case we are looking at an N^2-codimensional section of the cone as opposed to the 1-codimensional base. The crucial point is that, in either case, the codimension is much smaller than the dimension, which is $N^4 - N^2$. The following technical result will imply that then, under relatively mild additional assumptions assuring that the base of the cone is reasonably balanced (which will be the case for all the cones we studied), the volume radius of the section will be very close to that of the entire base.

Proposition 6 *Let K be a convex body in an m-dimensional Euclidean space with centroid at a, and let H be a k-dimensional affine subspace passing through a. Let $r = r_K$ and $R = R_K$ be the in-radius and out-radius of K. Then*

$$\left(\mathrm{vrad}(K)\ R^{-\frac{m-k}{m}}\ b(m,k)\right)^{\frac{m}{k}} \le \mathrm{vrad}(K\cap H) \le \left(\mathrm{vrad}(K)\ r^{-\frac{m-k}{m}}\ b(m,k)\binom{m}{k}^{\frac{1}{m}}\right)^{\frac{m}{k}}, \tag{26}$$

where $b(m,k) := \left(\frac{vol_m(B_2^m)}{vol_k(B_2^k)vol_{m-k}(B_2^{m-k})}\right)^{\frac{1}{m}}$.

The proof of the Proposition is relegated to Appendix 6.3; now we explain its consequences. First, let us analyze the parameters that appear in (26). By Corallary 3, for all bases of cones that we consider here we have $r = 1/\sqrt{d-1}$ and $R = \sqrt{d-1}$, where $d = N^2$. Next, we have $m = d^2 - 1 = N^4 - 1$, $k = d^2 - d = N^4 - N^2$ and $m - k = d - 1 = N^2 - 1$, in particular $\frac{m}{k} = 1 + \frac{1}{N^2} = 1 + \frac{1}{d}$ and $\frac{m-k}{m} = \frac{1}{N^2+1} = \frac{1}{d+1}$. Further, the quantity $b(m,k) = \left(\frac{\Gamma(k/2+1)\Gamma((m-k)/2+1)}{\Gamma(m/2+1)}\right)^{\frac{1}{m}}$ (related to the Beta function) is easily shown to satisfy $1/\sqrt{2} < b(m,k) < 1$ (for our values of m,k it is actually $1 - O(\frac{\log N}{N^2})$). Similarly, $1 \le \binom{m}{k}^{\frac{1}{m}} \le 2$ for all k,m and $1 + O(\frac{\log N}{N^2})$ for our values of m,k. Consequently, if $\mathrm{vrad}(K)$ is subexponential in d (in our applications it is a low power of N, hence of d), then $\mathrm{vrad}(K\cap H)/\mathrm{vrad}(K) \to 1$ as $N \to \infty$.

This leads to

Theorem 7 *We have the following asymptotic relations*

(i) $\lim_{N\to\infty} \mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{TP}}\big) = e^{-1/4}$

(ii) $\frac{1}{4} \le \liminf_N \frac{\mathrm{vrad}\left(\mathcal{P}_N^{\mathrm{TP}}\right)}{N^{1/2}} \le \limsup_N \frac{\mathrm{vrad}\left(\mathcal{P}_N^{\mathrm{TP}}\right)}{N^{1/2}} \le 6$

(iii) $\frac{1}{6} \leq \liminf_N \frac{\mathrm{vrad}\left(\mathcal{SP}_N^{\mathrm{TP}}\right)}{N^{-1/2}} \leq \limsup_N \frac{\mathrm{vrad}\left(\mathcal{SP}_N^{\mathrm{TP}}\right)}{N^{-1/2}} \leq 4$

(iv) $\frac{e^{1/4}}{2} \leq \liminf_N \frac{\mathrm{vrad}\left(\mathcal{T}_N^{\mathrm{TP}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)}$

(v) $\limsup_N \frac{\mathrm{vrad}\left(\mathcal{D}_N^{\mathrm{TP}}\right)}{\mathrm{vrad}\left(\mathcal{CP}_N^{\mathrm{b}}\right)} \leq 2e^{1/4}$

Upper and lower bounds in the spirit of Theorem 5 (i.e., valid for all N) can be likewise obtained.

The reader may wonder why we perform our initial analysis on bases of cones rather than working directly with the smaller sets of trace preserving maps. The reason for this is two-fold. First, the bases being homothetic to various sets of states, any information about them is at the same time more readily available and interesting by itself. Second, while we do have – as a consequence of Lemma 1 – nice duality relations between bases of cones, similar results for sets of trace preserving maps are just not true. As a demonstration of that phenomenon we show in Appendix 6.4 that, in contrast to the bases $\mathcal{CP}^{\mathrm{b}}$, the sets $\mathcal{CP}^{\mathrm{TP}}$ are very far from being self-dual in the sense of (22).

5 Conclusions

We derived tight explicit bounds for the effective radius (in the sense of Hilbert-Schmidt volume), or volume radius, of the set of quantum operations acting on density matrices of size N, and for other convex sets of trace preserving maps acting such matrices such as positive, decomposable, PPT inducing or superpositive maps. The novelty of our approach depends on systematic use of duality to derive quantitative estimates, and on technical tools, some of which are not very familiar even in convex analysis.

Since the volume radii of the sets of trace preserving maps that are positive display a different dependendce on the dimensionality than those of the smaller set of decomposable maps, the ratio of the volumes of the latter and the former set tends rapidly to 0 as the dimension increases. In other words, a generic positive trace preserving map is not decomposable and, *a fortiori*, not completely positive. Thus we were able to prove a stronger statement than the one advertised in the title of the paper. Similarly, a generic PPT inducing quantum operation (and, *a fortiori*, a generic quantum operation) is not superpositive. Analogous relations (some of which were known) exist between the sets of states related to those of maps via the Jamiołkowski isomorphism.

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6 Appendices

6.1 Better constants in Theorem 5: mean width, Urysohn inequality and related tools

The arguments given in the preceding section did not yield the asserted values of the constant $\frac{1}{4}$ in part (ii), the constants $\frac{e^{1/4}}{2}$ and $\frac{1}{4}$ in part (iv), and the constants 8 and $2e^{1/4}$ in part (v) of Theorem 5. We will now present the somewhat more involved line of reasoning that does yield these constants.

The following concepts will be helpful in our analysis. If $K \subset \mathbb{R}^m$ is a convex body containing the origin in its interior, one defines the *gauge of* K via

$$\|x\|_K := \inf\{t \geq 0 \,:\, x \in tK\}.$$

Roughly, $\|x\|_K$ is the norm, for which K is the unit ball, except that there is no symmetry requirement. Next, the *mean width* of K (or, more precisely, the mean *half-width*) is defined by

$$w(K) := \int_{S^{m-1}} \|x\|_{K^\circ} dx = \int_{S^{m-1}} \max_{y\in K} \langle x, y\rangle dx$$

(integration with respect to the normalized Lebesgue measure on S^{m-1}). A classical result known as Urysohn's inequality (see, e.g., [41]) asserts then that

$$\mathrm{vrad}(K) \leq w(K). \tag{27}$$

A companion inequality, which is even more elementary, is

$$\mathrm{vrad}(K) \geq w(K^\circ)^{-1}. \tag{28}$$

The proof of (28) is based on expressing the volume as an integral in polar coordinates and then using twice Hölder inequality: $\mathrm{vrad}(K) = \left(\int_{S^{m-1}} \|x\|_K^{-m} dx\right)^{1/m} \geq \int_{S^{m-1}} \|x\|_K^{-1} dx \geq \left(\int_{S^{m-1}} \|x\|_K dx\right)^{-1} = w(K^\circ)^{-1}$.

Applying (27) in our setting of the $N^4 - 1$-dimensional space H^{b} and for $K = \mathcal{D}_N^{\mathrm{b}}$, we obtain

$$\mathrm{vrad}(\mathcal{D}_N^{\mathrm{b}}) \leq w(\mathcal{D}_N^{\mathrm{b}}) = w(\mathrm{conv}(\mathcal{CP}_N^{\mathrm{b}} \cup \mathcal{C}c\mathcal{P}_N^{\mathrm{b}})) \leq w(\mathcal{CP}_N^{\mathrm{b}} + \mathcal{C}c\mathcal{P}_N^{\mathrm{b}}) = 2w(\mathcal{CP}_N^{\mathrm{b}}) \leq 4, \tag{29}$$

because $w(\cdot)$ commutes with the Minkowski addition (of sets), and because $w(\mathcal{CP}_N^{\mathrm{b}}) = w(\mathcal{C}c\mathcal{P}_N^{\mathrm{b}}) \leq 2$. The latter is a consequence of similar estimates for the set of all states (which is equivalent to $\mathcal{CP}_N^{\mathrm{b}}$ up to a homothety), see [14, 16]. (We note that while the *limit relation* $w(\mathcal{CP}_N^{\mathrm{b}}) \to 2$ as $N \to \infty$ follows easily from well-known facts about random matrices, the *estimate valid for all* N requires finer arguments such as those presented in appendices of [14]). Combining the above estimate with part (i) of Theorem 3 we obtain the upper estimate in part (v) with the asserted constant 8.

The same bound $w(\mathcal{D}_N^{\rm b}) \leq 4$ combined with (28) (applied this time with $K = \mathcal{T}_N^{\rm b}$) and with part (i) leads to the lower bound $\frac{1}{4}$ in part (iv).

To obtain the asymptotic bounds from part (iv) and (v) with the required constants $\frac{e^{1/4}}{2}$ and $2e^{1/4}$ we argue similarly, but instead of the universal estimate $w(\mathcal{D}_N^{\rm b}) \leq 4$ we use a tighter asymptotic bound $\limsup_N w(\mathcal{D}_N^{\rm b}) \leq 2$ This bound is a consequence of classical isoperimetric inequalities and the measure concentration phenomenon that they induce (see, e.g., [40]): a Lipschitz function on S^{m-1} is strongly concentrated around its mean. In particular, if the out-radius of K is at most R, then $\int_{S^{m-1}} |\|x\|_{K^\circ} - w(K)|\, dx = O(R/m^{1/2})$. If $K = \mathcal{CP}_N^{\rm b}$ or $\mathcal{C}c\mathcal{P}_N^{\rm b}$, then, by Corollary 3, $R = (N^2-1)^{1/2}$ while $m = N^4 - 1$, hence $R/m^{1/2} = (N^2+1)^{-1/2} < N^{-1}$. It is then an elementary exercise to show that

$$\begin{aligned} w(\mathcal{D}_N^{\rm b}) &= \int_{S^{m-1}} \max\left(\|x\|_{(\mathcal{CP}_N^{\rm b})^\circ}, \|x\|_{(\mathcal{C}c\mathcal{P}_N^{\rm b})^\circ}\right) dx \\ &\leq \max\left(w(\mathcal{CP}_N^{\rm b}), w(\mathcal{C}c\mathcal{P}_N^{\rm b})\right) + O(N^{-1}) \\ &\leq 2 + O(N^{-1}), \end{aligned}$$

whence $\limsup_N w(\mathcal{D}_N^{\rm b}) \leq 2$, as required. Universal (as opposed to asymptotic) upper bounds on $\mathrm{vrad}(\mathcal{D}_N^{\rm b})$ better than 4 obtained in (29) can also be derived this way, most efficiently by converting spherical integrals to Gaussian integrals and using the Gaussian isoperimetric inequality. (This would also improve somewhat the bounds $\frac{1}{4}$ and 8 in parts (iv) and (v), but we will not pursue this direction here as the payoff doesn't seem to justify the effort.)

Finally, to obtain the *lower* bound on $\mathrm{vrad}(\mathcal{P}_N^{\rm b})$ from part (ii) of the Theorem, we note that the upper bound $4N^{-1/2}$ for $\mathrm{vrad}(\mathcal{SP}_N^{\rm b})$ (stated in part (iii)) was *de facto* (see [16]) deduced from the stronger estimate $w(\mathcal{SP}_N^{\rm b}) \leq 4N^{-1/2}$. It then remains to apply (28) and the duality between $\mathcal{P}_N^{\rm b}$ and $\mathcal{SP}_N^{\rm b}$.

6.2 Symmetrized bodies and order intervals

We will now analyze the polar of the symmetrized body $\mathcal{C}^{\rm sym}$. Recall the notation of section 3.2: $\mathcal{V}$ is a real Hilbert space, $\mathcal{C} \subset \mathcal{V}$ a closed convex cone, $\mathcal{C}^*$ the dual cone. Next, $e \in \mathcal{C} \cap \mathcal{C}^*$ is a unit vector, $V^{\rm b} := \{x \in \mathcal{V} \,:\, \langle x, e\rangle = 1\}$ is an affine subspace of $\mathcal{V}$ and $\mathcal{C}^{\rm b} = \mathcal{C} \cap V^{\rm b}$ is the base of the cone $\mathcal{C}$. Finally, the symmetrized body is defined as $\mathcal{C}^{\rm sym} := \mathrm{conv}(-\mathcal{C}^{\rm b} \cup \mathcal{C}^{\rm b})$; the minus sign referring to the symmetric image with respect to 0. An important point, following from classical results [39, 44] and explained in Appendix C of [14], is that under mild assumptions which are satisfied for all the cones we consider, the volume radii of $\mathcal{C}^{\rm b}$ and of $\mathcal{C}^{\rm sym}$ differ by a factor smaller than 2.

Our main assertion (equation (23) in section 3.2) is that

$$(\mathcal{C}^{\mathrm{sym}})^\circ = (e - \mathcal{C}^*) \cap (-e + \mathcal{C}^*),$$

where the polarity has the standard meaning (i.e., inside the entire space $\mathcal{V}$ *and* with respect to the origin). That is, $y \in (\mathcal{C}^{\mathrm{sym}})^\circ$ iff both $y + e$ and $e - y$ are in $\mathcal{C}^*$ or, in other words, iff y belongs to $[-e, e]$, the order interval in the sense of the order induced by the cone $\mathcal{C}^*$. For example, if we want to investigate $\mathcal{P}^{\mathrm{b}}$ and $\mathcal{P}^{\mathrm{sym}} = \mathrm{conv}\left(-\mathcal{P}^{\mathrm{b}} \cup \mathcal{P}^{\mathrm{b}}\right)$, we may specify the framework above to $\mathcal{C} = \mathcal{P}$, obtaining

$$(\mathcal{P}^{\mathrm{sym}})^\circ = \big(-\Phi_* + \mathcal{S}\mathcal{P}\big) \cap \big(\Phi_* - \mathcal{S}\mathcal{P}\big).$$

To prove the assertion, denote $V^- := \{x \in \mathcal{V} \ : \ \langle x, e\rangle \le 1\}$ (one of the half-spaces determined by V^{b}) and $\mathcal{C}^{\mathrm{E}} = \mathcal{C} \cap V^-$ (cf. Figure 2 in section 3). Then

$$\mathcal{C}^{\mathrm{sym}} = \mathrm{conv}(-\mathcal{C}^{\mathrm{b}} \cup \mathcal{C}^{\mathrm{b}}) = \mathrm{conv}\left(-\mathcal{C}^{\mathrm{E}} \cup \mathcal{C}^{\mathrm{E}}\right).$$

Hence, using standard rules for polar operations (see, e.g., [41]),

$$(\mathcal{C}^{\mathrm{sym}})^\circ = \big(-\mathcal{C}^{\mathrm{E}}\big)^\circ \cap \big(\mathcal{C}^{\mathrm{E}}\big)^\circ.$$

Next,

$$\big(\mathcal{C}^{\mathrm{E}}\big)^\circ = \big(\mathcal{C} \cap V^-\big)^\circ = \overline{\mathrm{conv}}\left((V^-)^\circ \cup \mathcal{C}^\circ\right) = \overline{\mathrm{conv}}\left((-\infty, 1] \cdot e \ \cup \ -\mathcal{C}^*\right) = e - \mathcal{C}^*,$$

where the bar stands for the closure. Combining this with the preceding formula and again using the standard rules gives

$$(\mathcal{C}^{\mathrm{sym}})^\circ = (e - \mathcal{C}^*) \cap (-e + \mathcal{C}^*)$$

or the intersection of two cones with vertices at e and $-e$. Clearly this does not equal $(\mathcal{C}^*)^{\mathrm{sym}}$ except in dimension 1. However, the two bodies are closely related. For example, if e is the point of symmetry of $\mathcal{C}^{\mathrm{b}}$, then $(\mathcal{C}^*)^{\mathrm{sym}}$ is a cylinder with the base $(\mathcal{C}^*)^{\mathrm{b}}$ and the axis $[-e, e]$, while $(\mathcal{C}^{\mathrm{sym}})^\circ$ is a union of two cones whose common base is $(\mathcal{C}^*)^{\mathrm{b}} - e$, the central section of the cylinder, and the vertices are $-e$ and e. The two bodies only differ in one dimension; if thought of as unit balls with respect to the corresponding norms, the two norms coincide on the hyperspace $\mathcal{V}_0 := \{x \in \mathcal{V} \ : \ \langle x, e\rangle = 0\}$ and on the complementary one-dimensional space $\mathbb{R}e$, but on the entire space we have in the first case the direct sum in the ℓ_∞ sense, while in the second case in the ℓ_1 sense. If the base $\mathcal{C}^{\mathrm{b}}$ is non-symmetric, the situation is more complicated. For example, the section $\mathcal{V}_0 \cap (\mathcal{C}^{\mathrm{sym}})^\circ$ is congruent to the intersection of $(\mathcal{C}^*)^{\mathrm{b}}$ with its symmetric image with respect to e, but (see [42]) the volume radii of the two bodies are comparable if, for example, e is the only point that is fixed under isometries of $(\mathcal{C}^*)^{\mathrm{b}}$ (as is the case in all our applications), or just the centroid of $(\mathcal{C}^*)^{\mathrm{b}}$.

6.3 Proof of Proposition 2: for "balanced" cones, $\mathcal{C}^{\mathrm{b}}$ and $\mathcal{C}^{\mathrm{TP}}$ have comparable volume radius

We may assume that $a = 0$ (otherwise consider $K - a$). By hypothesis, we have then

$$rB_2^m \subset K \subset RB_2^m, \tag{30}$$

where B_2^m is the m-dimensional unit Euclidean ball. For a subspace E, denote by P_E the orthogonal projection onto E. Then (see [42, 43]),

$$\mathrm{vol}_m(K) \leq \mathrm{vol}_k(K \cap H) \ \ \mathrm{vol}_s(P_{H^\perp} K), \tag{31}$$

where $s = m - k$ and $H^\perp$ is the $m-k$-dimensional space orthogonal to the k-dimensional subspace H. Therefore

$$\frac{\mathrm{vol}_m(K)}{\mathrm{vol}_m(B_2^m)} \leq \frac{\mathrm{vol}_k(K \cap H)}{\mathrm{vol}_k(B_2^k)} \ \frac{\mathrm{vol}_s(P_{H^\perp} K)}{\mathrm{vol}_s(B_2^s)} \ \frac{\mathrm{vol}_k(B_2^k)\mathrm{vol}_s(B_2^s)}{\mathrm{vol}_m(B_2^m)}$$

Hence, using (30),

$$\mathrm{vrad}(K)^m \ \leq \ \mathrm{vrad}(K \cap H)^k \ R^s \ \frac{\mathrm{vol}_k(B_2^k) \ \mathrm{vol}_s(B_2^s)}{\mathrm{vol}_m(B_2^m)},$$

which is the first inequality in (26). For the second inequality, we start with the even more classical result (see [44] or [45]; same notation as (31))

$$\mathrm{vol}_m(K) \geq \binom{m}{k}^{-1} \mathrm{vol}_k(K \cap H) \ \mathrm{vol}_s(P_{H^\perp} K), \tag{32}$$

which doesn't even require that H passes through the centroid of K. As above, this can be rewritten in terms of volume radii as

$$\binom{m}{k} \mathrm{vrad}(K)^m \ \geq \ \mathrm{vrad}(K \cap H)^k \ r^s \ \frac{\mathrm{vol}_k(B_2^k) \ \mathrm{vol}_s(B_2^s)}{\mathrm{vol}_d(B_2^m)},$$

which is the second inequality in (26).

6.4 "No duality" for $\mathcal{CP}_N^{\mathrm{TP}}$

The purpose of this Appendix is to show that, in contrast to the bases $\mathcal{CP}^{\mathrm{b}}$, the sets $\mathcal{CP}^{\mathrm{TP}}$ are very far from being self-dual in the sense of (22), that is, that the polar of $\mathcal{CP}^{\mathrm{TP}}$ inside the space defined by the trace preserving condition (5) considered as a vector space with Φ_* as the origin is quite different from the reflection of $\mathcal{CP}^{\mathrm{TP}}$ with respect to Φ_*.

Generally, if $K \subset \mathbb{R}^m$ is a convex body containing the origin in its interior and $H \subset \mathbb{R}^m$ is a vector subspace, $K^\circ \cap H$ is always contained in the polar of $K \cap H$ inside H, and the discrepancy between the two (i.e., the smallest constant $\lambda \geq 1$ such that the polar of $K \cap H$ is contained in $\lambda(K^\circ \cap H)$) is the same as the discrepancy between $K \cap H$ and the orthogonal projection of K onto H. That discrepancy is also equal to the maximal ratio between

$$\max_{x \in K} \langle u, x \rangle \quad \text{and} \quad \max_{y \in K \cap H} \langle u, y \rangle \tag{33}$$

over nonzero vectors $u \in H$.

In our case $K = \mathcal{CP}_N^{\mathrm{b}}$ and $K \cap H = \mathcal{CP}_N^{\mathrm{TP}}$. As a vector space, H may be identified with maps whose dynamical matrix has partial trace equal to 0. We will argue in the language of dynamical (Choi) matrices considered as "flat" block matrices. In these terms, membership in H is equivalent to each block being of trace 0. We will choose as u the block matrix whose ${}_{11}$-th block is $U = E_{11} - N^{-1} I_N$ and the remaining blocks are 0. Further, we will choose as x the matrix whose ${}_{11}$-th block is $X = N E_{11}$ and the remaining blocks are 0; then the scalar product corresponding to $\langle u, x \rangle$ is $\mathrm{tr}(UX) = N - 1$. On the other hand, if Y is the ${}_{11}$-th block of the Choi matrix of any element of $\mathcal{CP}_N^{\mathrm{TP}}$, then Y is a state and so the scalar product corresponding to $\langle u, y \rangle$ is $\mathrm{tr}(UY) = \mathrm{tr}(E_{11} Y) - N^{-1} \mathrm{tr} Y \leq \mathrm{tr} Y - N^{-1} \mathrm{tr} Y = 1 - N^{-1}$. Accordingly, the discrepancy between the two maxima in (33) is at least $(N-1)/(1-N^{-1}) = N$.

6.5 Volume radius of the set of trace non increasing maps

We want to determine the asymptotic order of the volume radius of the set of all completely postive, trace non increasing maps $\Phi : \mathcal{M}_N \to \mathcal{M}_N$ i.e. the set

$$\mathcal{CP}_N^{\mathrm{TNI}} := \{\Phi \in \mathcal{CP}_N : \ \mathrm{Tr}\, \Phi(\rho) \leq \mathrm{Tr}\, \rho \ \text{ for all } \ \rho \geq 0\}.$$

As pointed out earlier, an exact formula for that volume was very recently found (independently from this work and by a different method) in [30]. However, an argument using the approach of this paper is conceptually very simple and so we include it. We have

Proposition 8 *We have, for all N,*

$$\big(e\, N^{5/2}\big)^{-N^2} \leq \frac{\mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{TNI}}\big)}{\mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{TP}}\big) \times \mathrm{vol}\big(\{M \in \mathcal{M}_N : 0 \leq M \leq I_N\}\big)} \leq N^{-N^2/2} \tag{34}$$

To derive estimates on $\mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{TNI}}\big)$ from the Proposition, one needs to use the readily available information on the two factors in the denominator of the middle term of (34). First, the asymptotic order of the volume radius of $\mathcal{CP}_N^{\mathrm{TP}}$ was determined in Theorem

7(i). Next, the set $\mathcal{A} := \{M \in \mathcal{M}_N : 0 \leq M \leq I_N\}$ is a ball of radius 1/2 (in the operator norm) centered at $I_N/2$, and so its volume radius admits easy bounds given by the in- and outradius: 1/2 and $\sqrt{N}/2$ (actually a much tighter lower bound $\sqrt{N}/4$ can be obtained via a slight modification of the argument from Theorem 5(i), see Appendix 6.1, but for our purposes the trivial bounds suffice). A straighforward calculation leads then to

Corollary 9

$$\lim_{N\to\infty} \mathrm{vrad}\big(\mathcal{CP}_N^{\mathrm{TNI}}\big) = e^{-1/4}.$$

The key point is that in order to calculate the volume radius we need to raise the volume to the power $1/N^4$. Thus the factors such as $(e\,N^{5/2})^{-N^2}$ on the left hand side of (34) are inconsequential since it leads to an expression of the form $1 - O(\frac{\log N}{N^2})$. For the same reason, the effects of vol$(\mathcal{A})$ and of the $b(m,k)$-type factor, which also enters the calculation (cf. Proposition 6 and the comments following it), tend to 0 as $N \to \infty$.

For the proof of Proposition 8 we note first that $\mathcal{CP}_N^{\mathrm{TNI}}$ is canonically isometric to the set of *subunital* maps

$$\mathcal{CP}_N^{\mathrm{SU}} := \{\Phi \in \mathcal{CP} : \ \Phi(I_N) \leq I_N\}.$$

In what follows we will work with the latter set. The isometry, which assigns to $\Phi : \mathcal{M}_N \to \mathcal{M}_N$ the dual (in the linear algebra, or Banach space sense) map Φ^*, sends $\mathcal{CP}_N^{\mathrm{TP}}$ to the set of *unital* maps $\mathcal{CP}_N^{\mathrm{U}} := \{\Phi \in \mathcal{CP}_N : \ \Phi(I_N) = I_N\}$. The set $\mathcal{CP}_N^{\mathrm{SU}}$ admits a natural fibration: with every $M \in \mathcal{A}$, we may associate

$$F_M = \{\Phi \in \mathcal{CP} : \ \Phi(I_N) = M\}; \tag{35}$$

in particular $F_{I_N} = \mathcal{CP}_N^{\mathrm{U}}$. In the language of Choi (dynamical) matrices the codition from (35) translates to $\mathrm{tr}_B D_\Phi = M$, or to $\sum_j D_{jj} = M$ if we think of D_Φ as a block matrix $D_\Phi = (D_{jk}) = \big(\Phi(E_{jk})\big)$. Since all fibers F_M are parallel to the subspace $\mathcal{N}$ defined by $\mathrm{tr}_B D_\Phi = 0$ (or $\sum_j D_{jj} = 0$), one can express the volume as an integral

$$\mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{TNI}}\big) = \mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{SU}}\big) = N^{-N^2/2} \int_{\mathcal{A}} \mathrm{vol}\big(F_M\big)\, dM \tag{36}$$

The reason for the factor $N^{-N^2/2}$ is that while the fibration is naturally parametrized by the elements of $\mathcal{A}$, the projection of F_M onto $\mathcal{N}^\perp$ is actually the map $\rho \to N^{-1}\mathrm{tr}(\rho)M$, whose Choi matrix is $N^{-1}I_N \otimes M$ (or a block matrix whose all diagonal blocks are M/N and off-diagonal blocks are 0). Now, the Hilbert-Schmidt norm of $N^{-1}I_N$ is $N^{-1/2}$, and so the projection of $\mathcal{CP}_N^{\mathrm{SU}}$ onto $\mathcal{N}^\perp$ is isometric to $N^{-1/2}\mathcal{A}$.

The second inequality in (34) is now an immediate consequence of (36) and the bound $\mathrm{vol}\big(F_M\big) \leq \mathrm{vol}\big(F_{I_N}\big) = \mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{U}}\big) = \mathrm{vol}\big(\mathcal{CP}_N^{\mathrm{TP}}\big)$, valid for all $M \in \mathcal{A}$, which in turn follows, e.g., from F_M being the image of F_{I_N} under the contraction $g_M : \Phi(\cdot) \to M^{\frac{1}{2}}\Phi(\cdot)M^{\frac{1}{2}}$. (On the level of dynamical matrices, the action of g_M is given by $D_\Phi \to \big(I_N \otimes M^{\frac{1}{2}}\big)D_\Phi\big(I_N \otimes M^{\frac{1}{2}}\big)$ or, in the language of block matrices, by $(D_{jk}) \to (M^{\frac{1}{2}}D_{jk}M^{\frac{1}{2}})$.) The fact that $g_M\big(F_{I_N}\big) \subset F_M$ is obvious from the definition; surjectivity for invertible M's follows by considering the inverse $g_M^{-1} = g_{M^{-1}}$, and for singular M's by looking at invertible approximants.

For the first inequality in (34) we may use (32) with $K = \mathcal{CP}_N^{\mathrm{SU}}$ and the section $K \cap H = \mathcal{CP}_N^{\mathrm{U}}$. As pointed out earlier, the set $P_{H^\perp}K = P_{\mathcal{N}^\perp}K$ is then isometric to $N^{-1/2}\mathcal{A}$, and it remains to use the elementary bound $\binom{m}{k} = \binom{m}{s} \le (em/s)^s$. An alternative argument is to restrict the integration in (36) to $\{M : tI_N \le M \le I_N\}$, which is a ball in the operator norm of radius $(1-t)/2$, then use the fact that for such M the function cg_M^{-1} is a contraction, and finally optimize over $t \in (0,1)$. This approach allows in fact to express the Jacobian of g_M in terms of eigenvalues of M and, subsequently, to express the ratio under consideration as a multiple integral over $[0,1]^N$, but we will not pursue this path further.

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