Polytopes with Vertices Chosen Randomly from the Boundary of a Convex Body

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Summary. Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}_+$ be a continuous, positive function with $\int_{\partial K} f(x) \mathrm{d}\mu_{\partial K}(x) = 1$ where $\mu_{\partial K}$ is the surface measure on ∂K. Let $\mathbb{P}_f$ be the probability measure on ∂K given by $\mathrm{d}\mathbb{P}_f(x) = f(x)\mathrm{d}\mu_{\partial K}(x)$. Let κ be the (generalized) Gauß-Kronecker curvature and $\mathbb{E}(f, N)$ the expected volume of the convex hull of N points chosen randomly on ∂K with respect to $\mathbb{P}_f$. Then, under some regularity conditions on the boundary of K

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f,N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x),$$

where c_n is a constant depending on the dimension n only.

The minimum at the right-hand side is attained for the normalized affine surface area measure with density

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)}.$$

Contents

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1 Introduction

1.1 Notation and Background. The Main Theorem

How well can a convex body be approximated by a polytope?

This is a central question in the theory of convex bodies, not only because it is a natural question and interesting in itself but also because it is relevant in many applications, for instance in computervision ([SaT1], [SaT2]), tomography [Ga], geometric algorithms [E].
We recall that a convex body K in $\mathbb{R}^n$ is a compact, convex subset of $\mathbb{R}^n$ with non-empty interior and a polytope P in $\mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$.

As formulated above, the question is vague and we need to make it more precise.
Firstly, we need to clarify what we mean by "approximated". There are many metrics which can and have been considered. For a detailed account concerning these metrics we refer to the articles by Gruber [Gr1],[Gr3]. We will concentrate here on the symmetric difference metric d_s which measures the distance between two convex bodies C and K through the volume of the difference set

$$d_s(C,K) = \mathrm{vol}_n(C \triangle K) = \mathrm{vol}_n((C \setminus K) \cup (K \setminus C)).$$

Secondly, various assumptions can be made and have been made on the approximating polytopes P. For instance, one considers only polytopes contained in K or only polytopes containing K, polytopes with a fixed number of verices, polytopes with a fixed number of facets, etc. Again we refer to the articles [Gr1],[Gr3] for details.

We will concentrate here on the question of approximating a convex body K in $\mathbb{R}^n$ by inscribed polytopes P_N with a fixed number of vertices N in the d_s metric. As we deal with inscribed polytopes the d_s metric reduces to the volume difference

$$\mathrm{vol}_n(K) - \mathrm{vol}_n(P_N)$$

and we ask how the (optimal) dependence is in this metric on the various paramenters involved like the dimension n, the number of vertices N and so on.

As a first result in this direction we want to mention a result by Bronshteyn and Ivanov [BrI].
There is a numerical constant $c > 0$ such that for every convex body K in $\mathbb{R}^n$ which is contained in the Euclidean unit ball and for every $N \in \mathbb{N}$ there exists a polytope $P_N \subseteq K$ with N vertices such that

$$\mathrm{vol}_n(K) - \mathrm{vol}_n(P_N) \leq c \, \frac{n \ \mathrm{vol}_n(K)}{N^{\frac{2}{n-1}}}.$$

The dependence on N and n in this result is optimal. This can be seen from the next two results. The first is due to Macbeath and says that the Euclidean unit ball B_2^n is worst approximated in the d_s metric by polytopes or more precisely [Ma]:
For every convex body K in $\mathbb{R}^n$ with $vol_n(K) = vol_n(B_2^n)$ we have

$$\inf \{d_s(K, P_N) : P_N \subseteq K \text{ and } P_N \text{ has at most N vertices}\} \leq$$

$$\inf \{d_s(B_2^n, P_N) : P_N \subseteq B_2^n \text{ and } P_N \text{ has at most N vertices}\}.$$

Notice that $\inf \{d_s(K, P_N) : P_N \subseteq K$ and P_N has at most N vertices$\}$ is the d_s-distance of the best approximating inscribed polytope with N vertices to K. By a compactness argument such a best approximating polytope exists always.

Hence to get an estimate from below for the Bronshteyn Ivanov result, it is enough to check the Euclidean unit ball which was done by Gordon, Reisner and Schütt [GRS1], [GRS2].
There are two positive constants a and b such that for all $n \geq 2$, every $N \geq (bn)^{\frac{2}{n+1}}$, every polytope $P_N \subseteq B_2^n$ with at most N vertices one has

$$\mathrm{vol}_n(B_2^n) - \mathrm{vol}_n(P_N) \geq a \, \frac{n \ \mathrm{vol}_n(B_2^n)}{N^{\frac{2}{n-1}}}.$$

Thus the optimal dependence on the dimension is n and on N it is $N^{\frac{2}{n-1}}$. The next result is about best approximation for large N.

Let K be a convex body in $\mathbb{R}^n$ with C^2-boundary ∂K and everywhere strictly positive curvature κ. Then

$$\lim_{N\to\infty} \frac{\inf\{d_s(K,P_N)|P_N \subseteq K \text{ and } P_N \text{ has at most N vertices}\}}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}}$$

$$= \tfrac{1}{2}\mathrm{del}_{n-1}\left(\int_{\partial K} \kappa(x)^{\frac{1}{n-1}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}.$$

This theorem was proved by McClure and Vitale [McV] in dimension 2 and by Gruber [Gr2] for general n. On the right hand side of the above equation we find the expression $\int_{\partial K} \kappa(x)^{\frac{1}{n-1}} d\mu_{\partial K}(x)$ which is an affine invariant, the so called affine surface area of K which "measures" the boundary behaviour of K. It is natural that such a term should appear in questions of approximation of convex bodies by polytopes. Intuitively we expect that more vertices of the approximating polytope should be put where the boundary of K is very curved and fewer points should be put where the boundary of K is flat to get a good approximation in the d_s-metric. In Section 1.3 we will discuss the affine surface in more detail.

del_{n-1}, which also appears on the right hand side of the above formula, is a constant that depends on n only. The value of this constant is known for for $n = 2, 3$. Putting for K the Euclidean unit ball in the last mentioned theorem, it follows from the result above by Gordon, Reisner and Schütt [GRS1], [GRS2] that del_{n-1} is of the order n. del_{n-1} was determined more precisely by Mankiewicz and Schütt [MaS1], [MaS2]. We refer to Section 1.4. for the exact statements.

Now we want to come to approximation of convex bodies by random polytopes.

A random polytope is the convex hull of finitely many points that are chosen from K with respect to a probability measure $\mathbb{P}$ on K. The expected volume of a random polytope of N points is

$$\mathbb{E}(\mathbb{P}, N) = \int_K \cdots \int_K \mathrm{vol}_n([x_1, \dots, x_N])\mathrm{d}\mathbb{P}(x_1)\dots\mathrm{d}\mathbb{P}(x_N)$$

where $[x_1, \dots, x_N]$ is the convex hull of the points $x_1, \dots, x_N$. Thus the expression $\mathrm{vol}_n(K) - \mathbb{E}(\mathbb{P}, N)$ measures how close a random polytope and the convex body are in the symmetric difference metric. Rényi and Sulanke [ReS1], [ReS2] have investigated this expression for large numbers N of chosen points. They restricted themselves to dimension 2 and the case that the probability measure is the normalized Lebesgue measure on K.

Their results were extended to higher dimensions in case that the probability measure is the normalized Lebesgue measure. Wieacker [Wie] settled the case of the Euclidean ball in dimenision n. Bárány proved the result for convex bodies with C^3-boundary and everywhere positive curvature [Ba1]. This result was generalized to arbitrary convex bodies in [Sch1] (see also Section 1.4):

Let K be a convex body in $\mathbb{R}^n$. Then

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(\mathbb{P}_m, N)}{\left(\frac{\mathrm{vol}_n(K)}{N}\right)^{\frac{2}{n+1}}} = c_1(n) \int_{\partial K} \kappa(x)^{\frac{1}{n-1}} \mathrm{d}\mu_{\partial K}(x),$$

where $c_1(n)$ *is a constant that depends on* n*.*

We can use this result to obtain an approximation of a convex body by a polytope with at most N vertices. Notice that this does not give the optimal dependence on N. One of the reasons is that not all the points chosen at random from K appear as vertices of the approximating random polytope. We will get back to this point in Section 1.4.

One avoids this problem that not all points chosen appear as vertices of the random polytope by choosing the points at random directly on the boundary of the convex body K.

This is what we do in this paper. We consider convex bodies in dimension n and probability measures that are concentrated on the boundary of the convex body. It is with respect to such probability measures that we choose the points at random on the boundary of K and all those points will then be vertices of the random polytope. This had been done before only in the case of the Euclidean ball by Müller [Mü] who proved the asymptotic formula for the Euclidean ball with the normalized surface measure as probability measure.

Here we treat much more general measures $\mathbb{P}_f$ defined on the boundary of K where we only assume that the measure has a continuous density f with respect to the surface measure $\mu_{\partial K}$ on ∂K. Under some additional technical assumptions we prove an asymptotic formula. This is the content of Theorem 1.1.

In the remainder of Section 1.1 we will introduce further notation used throughout the paper. We conclude Section 1.1 by stating the Theorem 1.1. The whole paper is devoted to prove this main theorem. In doing that, we develop tools that should be helpful in further investigations.

In Section 1.2 we compute which is the optimal f to give the least value in the volume difference

$$\mathrm{vol}_n(K) - \mathbb{E}(\mathbb{P}_f, N).$$

It will turn out that the affine surface area density gives the optimal measure: Choosing points according to this measure gives random polytopes of greatest possible volume. Again, this is intuitively clear: An optimal measure should put more weight on points with higher curvature. Moreover, and this is a crucial observation, if the optimal measure is unique then it must be affine invariant. There are not too many such measures and the affine surface measure is the first that comes to ones mind. This measure satisfies two necessary properties: It is affine invariant and it puts more weight on points with greater curvature.

In Section 1.5 we compare random approximation with best approximation and observe a remarkable fact. Namely, it turns out that -up to a numerical constant- random approximation with the points chosen $\mathbb{P}_f$-randomly from the boundary of K with the optimal f is as good as best approximation.

In Section 1.3 we propose an extension of the p-affine surface area which was introduced by Lutwak [Lu] and Hug [Hu]. We also give a geometric interpretation of the p-affine surface area in terms of random polytopes.

It was a crucial step in the proof of Theorem 1.1 to relate the random polytope to a geometric object. The appropriate geometric object turned out to be the surface body which we introduce in Chapter 2.

In Chapter 3 we review J. Müller's proof for the case of the Euclidean ball. We use his result in our proof.

Chapter 4 is devoted to prove probabilistic inequalities needed for the proof of Theorem 1.1 and finally Chapter 5 gives the proof of Theorem 1.1.

Now we introduce further notations used throughout the paper.

$B_2^n(x,r)$ is the Euclidean ball in $\mathbb{R}^n$ centered at x with radius r. We denote $B_2^n = B_2^n(0,1)$. S^{n-1} is the boundary ∂B_2^n of the Euclidean unit ball. The norm $\|\cdot\|$ is the Euclidean norm.

The distance $d(A,B)$ of two sets in $\mathbb{R}^n$ is

$$d(A,B) = \inf\{\|x-y\| | x \in A, y \in B\}.$$

For a convex body K the metric projection $p: \mathbb{R}^n \to K$ maps x onto the unique point $p(x) \in K$ with

$$\|x - p(x)\| = \inf_{y \in K} \|x-y\|.$$

The uniqueness of the point $p(x)$ follows from the convexity of K. If $x \in K$ then $p(x) = x$.

For x, ξ in $\mathbb{R}^n$, $\xi \neq 0$, $H(x,\xi)$ denotes the hyperplane through x and orthogonal to ξ. The two closed halfspaces determined by this hyperplane are denoted by $H^-(x,\xi)$ and $H^+(x,\xi)$. $H^-(x,\xi)$ is usually the halfspace that contains $x+\xi$. Sometimes we write H, H^+ and H^-, if it is clear which are the vectors x and ξ involved.

For points $x_1, \dots x_N \in \mathbb{R}^n$ we denote by

$$[x_1, \dots x_N] = \left\{ \lambda_1 x_1 + \cdots + \lambda_N x_N \,\middle|\, 0 \leq \lambda_i \leq 1, 1 \leq i \leq N, \sum_{i=1}^N \lambda_i = 1 \right\}$$

the convex hull of these points. In particular, the closed line segment between two points x and y is

$$[x,y] = \{\lambda x + (1-\lambda)y | \ 0 \leq \lambda \leq 1\}.$$

The open line segment is denoted by

$$(x,y) = \{\lambda x + (1-\lambda)y | \ 0 < \lambda < 1\}.$$

$\mu_{\partial K}$ is the surface area measure on ∂K. It equals the restriction of the $n-1$-dimensional Hausdorff measure to ∂K. We write in short μ if it is clear which is the body K involved. Let $f : \partial K \to \mathbb{R}$ be a integrable, nonnegative function with

$$\int_{\partial K} f(x) \mathrm{d}\mu = 1.$$

Then we denote by $\mathbb{P}_f$ the probability measure with $\mathrm{d}\mathbb{P}_f = f \mathrm{d}\mu_{\partial K}$ and $\mathbb{E}(f, N) = \mathbb{E}(\mathbb{P}_f, N)$. If f is the constant function $(\mathrm{vol}_{n-1}(\partial K))^{-1}$ then we write $\mathbb{E}(\partial K, N) = \mathbb{E}(\mathbb{P}_f, N)$. For a measurable subset A of ∂K we write $\mathrm{vol}_{n-1}(A)$ for $\mu_{\partial K}(A)$.

Let K be a convex body in $\mathbb{R}^n$ with boundary ∂K. For $x \in \partial K$ we denote the outer unit normal by $N_{\partial K}(x)$. We write in short $N(x)$ if it is clear which is the body K involved. The normal $N(x)$ may not be unique. $\kappa_{\partial K}(x)$ is the (generalized) Gauß curvature at x (see also Section 1.5 for the precise definition). By a result of Aleksandrov [Al] it exists almost everywhere. Again, we write in short $\kappa(x)$ if it is clear which is the body K involved. The centroid or center of mass *cen* of K is

$$\mathrm{cen} = \frac{\int_K x \mathrm{d}x}{\mathrm{vol}_n(K)}.$$

We conclude Section 1.1 with the main theorem.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$ such that there are r and R in $\mathbb{R}$ with $0 < r \leq R < \infty$ so that we have for all $x \in \partial K$*

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

and let $f : \partial K \to \mathbb{R}_+$ be a continuous, positive function with $\int_{\partial K} f(x) \mathrm{d}\mu_{\partial K}(x) = 1$. Let $\mathbb{P}_f$ be the probability measure on ∂K given by $\mathrm{d}\mathbb{P}_f(x) = f(x) \mathrm{d}\mu_{\partial K}(x)$. Then we have

$$\lim_{N \to \infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} \mathrm{d}\mu_{\partial K}(x)$$

where κ is the (generalized) Gauß-Kronecker curvature and

$$c_n = \frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n + 1 + \frac{2}{n-1}\right)}{2(n+1)! (\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}.$$

The minimum at the right-hand side is attained for the normalized affine surface area measure with density

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)}.$$

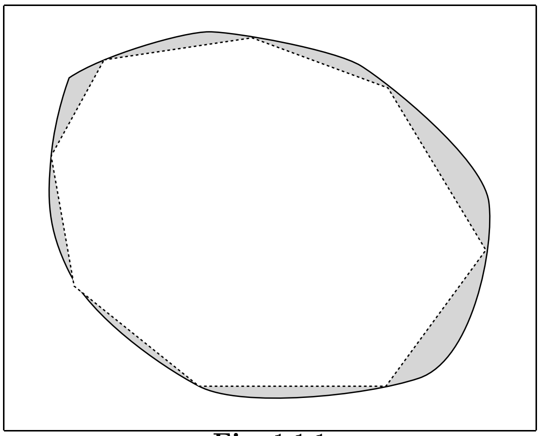

Fig. 1.1.1

The condition: There are r and R in $\mathbb{R}$ with $0 < r \leq R < \infty$ so that we have for all $x \in \partial K$

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

is satisfied if K has a C^2-boundary with everywhere positive curvature. This follows from Blaschke's rolling theorem ([Bla2] , p.118) and a generalization of it ([Lei], Remark 2.3). Indeed, we can choose

$$r = \min_{x \in \partial K} \min_{1 \leq i \leq n-1} r_i(x) \qquad\qquad R = \max_{x \in \partial K} \max_{1 \leq i \leq n-1} r_i(x)$$

where $r_i(x)$ denotes the i-th principal curvature radius.

By a result of Aleksandrov [Al] the generalized curvature κ exists a.e. on every convex body. It was shown in [SW1] that $\kappa^{\frac{1}{n+1}}$ is an integrable function. Therefore the density

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)}.$$

exists provided that $\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x) > 0$. This is certainly assured by the assumption on the boundary of K.

1.2 Discussion of some Measures $\mathbb{P}_f$ and the Optimality of the Affine Surface Area Measure

We want to discuss some measures that are of interest.

1. The most interesting measure is the normalized affine surface area measure as given in the theorem. This measure is affine invariant, i.e. for an affine, volume preserving map T and all measurable subsets A of ∂K

$$\int_A \kappa_{\partial K}^{\frac{1}{n+1}}(x) \mathrm{d}\mu_{\partial K}(x) = \int_{T(A)} \kappa_{\partial T(K)}^{\frac{1}{n+1}}(x) \mathrm{d}\mu_{\partial T(K)}(x).$$

Please note that if the optimal measure is unique it should be affine invariant since the image measure induced by T must also be optimal.

We show that the measure is affine invariant. To do so we introduce the convex floating body. For $t \in \mathbb{R}$, $t > 0$ sufficiently small, the convex floating body $C_{[t]}$ of a convex body C [SW1] is the intersection of all halfspaces whose defining hyperplanes cut off a set of n-dimensional volume t from C. By [SW1] we have for all convex bodies C

$$\lim_{t\to 0} \frac{\mathrm{vol}_n(C) - \mathrm{vol}_n(C_{[t]})}{t^{\frac{2}{n+1}}} = d_n \int_{\partial C} \kappa_{\partial C}(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial C}(x),$$

where $d_n = \frac{1}{2}\left(\frac{n+1}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{2/(n+1)}$. For an affine, volume preserving map T we have

$$\mathrm{vol}_n(C) = \mathrm{vol}_n(T(C)) \quad \text{and} \quad \mathrm{vol}_n(C_{[t]}) = \mathrm{vol}_n(T(C_{[t]})). \tag{1}$$

Thus the expression

$$\int_{\partial C} \kappa_{\partial C}(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial C}(x)$$

is affine invariant. For a closed subset A of ∂K where K is a convex body, we define the convex body C as the convex hull of A. For a point $x \in \partial C$ with $x \notin A$ we have that the curvature must be 0 if it exists. Thus we get by the affine invariance (1) for all closed sets A

$$\int_A \kappa_{\partial C}(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial C}(x) = \int_{\partial T(A)} \kappa_{\partial T(C)}(y)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial T(C)}(y).$$

This formula extends to all measurable sets. For the affine surface measure we get

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f,N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}. \tag{2}$$

We show now that the expression for any other measure given by a density f is greater than or equal to (2). Since $\int_{\partial K} f(x)\mathrm{d}\mu_{\partial K}(x) = 1$, we have

$$\begin{aligned}
&\left(\frac{1}{\mathrm{vol}_{n-1}(\partial K)} \int_{\partial K} \left|\frac{\kappa(x)}{f(x)^2}\right|^{\frac{1}{n-1}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{1}{n+1}} \\
&= \left(\frac{1}{\mathrm{vol}_{n-1}(\partial K)} \int_{\partial K} \left|\left(\frac{\kappa(x)}{f(x)^2}\right)^{\frac{1}{n^2-1}}\right|^{n+1} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{1}{n+1}} \times \\
&\left(\frac{1}{\mathrm{vol}_{n-1}(\partial K)} \int_{\partial K} \left|f(x)^{\frac{2}{n^2-1}}\right|^{\frac{n^2-1}{2}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{2}{n^2-1}} (\mathrm{vol}_{n-1}(\partial K))^{\frac{2}{n^2-1}}.
\end{aligned}$$

We have $\frac{1}{n+1}+\frac{2}{n^2-1}=\frac{1}{n-1}$ and we apply Hölder inequality to get

$$\left(\frac{1}{\mathrm{vol}_{n-1}(\partial K)}\int_{\partial K}\left|\frac{\kappa(x)}{f(x)^2}\right|^{\frac{1}{n-1}}\mathrm{d}\mu_{\partial K}(x)\right)^{\frac{1}{n+1}}$$
$$\geq\left(\frac{1}{\mathrm{vol}_{n-1}(\partial K)}\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\mathrm{d}\mu_{\partial K}(x)\right)^{\frac{1}{n-1}}(\mathrm{vol}_{n-1}(\partial K))^{\frac{2}{n^2-1}},$$

which gives us

$$\int_{\partial K}\left|\frac{\kappa(x)}{f(x)^2}\right|^{\frac{1}{n-1}}\mathrm{d}\mu_{\partial K}(x)\geq\left(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}.$$

2. The second measure of interest is the surface measure given by the constant density

$$f(x)=\frac{1}{\mathrm{vol}_{n-1}(\partial K)}.$$

This measure is not affine invariant and we get

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{\left(\frac{\mathrm{vol}_{n-1}(\partial K)}{N}\right)^{\frac{2}{n-1}}}=c_n\int_{\partial K}\kappa(x)^{\frac{1}{n-1}}\mathrm{d}\mu_{\partial K}(x).$$

3. The third measure is obtained in the following way. Let K be a convex body, *cen* its centroid and A a subset of ∂K. Let

$$\mathbb{P}(A)=\frac{\mathrm{vol}_n([\mathrm{cen},A])}{\mathrm{vol}_n(K)}.$$

If the centroid is the origin, then the density is given by

$$f(x)=\frac{<x,N_{\partial K}(x)>}{\int_{\partial K}<x,N_{\partial K}(x)>\mathrm{d}\mu_{\partial K}(x)}$$

and the measure is invariant under linear, volume preserving maps. We have $\frac{1}{n}\int_{\partial K}<x,N(x)>\mathrm{d}\mu_{\partial K}(x)=\mathrm{vol}_n(K)$ and thus

$$f(x)=\frac{<x,N_{\partial K}(x)>}{n\ \mathrm{vol}_n(K)}.$$

We get

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{\left(\frac{n\ \mathrm{vol}_n(K)}{N}\right)^{\frac{2}{n-1}}}=c_n\int_{\partial K}\frac{\kappa(x)^{\frac{1}{n-1}}}{<x,N_{\partial K}(x)>^{\frac{2}{n-1}}}\mathrm{d}\mu_{\partial K}(x).$$

We recall that for $p>0$ the p-affine surface area $O_p(K)$ [Lu], [Hu] of a convex body K is defined as (see 1.3 below for more details)

$$O_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N_{\partial K}(x) >^{\frac{n(p-1)}{n+p}}} \mathrm{d}\mu_{\partial K}(x).$$

Note that then for $n > 2$ the right hand expression above is a p-affine surface area with $p = n/(n-2)$.

4. More generally, let K be a convex body in $\mathbb{R}^n$ with centroid at the origin and satisfying the assumptions of Theorem 1.1. Let α and β be real numbers. Let the density be given by

$$f_{\alpha,\beta}(x) = \frac{< x, N_{\partial K}(x) >^{\alpha} \kappa(x)^{\beta}}{\int_{\partial K} < x, N_{\partial K}(x) >^{\alpha} \kappa(x)^{\beta} d\mu_{\partial K}(x)}.$$

Then by Theorem 1.1

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{\alpha,\beta}, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} =$$

$$c_n \left(\int_{\partial K} \frac{\kappa(x)^{\frac{1-2\beta}{n-1}} \mathrm{d}\mu_{\partial K}(x)}{< x, N_{\partial K}(x) >^{\frac{2\alpha}{n-1}}} \right) \left(\int_{\partial K} < x, N_{\partial K}(x) >^{\alpha} \quad \kappa(x)^{\beta} \mathrm{d}\mu_{\partial K}(x) \right)^{\frac{2}{n-1}}.$$

The second expression on the right hand side of this equation is a p-affine surface area iff

$$\alpha = -\frac{n(p-1)}{n+p} \qquad \text{and} \qquad \beta = \frac{p}{n+p}.$$

Then

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{O_p(K)}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \kappa(x)^{\frac{n-p}{(n-1)(n+p)}} < x, N_{\partial K}(x) >^{\frac{2n(p-1)}{(n-1)(n+p)}} \mathrm{d}\mu_{\partial K}(x).$$

Note that the right hand side of this equality is a q-affine surface area with $q = \frac{n-p}{n+p-2}$.

5. Another measure of interest is the measure induced by the Gauß map. The Gauß map $N_{\partial K} : \partial K \to \partial B_2^n$ maps a point x to its normal $N_{\partial K}(x)$. As a measure we define

$$\mathbb{P}(A) = \sigma\{N_{\partial K}(x) | x \in A\}$$

where σ is the normalized surface measure on ∂B_2^n. This can also be written as

$$\mathbb{P}(A) = \frac{\int_A \kappa(x) \mathrm{d}\mu_{\partial K}(x)}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

This measure is not invariant under linear transformations with determinant 1. This can easily be seen by considering the circle with radius 1 in $\mathbb{R}^2$. An affine transformation changes the circle into an ellipse. We consider a small neighborhood of an apex with small curvature. This is the affine image of a small set whose image under the Gauß map is larger. We get

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f,N)}{\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \kappa(x)^{-\frac{1}{n-1}} \mathrm{d}\mu_{\partial K}(x).$$

1.3 Extensions of the p-Affine Surface Area

The p-affine surface area $O_p(K)$ was introduced by Lutwak [Lu], see also Hug [Hu]. For $p=1$ we get the affine surface area which is related to curve evolution and computer vision [SaT1, SaT2]. Meyer and Werner [MW1, MW2] gave a geometric interpretation of the p-affine surface area in terms of the Santaló bodies. They also observed that -provided the integrals exist- the definition of Lutwak for the p-affine surface area makes sense for $-n < p \leq 0$ and their geometric interpretation in terms of the Santaló bodies also holds for this range of p. They also gave a definition of the p-affine surface area for $p=-n$ together with its geometric interpretation.

In view of 1.2.4 we propose here to extend the p-range even further, namely to $-\infty \leq p \leq \infty$. Theorem 1.1 then provides a geometric interpretation of the p-affine surface area for this whole p-range. See also [SW2] for another geometric interpretation.

Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. For p with $p \neq -n$ and $-\infty \leq p \leq \infty$ we put

$$O_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa(x)}{< x, N_{\partial K}(x) >^n} \mathrm{d}\mu_{\partial K}(x)$$

and

$$O_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N_{\partial K}(x) >^{\frac{n(p-1)}{n+p}}} \mathrm{d}\mu_{\partial K}(x),$$

provided the integrals exist.
If 0 is an interior point of K then there are strictly positive constants a and b such that

$$a \leq < x, N_{\partial K}(x) > \leq b.$$

Assume now that K is such that the assumptions of Theorem 1.1 hold. Then the above integrals are finite. We consider the densities

$$f_{\pm\infty}(x) = \frac{1}{O_{\pm\infty}(K)} \frac{\kappa(x)}{< x, N_{\partial K}(x) >^n}$$

and for $-\infty < p < \infty$, $p \neq -n$

$$f_p(x) = \frac{1}{O_p(K)} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N_{\partial K}(x) >^{\frac{n(p-1)}{n+p}}}.$$

As a corollary to Theorem 1.1 we get the following geometric interpretation of the p-affine surface area.

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{\pm\infty}, N)}{\left(\frac{O_{\pm\infty}(K)}{N}\right)^{\frac{2}{n-1}}} =$$
$$c_n \int_{\partial K} \kappa(x)^{-\frac{1}{n-1}} < x, N_{\partial K}(x) >^{\frac{2n}{n-1}} \mathrm{d}\mu_{\partial K}(x) = O_{-1}(K)$$

and

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_p, N)}{\left(\frac{O_p(K)}{N}\right)^{\frac{2}{n-1}}} =$$
$$c_n \int_{\partial K} \kappa(x)^{\frac{n-p}{(n-1)(n+p)}} < x, N_{\partial K}(x) >^{\frac{2n(p-1)}{(n-1)(n+p)}} \mathrm{d}\mu_{\partial K}(x) = O_q(K)$$

where $q = \frac{n-p}{n+p-2}$.

Thus each density f_p gives us a q-affine surface area O_q with $q = \frac{n-p}{n+p-2}$ as the expected difference volume. Note that for the density f_{-n+2} we get $O_{\pm\infty}(K)$. Conversely, for each q-affine surface area O_q, $-\infty \le q \le +\infty$, $q \ne -n$, there is a density f_p with $p = \frac{n-nq+2q}{q+1}$ such that

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_p, N)}{\left(\frac{O_p(K)}{N}\right)^{\frac{2}{n-1}}} = c_n O_q(K).$$

1.4 Random Polytopes of Points Chosen from the Convex Body

Whereas random polytopes of points chosen from the boundary of a convex body have up to now only been considered in the case of the Euclidean ball [Mü], random polytopes of points chosen from the convex body and not only from the boundary have been investigated in great detail. This has been done by Rényi and Sulanke [ReS1, ReS2] in dimension 2. Wieacker [Wie] computed the expected difference volume for the Euclidean ball in $\mathbb{R}^n$. Bárány [Ba1] showed for convex bodies K in $\mathbb{R}^n$ with C^3-boundary and everywhere positive curvature that

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(\mathbb{P}, N)}{(\frac{\mathrm{vol}_n(K)}{N})^{\frac{2}{n+1}}} = c_1(n) \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)$$

where $\mathbb{P}$ is the normalized Lebesgue measure on K, $\kappa(x)$ is the Gauß-Kronecker curvature, and

$$c_1(n) = \frac{(n+1)^{\frac{2}{n+1}}(n^2+n+2)(n^2+1)\Gamma(\frac{n^2+1}{n+1})}{2(n+3)(n+1)!\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n+1}}}.$$

Schütt [Sch1] verified that this formula holds for all convex bodies, where $\kappa(x)$ is the generalized Gauß-Kronecker curvature.

The order of best approximation of convex bodies by polytopes with a given number of vertices N is $N^{-\frac{2}{n-1}}$ (see above). The above formula for random polytopes chosen from the body gives $N^{-\frac{2}{n+1}}$. Thus random approximation by choosing the points from K does not give the optimal order. But one has to take into account that not all points chosen from the convex body turn out to be vertices of a random polytope. Substituting N by the number of expected vertices we get the optimal order [Ba2] for the exponent of N in the case of a convex body with C^3-boundary and everywhere positive curvature. Indeed, for all convex bodies with a C^3-boundary and everywhere positive curvature the expected number of i-dimensional faces is of the order $N^{\frac{n-1}{n+1}}$ [Ba2].

1.5 Comparison between Best and Random Approximation

Now we want to compare random approximation with best approximation in more detail. We will not only consider the exponent of N but also the other factors. It turns out that random approximation and best approximation with the optimal density are very close.

McClure and Vitale [McV] obtained an asymptotic formula for best approximation in the case $n = 2$. Gruber [Gr2] generalized this to higher dimensions. The metric used in these results is the symmetric difference metric d_S. Then these asymptotic best approximation results are (see above for the precise formulation):

If a convex body K in $\mathbb{R}^n$ has a C^2-boundary with everywhere positive curvature, then

$$\inf\{d_S(K, P_N) | P_N \subset K \text{ and } P_N \text{ is a polytope with at most N vertices}\}$$

is asymptotically the same as

$$\tfrac{1}{2}\mathrm{del}_{n-1}\left(\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}\mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}}\left(\frac{1}{N}\right)^{\frac{2}{n-1}}.$$

where del_{n-1} is a constant that is related to the Delone triangulations and depends only on the dimension n. Equivalently, the result states that if we divide one expression by the other and take the limit for N to ∞ we obtain 1. It was shown by Gordon, Reisner and Schütt in [GRS1, GRS2] that the constant del_{n-1} is of the order of n, which means that there are numerical constants a and b such that we have for all $n \in \mathbb{N}$

$$an \le \mathrm{del}_{n-1} \le bn.$$

It is clear from Theorem 1.1 that we get the best random approximation if we choose the affine surface area measure. Then the order of magnitude for random approximation is

$$\frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}} \left(\frac{1}{N}\right)^{\frac{2}{n-1}}.$$

Since

$$(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}} \sim \frac{1}{n} \quad \text{and} \quad \Gamma\left(n+1+\frac{2}{n-1}\right) \sim \Gamma(n+1)(n+1)^{\frac{2}{n-1}}$$

random approximation (with randomly choosing the points from the boundary of K) is of the same order as

$$n\left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \mathrm{d}\mu_{\partial K}(x)\right)^{\frac{n+1}{n-1}} \left(\frac{1}{N}\right)^{\frac{2}{n-1}},$$

which is the same order as best approximation.

In two papers by Mankiewicz and Schütt the constant del_{n-1} has been better estimated [MaS1, MaS2]. It was shown there

$$\tfrac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-\frac{2}{n-1}} \le \mathrm{del}_{n-1} \le (1+\tfrac{c\ln n}{n})\tfrac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-\frac{2}{n-1}},$$

where c is a numerical constant. In particular, $\lim_{n\to\infty} \frac{\mathrm{del}_{n-1}}{n} = \frac{1}{2\pi e} = 0.0585498\ldots$. Thus

$$\begin{aligned}\left(1-c\frac{\ln n}{n}\right) \lim_{N\to\infty} &\frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{as}, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} \\ &\le \lim_{N\to\infty} N^{\frac{2}{n-1}} \inf\{d_S(K,P_N) | P_N \subset K \text{ and } P_N \\ &\qquad\qquad \text{is a polytope with at most N vertices}\}.\end{aligned}$$

In order to verify this we have to estimate the quotient

$$\frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} (\tfrac{1}{2}\mathrm{del}_{n-1})^{-1}.$$

Since $\frac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-\frac{2}{n-1}} \le \mathrm{del}_{n-1}$ the quotient is less than $\frac{1}{n!}\Gamma(n+1+\frac{2}{n-1})$. Now we use Stirlings formula to get

$$\frac{\Gamma(n+1+\frac{2}{n-1})}{n!} \le 1 + c\frac{\ln n}{n}.$$

1.6 Subdifferentials and Indicatrix of Dupin

Let $\mathcal{U}$ be a convex, open subset of $\mathbb{R}^n$ and let $f:\mathcal{U}\to\mathbb{R}$ be a convex function. $\mathrm{d}f(x)\in\mathbb{R}^n$ is called subdifferential at the point $x_0\in\mathcal{U}$, if we have for all $x\in\mathcal{U}$

$$f(x_0)+<\mathrm{d}f(x_0),x-x_0>\le f(x).$$

A convex function has a subdifferential at every point and it is differentiable at a point if and only if the subdifferential is unique. Let $\mathcal{U}$ be an open, convex subset in $\mathbb{R}^n$ and $f:\mathcal{U}\to\mathbb{R}$ a convex function. f is said to be twice differentiable in a generalized sense in $x_0\in\mathcal{U}$, if there is a linear map $\mathrm{d}^2 f(x_0)$ and a neighborhood $\mathcal{U}(x_0)\subseteq\mathcal{U}$ such that we have for all $x\in\mathcal{U}(x_0)$ and for all subdifferentials $\mathrm{d}f(x)$

$$\|\mathrm{d}f(x)-\mathrm{d}f(x_0)-\mathrm{d}^2 f(x_0)(x-x_0)\|\le\Theta(\|x-x_0\|)\|x-x_0\|,$$

where Θ is a monotone function with $\lim_{t\to0}\Theta(t)=0$. $\mathrm{d}^2 f(x_0)$ is called generalized Hesse-matrix. If $f(0)=0$ and $\mathrm{d}f(0)=0$ then we call the set

$$\{x\in\mathbb{R}^n|x^t\mathrm{d}^2 f(0)x=1\}$$

the indicatrix of Dupin at 0. Since f is convex this set is an ellipsoid or a cylinder with a base that is an ellipsoid of lower dimension. The eigenvalues of $\mathrm{d}^2 f(0)$ are called principal curvatures and their product is called the Gauß-Kronecker curvature κ. Geometrically the eigenvalues of $\mathrm{d}^2 f(0)$ that are different from 0 are the lengths of the principal axes of the indicatrix raised to the power -2.

The following lemma can be found in e.g. [SW1].

Lemma 1.1. *Let $\mathcal{U}$ be an open, convex subset of $\mathbb{R}^n$ and $0\in\mathcal{U}$. Suppose that $f:\mathcal{U}\to\mathbb{R}$ is twice differentiable in the generalized sense at 0 and that $f(0)=0$ and $\mathrm{d}f(0)=0$.*
(i) Suppose that the indicatrix of Dupin at 0 is an ellipsoid. Then there is a monotone, increasing function $\psi:[0,1]\to[1,\infty)$ with $\lim_{s\to0}\psi(s)=1$ such that

$$\left\{(x,s)\left|x^t\mathrm{d}^2 f(0)x\le\frac{2s}{\psi(s)}\right.\right\}$$
$$\subseteq\{(x,s)|f(x)\le s\}\subseteq\{(x,s)|x^t\mathrm{d}^2 f(0)x\le2s\psi(s)\}.$$

(ii) Suppose that the indicatrix of Dupin is an elliptic cylinder. Then for every $\epsilon>0$ there is $s_0>0$ such that we have for all s with $s<s_0$

$$\left\{(x,s)\left|x^t\mathrm{d}^2 f(0)x+\epsilon\|x\|^2\le2s\right.\right\}\subseteq\{(x,s)|f(x)\le s\}.$$

Lemma 1.2. *Let K be a convex body in $\mathbb{R}^n$ with $0 \in \partial K$ and $N(0) = -e_n$. Suppose that the indicatrix of Dupin at 0 is an ellipsoid. Suppose that the principal axes $b_i e_i$ of the indicatrix are multiples of the unit vectors e_i, $i = 1, \dots, n-1$. Let $\mathcal{E}$ be the n-dimensional ellipsoid*

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{n-1} \frac{x_i^2}{b_i^2} + \frac{\left(x_n - \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}} \right)^2}{\left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}} \le \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}} \right. \right\}.$$

Then there is an increasing, continuous function $\phi : [0, \infty) \to [1, \infty)$ with $\phi(0) = 1$ such that we have for all t

$$\begin{aligned} &\left\{ \left(\frac{x_1}{\phi(t)}, \dots, \frac{x_{n-1}}{\phi(t)}, t \right) \middle| x \in \mathcal{E}, x_n = t \right\} \\ &\quad \subseteq K \cap H((0, \dots, 0, t), N(0)) \\ &\quad \subseteq \{ (\phi(t) x_1, \dots, \phi(t) x_{n-1}, t) | x \in \mathcal{E}, x_n = t \}. \end{aligned}$$

We call $\mathcal{E}$ the standard approximating ellipsoid .

Proof. Lemma 1.2 follows from Lemma 1.1. Let f be a function whose graph is locally the boundary of the convex body. Consider (x, s) with

$$x^t \mathrm{d}^2 f(0) x = 2s$$

which is the same as

$$\sum_{i=1}^{n-1} \frac{x_i^2}{b_i^2} = 2s.$$

Then

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{x_i^2}{b_i^2} + \frac{\left(x_n - \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}} \right)^2}{\left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}} \\ &\qquad = 2s + \frac{\left(s - \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}} \right)^2}{\left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}} = \frac{s^2}{\left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}} + \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}. \end{aligned}$$

□

Let us denote the lengths of the principal axes of the indicatrix of Dupin by b_i, $i = 1, \dots, n-1$. Then the lengths a_i, $i = 1, \dots, n$ of the principal axes of the standard approximating ellipsoid $\mathcal{E}$ are

$$a_i = b_i \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{1}{n-1}} \qquad i = 1, \dots, n-1 \qquad \text{and} \qquad a_n = \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}. \tag{3}$$

This follows immediately from Lemma 1.2. For the Gauß-Kronecker curvature we get

$$\prod_{i=1}^{n-1} \frac{a_n}{a_i^2}. \tag{4}$$

This follows as the Gauß-Kronecker curvature equals the product of the eigenvalues of the Hesse matrix. The eigenvalues are b_i^{-2}, $i = 1, \dots, n-1$. Thus

$$\prod_{i=1}^{n-1} b_i^{-2} = \left(\prod_{i=1}^{n-1} b_i \right)^2 \prod_{i=1}^{n-1} \left(b_i \left(\prod_{k=1}^{n-1} b_k \right)^{\frac{1}{n-1}} \right)^{-2} = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}.$$

In particular, if the indicatrix of Dupin is a sphere of radius $\sqrt{\rho}$ then the standard approximating ellipsoid is a Euclidean ball of radius ρ.

We consider the transform $T : \mathbb{R}^n \to \mathbb{R}^n$

$$T(x) = \left(\frac{x_1}{a_1} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \dots, \frac{x_{n-1}}{a_{n-1}} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, x_n \right). \tag{5}$$

This transforms the standard approximating ellipsoid $\mathcal{E}$ into a Euclidean ball $T(\mathcal{E})$ with radius $r = (\prod_{i=1}^{n-1} b_i)^{2/(n-1)}$. This is obvious since the principal axes of the standard approximating ellipsoid are given by (3). The map T is volume preserving.

Lemma 1.3. *Let*

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \left| \frac{x_i}{a_i} \right|^2 \le 1 \right\}$$

and let $H = H((a_n - \Delta)e_n, e_n)$. Then for all Δ with $\Delta \le \frac{1}{2} a_n$ the intersection $\mathcal{E} \cap H$ is an ellipsoid whose principal axes have lengths

$$\frac{a_i}{a_n} \left(2a_n \Delta - \Delta^2 \right)^{\frac{1}{2}} \qquad i = 1, \dots, n-1.$$

Moreover,

$$\begin{aligned} \mathrm{vol}_{n-1}(\mathcal{E} \cap H) &\le \mathrm{vol}_{n-1}(\partial \mathcal{E} \cap H^-) \\ &\le \sqrt{1 + \frac{2\Delta a_n^3}{(a_n - \Delta)^2 \min_{1 \le i \le n-1} a_i^2}}\, \mathrm{vol}_{n-1}(\mathcal{E} \cap H) \end{aligned}$$

and

$$\mathrm{vol}_{n-1}(\mathcal{E}\cap H)=\mathrm{vol}_{n-1}(B_2^{n-1})\left(\prod_{i=1}^{n-1}a_i\right)\left(\frac{2\Delta}{a_n}-\left|\frac{\Delta}{a_n}\right|^2\right)^{\frac{n-1}{2}}$$
$$=\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\sqrt{\kappa(a_n e_n)}}\left(2\Delta-\frac{\Delta^2}{a_n}\right)^{\frac{n-1}{2}},$$

where κ is the Gauß-Kronecker curvature.

Proof. The left hand inequality is trivial. We show the right hand inequality. Let p_{e_n} be the orthogonal projection onto the subspace orthogonal to e_n. We have

$$\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H^-)=\int_{\mathcal{E}\cap H}\frac{1}{<e_n,N_{\partial\mathcal{E}}(\bar{y})>}\mathrm{d}y \tag{6}$$

where $\bar{y}_i=y_i$, $i=1,\dots,n-1$, and

$$\bar{y}_n=a_n\sqrt{1-\sum_{i=1}^{n-1}\left|\frac{y_i}{a_i}\right|^2}.$$

Therefore we get

$$\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H^-)\leq\frac{\mathrm{vol}_{n-1}(\mathcal{E}\cap H)}{\min_{x\in\partial\mathcal{E}\cap H^-}<e_n,N_{\partial\mathcal{E}}(x)>}.$$

We have

$$N_{\partial\mathcal{E}}(x)=\frac{\left(\frac{x_i}{a_i^2}\right)_{i=1}^n}{\sqrt{\sum_{i=1}^n|\frac{x_i}{a_i^2}|^2}}.$$

Therefore we get

$$<e_n,N_{\partial\mathcal{E}}(x)>=\frac{\frac{x_n}{a_n^2}}{\sqrt{\sum_{i=1}^n|\frac{x_i}{a_i^2}|^2}}=\left(1+\frac{a_n^4}{x_n^2}\sum_{i=1}^{n-1}\frac{x_i^2}{a_i^4}\right)^{-\frac{1}{2}}$$
$$\geq\left(1+\frac{a_n^4}{x_n^2\min_{1\leq i\leq n-1}a_i^2}\sum_{i=1}^{n-1}\frac{x_i^2}{a_i^2}\right)^{-\frac{1}{2}}$$
$$=\left(1+\frac{a_n^4}{x_n^2\min_{1\leq i\leq n-1}a_i^2}\left(1-\left|\frac{x_n}{a_n}\right|^2\right)\right)^{-\frac{1}{2}}$$
$$=\left(1+\frac{a_n^2}{\min_{1\leq i\leq n-1}a_i^2}\left(\frac{a_n^2}{x_n^2}-1\right)\right)^{-\frac{1}{2}}.$$

The last expression is smallest for $x_n = a_n - \Delta$. We get

$$< e_n, N_{\partial\mathcal{E}}(x) > \geq \left(1 + \frac{a_n^2(2\Delta a_n - \Delta^2)}{(a_n - \Delta)^2 \min_{1\leq i\leq n-1} a_i^2}\right)^{-\frac{1}{2}}$$
$$\geq \left(1 + \frac{2\Delta a_n^3}{(a_n - \Delta)^2 \min_{1\leq i\leq n-1} a_i^2}\right)^{-\frac{1}{2}}.$$

The equalities are proved using

$$\kappa(a_n e_n) = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}.$$

□

Lemma 1.4. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin at x_0 exists and is an ellipsoid. Let $\mathcal{E}$ be the standard approximating ellipsoid at x_0. Then for all $\epsilon > 0$ there is Δ_0 such that for all $\Delta < \Delta_0$*

$$\mathrm{vol}_{n-1}(K \cap H(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))) \leq$$
$$\mathrm{vol}_{n-1}(\partial K \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))) \leq$$
$$(1+\epsilon)\sqrt{1 + \frac{2\Delta a_n^3}{(a_n - \Delta)^2 \min_{1\leq i\leq n-1} a_i^2}}\,\mathrm{vol}_{n-1}(K\cap H(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))),$$

where $a_1, \ldots, a_n$ are the lengths of the principal axes of $\mathcal{E}$.

Proof. We can assume that K is in such a position that $N_{\partial K}(x_0)$ coincides with the n-th unit vector e_n and that the equation of the approximating ellipsoid is

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \left|\frac{x_i}{a_i}\right|^2 \leq 1 \right\}.$$

Then the proof follows from Lemma 1.2 and Lemma 1.3. □

Lemma 1.5. *Let H be a hyperplane with distance p from the origin and s the area of the cap cut off by H from B_2^n. r denotes the radius of the $n-1$-dimensional Euclidean ball $H \cap B_2^n$. We have*

$$\frac{\mathrm{d}p}{\mathrm{d}s} = -\left(r^{n-3}\mathrm{vol}_{n-2}(\partial B_2^{n-1})\right)^{-1} = -\left((1-p^2)^{\frac{n-3}{2}}\mathrm{vol}_{n-2}(\partial B_2^{n-1})\right)^{-1}.$$

Proof. Using (6) and polar coordinates, we get for the surface area s of a cap of the Euclidean ball of radius 1

$$s = \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \int_0^r \frac{t^{n-2}}{(1-t^2)^{\frac{1}{2}}} \mathrm{d}t = \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \int_0^{\sqrt{1-p^2}} \frac{t^{n-2}}{(1-t^2)^{\frac{1}{2}}} \mathrm{d}t.$$

This gives

$$\frac{\mathrm{d}s}{\mathrm{d}p} = -\frac{\mathrm{vol}_{n-2}(\partial B_2^{n-1})(1-p^2)^{\frac{n-2}{2}}}{p} \frac{p}{\sqrt{1-p^2}} = -r^{n-3}\mathrm{vol}_{n-2}(\partial B_2^{n-1}).$$

□

Lemma 1.6. *(*Aleksandrov [Al]*) Let K be a convex body in $\mathbb{R}^n$. Then its boundary is almost everywhere twice differentiable in the generalized sense.*

For a proof of this result see [Ban], [EvG], [BCP].

At each point where ∂K is twice differentiable in the generalized sense the indicatrix of Dupin exists. Therefore the indicatrix of Dupin exists almost everywhere.

Lemma 1.7. *(*John [J]*) Let K be a convex body in $\mathbb{R}^n$ that is centrally symmetric with respect to the origin. Then there exists an ellipsoid $\mathcal{E}$ with center* 0 *such that*

$$\mathcal{E} \subseteq K \subseteq \sqrt{n}\ \mathcal{E}.$$

Lemma 1.8. *Let K and C be convex bodies in $\mathbb{R}^n$ such that $C \subseteq K$ and* 0 *is an interior point of C. Then we have for all integrable functions f*

$$\int_{\partial C} f(x) \mathrm{d}\mu_{\partial C}(x) = \int_{\partial K} f(x(y)) \frac{\|x(y)\|^n < y, N(y) >}{\|y\|^n < x(y), N(x(y)) >} \mathrm{d}\mu_{\partial K}(y)$$

where $\{x(y)\} = [0, y] \cap \partial C$.

2 The Surface Body

2.1 Definitions and Properties of the Surface Body

Let $0 < s$ and let $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function with $\int_{\partial K} f \mathrm{d}\mu = 1$.

The surface body $K_{f,s}$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $\mathbb{P}_f$-measure less than or equal to s from ∂K. More precisely,

$$K_{f,s} = \bigcap_{\mathbb{P}_f(\partial K \cap H^-) \le s} H^+.$$

We write usually K_s for $K_{f,s}$ if it is clear which function f we are considering. It follows from the Hahn-Banach theorem that $K_0 \subseteq K$. If in addition f is almost everywhere nonzero, then $K_0 = K$. This is shown in Lemma 2.1.(iv).

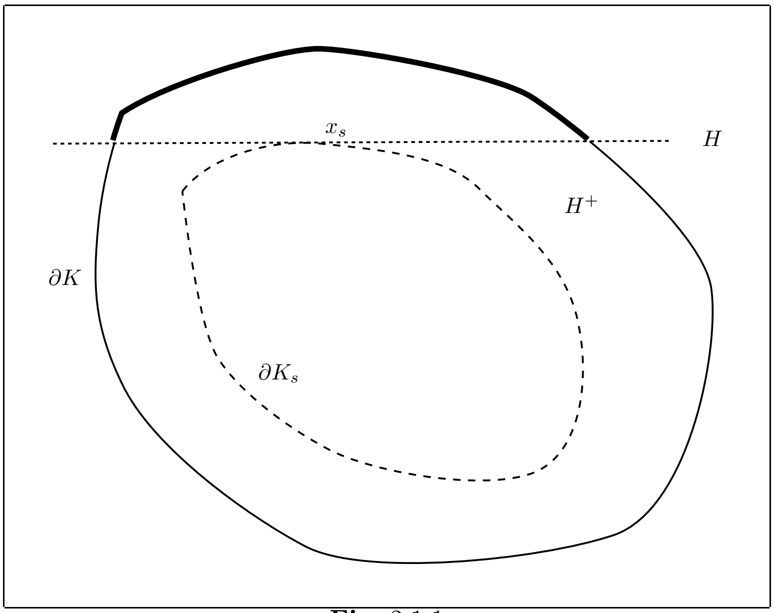

Fig. 2.1.1

We say that a sequence of hyperplanes H_i, $i \in \mathbb{N}$, in $\mathbb{R}^n$ converges to a hyperplane H if we have for all $x \in H$ that

$$\lim_{i \to \infty} d(x, H_i) = 0,$$

where $d(x,H) = \inf\{\|x-y\| : y \in H\}$. This is equivalent to: The sequence of the normals of H_i converges to the normal of H and there is a point $x \in H$ such that

$$\lim_{i \to \infty} d(x, H_i) = 0.$$

Lemma 2.1. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a a.e. positive, integrable function with $\int_{\partial K} f \mathrm{d}\mu = 1$. Let $\xi \in S^{n-1}$.*
(i) Let $x_0 \in \partial K$. Then

$$\mathbb{P}_f(\partial K \cap H^-(x_0 - t\xi, \xi))$$

is a continuous function of t on

$$\left[0, \max_{y \in K} < x_0 - y, \xi >\right).$$

(ii) Let $x \in \mathbb{R}^n$. Then the function

$$\mathbb{P}_f(\partial K \cap H^-(x - t\xi, \xi))$$

is strictly increasing on

$$\left[\min_{y \in K} < x - y, \xi >, \max_{y \in K} < x - y, \xi >\right].$$

(iii) Let H_i, $i \in \mathbb{N}$, be a sequence of hyperplanes that converge to the hyperplane H_0. Assume that the hyperplane H_0 intersects the interior of K. Then we have

$$\lim_{i \to \infty} \mathbb{P}_f(\partial K \cap H_i^-) = \mathbb{P}_f(\partial K \cap H_0^-).$$

(iv)

$$\overset{\circ}{K} \subseteq \bigcup_{0<s} K_s$$

In particular, $K = K_0$.

Proof. (i)

$$\mathrm{vol}_{n-1}(\partial K \cap H^-(x_0 - t\xi, \xi))$$

is a continuous function on

$$\left[0, \max_{y \in K} < x_0 - y, \xi >\right).$$

Then (i) follows as f is an integrable function.

(ii) Since $H^-(x, \xi)$ is the half space containing $x + \xi$ we have for $t_1 < t_2$

$$H^-(x - t_1\xi, \xi) \subsetneq H^-(x - t_2\xi, \xi).$$

If

$$\mathbb{P}_f(\partial K \cap H^-(x - t_1\xi, \xi)) = \mathbb{P}_f(\partial K \cap H^-(x - t_2\xi, \xi))$$

then f is a.e. 0 on $\partial K \cap H^-(x - t_2\xi, \xi) \cap H^+(x - t_1\xi, \xi)$. This is not true.

(iii) Let $H_i = H_i(x_i, \xi_i)$, $i = 0, 1, \ldots$. We have that

$$\lim_{i \to \infty} x_i = x_0 \qquad \lim_{i \to \infty} \xi_i = \xi_0,$$

where x_0 is an interior point of K. Therefore

$$\forall \epsilon > 0 \ \exists\ i_0 \ \forall\ i > i_0 :$$
$$\partial K \cap H^-(x_0 + \epsilon\xi_0, \xi_0) \subseteq \partial K \cap H^-(x_i, \xi_i) \subseteq \partial K \cap H^-(x_0 - \epsilon\xi_0, \xi_0).$$

This implies

$$\begin{aligned}\mathbb{P}_f\left(\partial K \cap H^-(x_0 + \epsilon\xi_0, \xi_0)\right) &\leq \mathbb{P}_f\left(\partial K \cap H^-(x_i, \xi_i)\right)\\ &\leq \mathbb{P}_f\left(\partial K \cap H^-(x_0 - \epsilon\xi_0, \xi_0)\right).\end{aligned}$$

Since x_0 is an interior point of K, for ϵ small enough $x_0 - \epsilon\xi_0$ and $x_0 + \epsilon\xi_0$ are interior points of K. Therefore,

$$H(x_0 - \epsilon\xi_0, \xi_0) \quad \text{and} \quad H(x_0 + \epsilon\xi_0, \xi_0)$$

intersect the interior of K. The claim now follows from (i).

(iv) Suppose the inclusion is not true. Then there is $x \in \overset{\circ}{K}$ with $x \notin \bigcup_{0<s} K_s$. Therefore, for every $s > 0$ there is a hyperplane H_s with $x \in H_s$ and

$$\mathbb{P}_f(\partial K \cap H_s^-) \leq s.$$

By compactness and by (iii) there is a hyperplane H with $x \in H$ and

$$\mathbb{P}_f(\partial K \cap H^-) = 0.$$

On the other hand, $\mathrm{vol}_{n-1}(\partial K \cap H^-) > 0$ which implies

$$\mathbb{P}_f(\partial K \cap H^-) > 0$$

since f is a.e. positive.

We have $K = K_0$ because K_0 is a closed set that contains $\overset{\circ}{K}$. □

Lemma 2.2. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a a.e. positive, integrable function with $\int_{\partial K} f \mathrm{d}\mu = 1$.*
(i) For all s such that $K_s \neq \emptyset$, and all $x \in \partial K_s \cap \overset{\circ}{K}$ there exists a supporting hyperplane H to ∂K_s through x such that $\mathbb{P}_f(\partial K \cap H^-) = s$.
(ii) Suppose that for all $x \in \partial K$ there is $R(x) < \infty$ so that

$$K \subseteq B_2^n(x - R(x)N_{\partial K}(x), R(x)).$$

Then we have for all $0 < s$ that $K_s \subset \overset{\circ}{K}$.

Proof. (i) There is a sequence of hyperplanes H_i with $K_s \subseteq H_i^+$ and $\mathbb{P}_f(\partial K \cap H_i^-) \leq s$ such that the distance between x and H_i is less than $\frac{1}{i}$. We check this.

Since $x \in \partial K_s$ there is $z \notin K_s$ with $\|x-z\| < \frac{1}{i}$. There is a hyperplane H_i separating z from K_s satisfying

$$\mathbb{P}_f(\partial K \cap H_i^-) \leq s \qquad \text{and} \qquad K_s \subseteq H_i^+.$$

We have

$$d(x, H_i) \leq \|x-z\| < \tfrac{1}{i}.$$

By compactness and by Lemma 2.1 .(iii) there is a subsequence that converges to a hyperplane H with $x \in H$ and $\mathbb{P}_f(\partial K \cap H^-) \leq s$.

If $\mathbb{P}_f(\partial K \cap H^-) < s$ then we choose a hyperplane $\tilde{H}$ parallel to H such that $\mathbb{P}_f(\partial K \cap \tilde{H}^-) = s$. By Lemma 2.1.(i) there is such a hyperplane. Consequently, x is not an element of K_s. This is a contradiction.

(ii) Suppose there is $x \in \partial K$ with $x \in K_s$ and $0 < s$. By $K \subseteq B_2^n(x - R(x)N_{\partial K}(x), R(x))$ we get

$$\mathrm{vol}_{n-1}(\partial K \cap H(x, N_{\partial K}(x))) = 0.$$

By Lemma 2.1.(i) we can choose a hyperplane H parallel to $H(x, N_{\partial K}(x))$ that cuts off a set with $\mathbb{P}_f(\partial K \cap \tilde{H}^-) = s$. This means that $x \notin K_s$. $\square$

Lemma 2.3. *Let K be a convex body in $\mathbb{R}^n$ and let $f: \partial K \to \mathbb{R}$ be a a.e positive, integrable function with $\int_{\partial K} f \mathrm{d}\mu = 1$.*
(i) Let s_i, $i \in \mathbb{N}$, be a strictly increasing sequence of positive numbers with $\lim_{i\to\infty} s_i = s_0$. *Then we have*

$$K_{s_0} = \bigcap_{i=1}^{\infty} K_{s_i}.$$

(ii) There exists T with $0 < T \leq \frac{1}{2}$ such that K_T is nonempty and $\mathrm{vol}_n(K_T) = 0$ *and* $\mathrm{vol}_n(K_t) > 0$ *for all $t < T$.*

Proof. (i) Since we have for all $i \in \mathbb{N}$ that $K_{s_0} \subseteq K_{s_i}$, we get

$$K_{s_0} \subseteq \bigcap_{i=1}^{\infty} K_{s_i}.$$

We show now that both sets are in fact equal. Let us consider $x \notin K_{s_0}$. If $x \notin K$, then $x \notin \bigcap_{i=1}^{\infty} K_{s_i}$, as

$$K = K_0 \supseteq \bigcap_{i=1}^{\infty} K_{s_i}.$$

If $x \in K$ and $x \notin K_{s_0}$ then there is a hyperplane H with $x \in \overset{\circ}{H^-}$, $K_{s_0} \subseteq H^+$, and

$$\mathbb{P}_f(K\cap H^-)\le s_0.$$

There is a hyperplane H_1 that is parallel to H and that contains x. There is another hyperplane H_2 that is parallel to both these hyperplanes and whose distance to H equals its distance to H_1. By Lemma 2.1.(ii) we get

$$0\le \mathbb{P}_f(\partial K\cap H_1^-)<\mathbb{P}_f(\partial K\cap H_2^-)<\mathbb{P}_f(\partial K\cap H^-)\le s_0.$$

Let $s_0^{'}=\mathbb{P}_f(\partial K\cap H_2^-)$. It follows that

$$x\notin \bigcap_{\mathbb{P}_f(H^-\cap\partial K)\le s_0^{'}} H^+=K_{s_0^{'}}.$$

Therefore $x\notin K_{s_i}$, for $s_i\ge s_0^{'}$.

(ii) We put

$$T=\sup\{s|\mathrm{vol}_n(K_s)>0\}.$$

Since the sets K_s are compact, convex, nonempty sets,

$$\bigcap_{\mathrm{vol}_n(K_s)>0} K_s$$

is a compact, convex, nonempty set. On the other hand, by (i) we have

$$K_T=\bigcap_{s<T}K_s=\bigcap_{\mathrm{vol}_n(K_s)>0}K_s.$$

Now we show that $\mathrm{vol}_n(K_T)=0$. Suppose that $\mathrm{vol}_n(K_T)>0$. Then there is $x_0\in\overset{\circ}{K}_T$. Let

$$t_0=\inf\{\mathbb{P}_f(\partial K\cap H^-)|x_0\in H\}.$$

Since we require that $x_0\in H$ we have that $\mathbb{P}_f(\partial K\cap H^-)$ is only a function of the normal of H. By Lemma 2.1.(iii), $\mathbb{P}_f(\partial K\cap H^-)$ is a continuous function of the normal of H. By compactness this infimum is attained and there is H_0 with $x_0\in H_0$ and

$$\mathbb{P}_f(\partial K\cap H_0^-)=t_0.$$

Moreover, $t_0>T$. If not, then $K_T\subseteq H_0^+$ and $x_0\in H_0$, which means that $x_0\in\partial K_T$, contradicting the assumption that $x_0\in\overset{\circ}{K}_T$.

Now we consider $K_{(1/2)(T+t_0)}$. We claim that x_0 is an interior point of this set and therefore

$$\mathrm{vol}_n(K_{\frac{1}{2}(T+t_0)})>0,$$

contradicting the fact that T is the supremum of all t with

$$\mathrm{vol}_n(K_t)>0.$$

We verify now that x_0 is an interior point of $K_{(1/2)(T+t_0)}$. Suppose x_0 is not an interior point of this set. Then in every neighborhood of x_0 there is $x \notin K_{\frac{1}{2}(T+t_0)}$. Therefore for every $\epsilon > 0$ there is a hyperplane H_ϵ such that

$$\mathbb{P}_f(\partial K \cap H_\epsilon^-) \leq \tfrac{1}{2}(T+t_0), \qquad x \in H_\epsilon \qquad \text{and} \qquad \|x - x_0\| < \epsilon.$$

By Lemma 2.1.(iii) we conclude that there is a hyperplane H with $x_0 \in H$ and

$$\mathbb{P}_f(\partial K \cap H^-) \leq \tfrac{1}{2}(T+t_0).$$

But this contradicts the definition of t_0. □

In the next lemma we need the Hausdorff distance d_H which for two convex bodies K and L in $\mathbb{R}^n$ is

$$d_H(K,L) = \max\left\{ \max_{x\in L} \min_{y \in K} \|x-y\|, \ \max_{y\in K} \min_{x \in L} \|x-y\| \right\}.$$

Lemma 2.4. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a positive, continuous function with $\int_{\partial K} f \mathrm{d}\mu = 1$.*
(i) Suppose that K has a C^1-boundary. Let s be such that $K_s \neq \emptyset$ and let $x \in \partial K_s \cap \overset{\circ}{K}$. Let H be a supporting hyperplane of K_s at x such that $\mathbb{P}_f(\partial K \cap H^-) = s$. Then x is the center of gravity of $\partial K \cap H$ with respect to the measure

$$\frac{f(y)\mu_{\partial K\cap H}(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >}$$

i.e.

$$x = \frac{\int_{\partial K \cap H} \frac{y f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{<N_{\partial K\cap H}(y), N_{\partial K}(y)>}}{\int_{\partial K \cap H} \frac{f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{<N_{\partial K\cap H}(y), N_{\partial K}(y)>}},$$

where $N_{\partial K}(y)$ is the unit outer normal to ∂K at y and $N_{\partial K \cap H}(y)$ is the unit outer normal to $\partial K \cap H$ at y in the plane H.
(ii) If K has a C^1-boundary and $K_s \subset \overset{\circ}{K}$, then K_s is strictly convex.
(iii) Suppose that K has a C^1-boundary and $K_T \subset \overset{\circ}{K}$. Then K_T consists of one point $\{x_T\}$ only. This holds in particular, if for every $x \in \partial K$ there are $r(x) > 0$ and $R(x) < \infty$ such that $B_2^n(x - r(x)N_{\partial K}(x), r(x)) \subseteq K \subseteq B_2^n(x - R(x)N_{\partial K}(x), R(x))$.
(iv) For all s with $0 \leq s < T$ and $\epsilon > 0$ there is $\delta > 0$ such that $d_H(K_s, K_{s+\delta}) < \epsilon$.

We call the point x_T of Lemma 2.4.(iii) the surface point. If K_T does not consist of one point only, then we define x_T to be the centroid of K_T.

Proof. (i) By Lemma 2.2.(i) there is a hyperplane H with $s = \mathbb{P}_f(\partial K \cap H^-)$. Let $\tilde{H}$ be another hyperplane passing through x and ϵ the angle between the two hyperplanes. Then we have

$$s = \mathbb{P}_f(\partial K \cap H^-) \le \mathbb{P}_f(\partial K \cap \tilde{H}^-).$$

Let ξ be one of the two vectors in H with $\|\xi\| = 1$ that are orthogonal to $H \cap \tilde{H}$. Then

$$\begin{aligned} 0 &\le \mathbb{P}_f(\partial K \cap \tilde{H}^-) - \mathbb{P}_f(\partial K \cap H^-) \\ &= \int_{\partial K \cap H} \frac{< y - x, \xi > f(y) \tan \epsilon}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >} \mathrm{d}\mu_{\partial K \cap H}(y) + o(\epsilon). \end{aligned}$$

We verify the latter equality. First observe that for $y \in \partial K \cap H$ the "height" is $< y - x, \xi > \tan \epsilon$. This follows from the following two graphics.

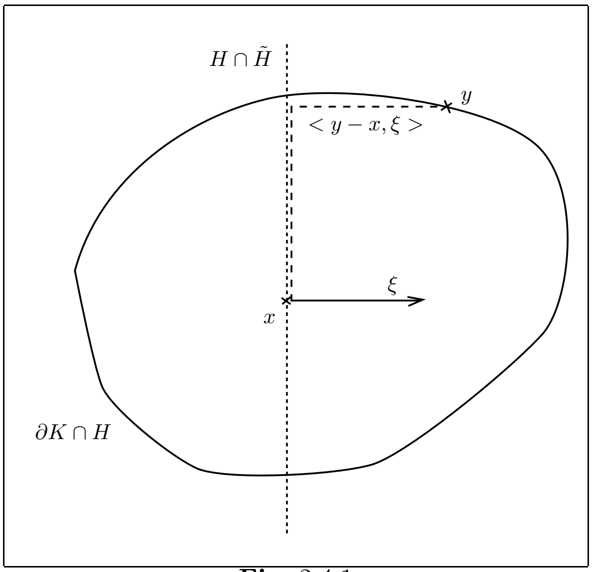

Fig. 2.4.1

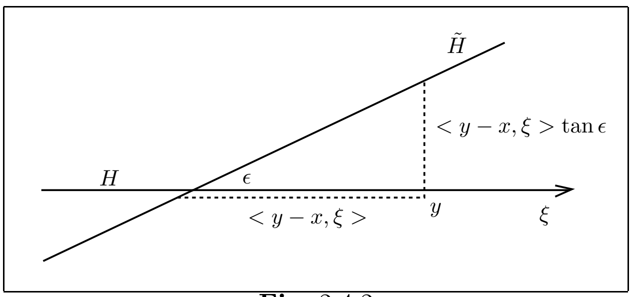

Fig. 2.4.2

A surface element at y equals, up to an error of order $o(\epsilon)$, the product of a volume element at y in $\partial K \cap H$ and the length of the tangential line segment between H and $\tilde{H}$ at y. The length of this tangential line segment is, up to an error of order $o(\epsilon)$,

$$\frac{< y-x, \xi > \tan \epsilon}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >}.$$

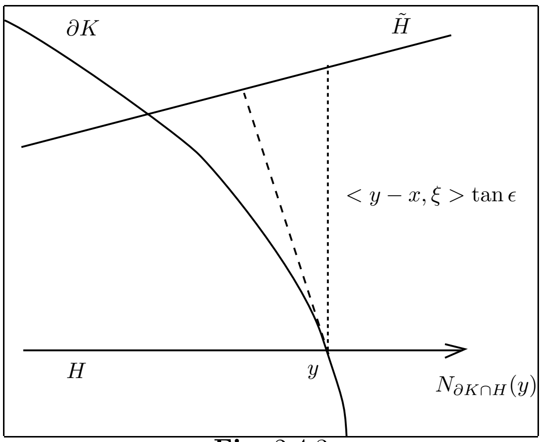

Fig. 2.4.3

Therefore,

$$0 \le \int_{\partial K \cap H} \frac{< y-x, \xi > f(y) \tan \epsilon}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >} \mathrm{d}\mu_{\partial K \cap H}(y) + o(\epsilon).$$

We divide both sides by ϵ and pass to the limit for ϵ to 0. Thus we get for all ξ

$$0 \le \int_{\partial K \cap H} \frac{< y-x, \xi > f(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >} \mathrm{d}\mu_{\partial K \cap H}(y).$$

Since this inequality holds for ξ as well as $-\xi$ we get for all ξ

$$0 = \int_{\partial K \cap H} \frac{< y-x, \xi > f(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >} \mathrm{d}\mu_{\partial K \cap H}(y)$$

or

$$0 = \left\langle \int_{\partial K \cap H} \frac{(y-x) f(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >} \mathrm{d}\mu_{\partial K \cap H}(y), \xi \right\rangle.$$

Therefore,

$$x = \frac{\int_{\partial K \cap H} \frac{y f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >}}{\int_{\partial K \cap H} \frac{f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >}}.$$

(ii) Suppose that K_s is not strictly convex. Then ∂K_s contains a line-segment $[u,v]$. Let $x \in (u,v)$. As $K_s \subseteq \overset{\circ}{K}$ it follows from Lemma 2.2.(i) that there exists a support-hyperplane $H = H(x, N_{K_s}(x))$ of K_s such that $\mathbb{P}_f(\partial K \cap H^-) = s$. Moreover, we have that $u, v \in H$.

By (i)

$$x = u = v = \frac{\int_{\partial K \cap H} \frac{y f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{<N_{\partial K \cap H}(y), N_{\partial K}(y)>}}{\int_{\partial K \cap H} \frac{f(y) \mathrm{d}\mu_{\partial K \cap H}(y)}{<N_{\partial K \cap H}(y), N_{\partial K}(y)>}}.$$

(iii) By Lemma 2.3.(ii) there is T such that K_T has volume 0. Suppose that K_T consists of more than one point. All these points are elements of the boundary of K_T since the volume of K_T is 0. Therefore ∂K_T contains a line-segment $[u,v]$ and cannot be strictly convex, contradicting (ii).

The condition: For every $x \in \partial K$ there is $r(x) < \infty$ such that $K \supseteq B_2^n(x - r(x)N_{\partial K}(x), r(x))$, implies that K has everywhere unique normals. This is equivalent to differentiability of ∂K. By Corollary 25.5.1 of [Ro] ∂K is continuously differentiable. The remaining assertion of (iii) now follows from Lemma 2.2.(ii). □

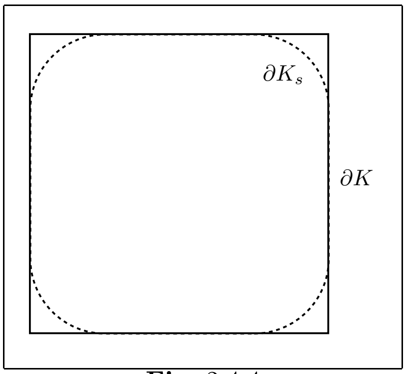

Fig. 2.4.4

We have the following remarks.

(i) The assertion of Lemma 2.2.(i) is not true if $x \in \partial K$. As an example consider the square S with sidelength 1 in $\mathbb{R}^2$ and $f(x) = \frac{1}{4}$ for all $x \in \partial S$. For $s = \frac{1}{16}$ the midpoints of the sides of the square are elements of $S_{1/16}$, but the tangent hyperplanes through these points contain one side and therefore cut off a set of $\mathbb{P}_f$-volume $\frac{1}{4}$ (compare Figure 2.4.4). The construction in higher dimensions for the cube is done in the same way.

This example also shows that the surface body is not necessarily strictly convex and it shows that the assertion of Lemma 2.2.(ii) does not hold without additional assumptions.

(ii) If K is a symmetric convex body and f is symmetric (i.e. $f(x) = f(-x)$ if the center of symmetry is 0), then the surface point x_T coincides with the center of symmetry.

If K is not symmetric then $T < \frac{1}{2}$ is possible. An example for this is a regular triangle C in $\mathbb{R}^2$. If the sidelength is 1 and $f = \frac{1}{3}$, then $T = \frac{4}{9}$ and $C_{\frac{4}{9}}$ consists of the barycenter of C.

(iii) In Lemma 2.4 we have shown that under certain assumptions the surface body reduces to a point. In general this is not the case. We give an example. Let K be the Euclidean ball B_2^n and

$$f = \frac{\chi_C + \chi_{-C}}{2\mathrm{vol}_{n-1}(C)}$$

where C is a cap of the Euclidean ball with surface area equal to $\frac{1}{4}\mathrm{vol}_{n-1}(\partial B_2^n)$. Then we get that for all s with $s < \frac{1}{2}$ that K_s contains a Euclidean ball with positive radius. On the other hand $K_{1/2} = \emptyset$.

2.2 Surface Body and the Indicatrix of Dupin

The indicatrix of Dupin was introduced in section 1.5.

Lemma 2.5. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a a.e. positive, integrable function with $\int_{\partial K} f \mathrm{d}\mu = 1$. Let $x_0 \in \partial K$. Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder). For all s such that $K_s \neq \emptyset$, let the point x_s be defined by*

$$\{x_s\} = [x_T, x_0] \cap \partial K_s.$$

Then for every $\epsilon > 0$ there is s_ϵ so that for all s with $0 < s \leq s_\epsilon$ the points x_s are interior points of K and for all normals $N_{\partial K_s}(x_s)$ (if not unique)

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \epsilon.$$

If x_0 is an interior point of an $(n-1)$-dimensional face, then, as in the example of the cube, there is $s_0 > 0$ such that we have for all s with $0 \leq s \leq s_0$ that $x_0 \in \partial K_s$. Thus $x_s = x_0$.

Proof. Let us first observe that for all s with $0 < s < T$ where T is given by Lemma 2.3.(ii) the point x_s is an interior point of K.

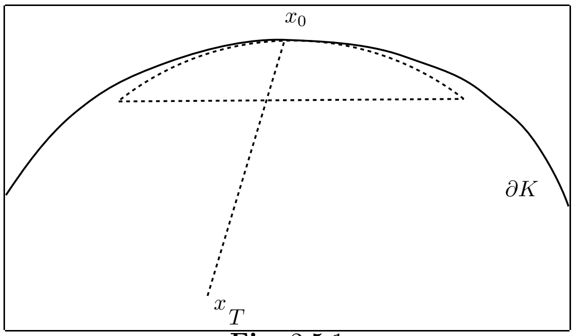

Fig. 2.5.1

First we observe that $x_0 \neq x_T$ since the indicatrix of Dupin at x_0 is an ellipsoid. Again (see Figure 2.5.1), since the indicatrix of Dupin at x_0 is an ellipsoid, (x_T, x_0) is a subset of the convex hull of a cap contained in K and x_T. Thus $(x_T, x_0) \subset \overset{\circ}{K}$. Lemma 2.1.(i) assures that

$$\mathbb{P}_f(\partial K \cap H(x_0 - tN_{\partial K}(x_0), N_{\partial K}(x_0)))$$

is a continuous function on $[0, \max_{y\in K} < x_0 - y, N_{\partial K}(x_0) >)$.

We claim now

$$\forall \delta > 0 \exists s_\delta > 0 \forall s, 0 \leq s \leq s_\delta :< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \delta.$$

Suppose that is not true. Then there is a sequence s_n, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} s_n = 0 \qquad \lim_{n\to\infty} N_{\partial K_{s_n}}(x_{s_n}) = \xi$$

where $\xi \neq N_{\partial K}(x_0)$. By Lemma 2.1.(iv) $\lim_{n\to\infty} x_{s_n} = x_0$. Thus we get

$$\lim_{n\to\infty} s_n = 0 \qquad \lim_{n\to\infty} x_{s_n} = x_0 \qquad \lim_{n\to\infty} N_{\partial K_{s_n}}(x_{s_n}) = \xi.$$

Since the normal at x_0 is unique and $\xi \neq N_{\partial K}(x_0)$ the hyperplane $H(x_0, \xi)$ contains an interior point of K. There is $y \in \partial K$ and a supporting hyperplane $H(y, \xi)$ to K at y that is parallel to $H(x_0, \xi)$. There is $\epsilon > 0$ and n_0 such that for all n with $n \geq n_0$

$$B_2^n(y, \epsilon) \cap H^+(x_{s_n}, N_{\partial K_{s_n}}(x_{s_n})) = \emptyset.$$

Thus we get

$$B_2^n(y, \epsilon) \cap \bigcup_{n \geq n_0} K_{s_n} = \emptyset.$$

On the other hand, by Lemma 2.1.(iv) we have

$$\bigcup_{s>0} K_s \supseteq \overset{\circ}{K}.$$

This is a contradiction. $\square$

Lemma 2.6. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a diagonal matrix with $a_i > 0$ for all $i = 1, \dots, n$. Then we have for all $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\| = 1$*

$$\left\| \frac{Ax}{\|Ax\|} - \frac{Ay}{\|Ay\|} \right\| \le 2 \left(\frac{\max_{1\le i\le n} a_i}{\min_{1\le i\le n} a_i} \right) \|x - y\|.$$

In particular we have

$$1 - \left\langle \frac{Ax}{\|Ax\|}, \frac{Ay}{\|Ay\|} \right\rangle \le 2 \left(\frac{\max_{1\le i\le n} a_i}{\min_{1\le i\le n} a_i} \right)^2 \|x - y\|^2.$$

Proof. We have

$$\|Ax - Ay\| \le (\max_{1\le i\le n} a_i)\|x - y\|$$

and

$$\begin{aligned}
\left\| \frac{Ax}{\|Ax\|} - \frac{Ay}{\|Ay\|} \right\| &\le \left\| \frac{Ax}{\|Ax\|} - \frac{Ay}{\|Ax\|} \right\| + \left\| \frac{Ay}{\|Ax\|} - \frac{Ay}{\|Ay\|} \right\| \\
&\le \frac{(\max_{1\le i\le n} a_i)\|x - y\|}{\|Ax\|} + \frac{|\|Ax\| - \|Ay\||}{\|Ax\|\|Ay\|}\|Ay\| \\
&\le 2\frac{(\max_{1\le i\le n} a_i)\|x - y\|}{\|Ax\|}.
\end{aligned}$$

Since $\|x\| = 1$ we have $\|Ax\| \ge \min_{1\le i\le n} |a_i|\|x\|$. □

By Lemma 2.5 the normal to ∂K_s at x_s differs little from the normal to K at x_0 if s is small. Lemma 2.7 is a strengthening of this result.

Lemma 2.7. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Let $f : \partial K \to \mathbb{R}$ be an integrable, a.e. positive function with $\int_{\partial K} f d\mu = 1$ that is continuous at x_0. Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder). For all s such that $K_s \ne \emptyset$, let x_s be defined by $\{x_s\} = [x_T, x_0] \cap \partial K_s$.*
(i) Then for every $\epsilon > 0$ there is s_ϵ so that for all s with $0 < s \le s_\epsilon$ the points x_s are interior points of K and

$$s \le \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K}(x_0))) \le (1 + \epsilon)s.$$

(ii) Then for every $\epsilon > 0$ there is s_ϵ so that for all s with $0 < s \le s_\epsilon$ and all normals $N_{\partial K_s}(x_s)$ at x_s

$$s \le \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \le (1 + \epsilon)s.$$

Proof. We position K so that $x_0 = 0$ and $N_{\partial K}(x_0) = e_n$. Let b_i, $i = 1, \dots, n-1$ be the lenghts of the principal axes of the indicatrix of Dupin. Then, by Lemma 1.2 and (3) the lengths of the principal axes of the standard approximating ellipsoid $\mathcal{E}$ at x_0 are given by

$$a_i = b_i \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{1}{n-1}} \qquad i = 1, \dots, n-1 \qquad \text{and} \qquad a_n = \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}.$$

We consider the transform $T : \mathbb{R}^n \to \mathbb{R}^n$ (5)

$$T(x) = \left(\frac{x_1}{a_1} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \dots, \frac{x_{n-1}}{a_{n-1}} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, x_n \right). \tag{7}$$

This transforms the standard approximating ellipsoid into a Euclidean ball with radius $r = (\prod_{i=1}^{n-1} b_i)^{2/(n-1)}$. T is a diagonal map with diagonal elements $\frac{\sqrt{a_n}}{b_1}, \dots, \frac{\sqrt{a_n}}{b_{n-1}}, 1$.

Let $\epsilon > 0$ be given. Let $\delta > 0$ be such that

$$\frac{(1+\delta)^{\frac{5}{2}}}{(1-\delta)(1-c^2\delta)^3} \leq 1 + \epsilon,$$

where

$$c = 2 \frac{\max \left\{ \max_{1 \leq i \leq n-1} \frac{b_i}{\sqrt{a_n}}, \ 1 \right\}}{\min \left\{ \min_{1 \leq i \leq n-1} \frac{b_i}{\sqrt{a_n}}, \ 1 \right\}}.$$

As f is continuous at x_0 there exists a neighborhood $B_2^n(x_0, \alpha)$ of x_0 such that for all $x \in B_2^n(x_0, \alpha) \cap \partial K$

$$f(x_0)\ (1-\delta) \leq f(x) \leq f(x_0)\ (1+\delta). \tag{8}$$

By Lemma 2.5, for all $\rho > 0$ there exists $s(\rho)$ such that for all s with $0 < s \leq s(\rho)$

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \rho \tag{9}$$

and the points x_s are interior points of K.

Therefore, for $\delta > 0$ given, it is possible to choose $s(\delta)$ such that for all s with $0 < s \leq s(\delta)$, $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$ differ so little that both of the following hold

$$\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \subseteq B_2^n(x_0, \alpha) \tag{10}$$

and

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \delta. \tag{11}$$

Indeed, in order to obtain (11) we have to choose ρ smaller than δ. In order to satisfy (10) we choose $s(\delta)$ so small that the distance of x_s to x_0 is less than one half of the height of the biggest cap of K with center x_0 that is contained in the set $K \cap B_2^n(x_0, \alpha)$. Now we choose ρ in (9) sufficiently small so that (10) holds.

As the points x_s are interior points of K, by Lemma 2.2.(i), for all s with $0 < s \leq s(\delta)$ there is $N_{\partial K_s}(x_s)$ such that

$$s = \mathbb{P}_f(\partial K \cap H(x_s, N_{\partial K_s}(x_s)). \tag{12}$$

Please note that

$$\frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \tag{13}$$

is the normal of the hyperplane

$$T(H(x_s, N_{\partial K_s}(x_s))).$$

We observe next that (9) implies that for all $\rho > 0$ there exists $s(\rho)$ such that for all $s \leq s(\rho)$

$$\left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle \geq 1 - c^2\rho, \tag{14}$$

where T^{-1t} is the transpose of the inverse of T and c the constant above.

Indeed, since

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \rho$$

we have

$$\|N_{\partial K}(x_0) - N_{\partial K_s}(x_s)\| \leq \sqrt{2\rho}.$$

Now we apply Lemma 2.6 to the map T^{-1t}. Since $N_{\partial K}(x_0) = e_n = T^{-1t}(e_n) = T^{-1t}(N_{\partial K}(x_0))$ we obtain with

$$c = 2\frac{\max\{\max_{1\leq i\leq n-1} \frac{b_i}{\sqrt{a_n}},\ 1\}}{\min\{\min_{1\leq i\leq n-1} \frac{b_i}{\sqrt{a_n}},\ 1\}}$$

that

$$\left\| N_{\partial K}(x_0) - \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\| \leq c\sqrt{2\rho}$$

which is the same as

$$1 - c^2\rho \leq \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle.$$

By Lemma 1.4, for δ given there exists t_1 such that for all t with $t \le t_1$

$$\begin{aligned} &\mathrm{vol}_{n-1}(K \cap H(x_0 - t\ N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ &\quad \le \mathrm{vol}_{n-1}(\partial K \cap H^-(x_0 - t\ N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ &\quad \le (1+\delta)\sqrt{1 + \frac{2ta_n^3}{(a_n - t)^2 \min_{1\le i\le n-1} a_i^2}} \\ &\qquad\qquad \times \mathrm{vol}_{n-1}(K \cap H(x_0 - t\ N(x_0), N(x_0))). \end{aligned} \tag{15}$$

Recall that r is the radius of the approximating Euclidean ball for $T(K)$ at $x_0 = 0$. For δ given, we choose $\eta = \eta(\delta)$ such that

$$\eta < \min\left\{ r\ \frac{1-(1-c^2\delta)^{\frac{2}{n-1}}}{1+(1-c^2\delta)^{\frac{2}{n-1}}},\ \delta \right\}. \tag{16}$$

Then, for such an η, by Lemma 1.2, there is $t_2 > 0$ so that we have for all t with $0 \le t \le t_2$

$$\begin{aligned} &B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H(x_0 - t\ N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ &\subseteq T(K) \cap T(H(x_0 - t\ N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ &\subseteq B_2^n(x_0 - (r+\eta)N_{\partial K}(x_0), r+\eta) \cap T(H(x_0 - t\ N_{\partial K}(x_0), N_{\partial K}(x_0))). \end{aligned} \tag{17}$$

Let $t_0 = \min\{t_1, t_2\}$.

By (14) we can choose $s(\eta)$ such that for all $s \le s(\eta)$, $N_{\partial K}(x_0)$ and the normal to $T(H(x_s, N_{\partial K_s}(x_s)))$ differ so little that both of the following hold

$$\left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle \ge 1 - c^2\eta \ge 1 - c^2\delta \tag{18}$$

and

$$\begin{aligned} \min\{y_n | y = (y_1, \dots, y_n) \in T(H(x_s, N_{\partial K_s}(x_s))) \\ \cap B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta)\} \ge -t_0. \end{aligned} \tag{19}$$

Then we get by (17) for all s with $0 < s \le s(\eta)$

$$\begin{aligned} &B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))) \\ &\quad \subseteq T(K) \cap T(H(x_s, N_{\partial K_s}(x_s))) \\ &\quad \subseteq B_2^n(x_0 - (r+\eta)N_{\partial K}(x_0), r+\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))). \end{aligned} \tag{20}$$

The set on the left hand side of (20) is a $(n-1)$-dimensional Euclidean ball whose radius is greater or equal

$$\sqrt{2(r-\eta)h_s - h_s^2} \tag{21}$$

where h_s is the distance of $T(x_s)$ to the boundary of the Euclidean ball $B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta)$. See Figure 2.7.1. The height of the cap

$$K \cap H^-(x_s, N_{\partial K}(x_0))$$

is denoted by Δ_s. It is also the height of the cap

$$K \cap H^-(T(x_s), N_{\partial K}(x_0))$$

because T does not change the last coordinate. Let θ be the angle between $x_0 - T(x_T)$ and $N_{\partial K}(x_0)$. Then we have by the Pythagorean theorem

$$((r-\eta) - h_s)^2 = ((r-\eta) - \Delta_s)^2 + (\Delta_s \tan\theta)^2$$

and consequently

$$h_s = (r-\eta)\left[1 - \sqrt{\left(1 - \frac{\Delta_s}{r-\eta}\right)^2 + \left(\frac{\Delta_s \tan\theta}{r-\eta}\right)^2}\,\right].$$

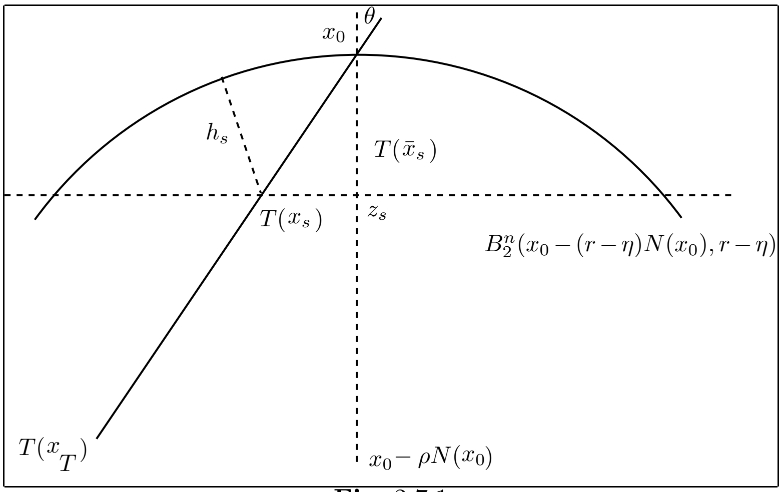

Fig. 2.7.1

x_0 and $T(x_s)$ are in the plane that can be seen in Figure 2.7.1. We use now $\sqrt{1-t} \le 1 - \frac{1}{2}t$ to get that

$$h_s \ge \Delta_s - \frac{1}{2}\frac{\Delta_s^2}{r-\eta}\left(1 + \tan^2\theta\right). \tag{22}$$

Now we prove (i). The inequality

$$s \le \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K}(x_0)))$$

holds because H passes through x_s. We show the right hand inequality. Let ϵ, δ and η be as above. We choose s_δ such that

1. $s_\delta \ \leq \ \min \, \{s(\delta), s(\eta) \ \}$
2. $\Delta_{s_\delta} \leq \min \left\{ t_0, \frac{a_n}{2}, (r-\eta), \frac{a_n^2 \delta}{8 \ \min_{1 \leq i \leq (n-1)} b_i}, \frac{4c^2\delta(r-\eta)}{(n-1)(1+\tan^2\theta)}, \right.$

$$\left. 2\left(r - \eta \frac{1+(1-c^2\delta)^{\frac{2}{n-1}}}{1-(1-c^2\delta)^{\frac{2}{n-1}}} \right) \right\}.$$

We have for all $s \leq s_\delta$

$$\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \geq \mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K_s}(x_s))).$$

Now note that

$$\mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K_s}(x_s))) = \frac{\mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s))))}{< N_{\partial K}(x_0), N_{\partial K_s}(x_s) >} \quad (23)$$

$$\geq \mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s)))) \quad (24)$$

where p_{e_n} is the orthogonal projection onto the first $n-1$ coordinates.

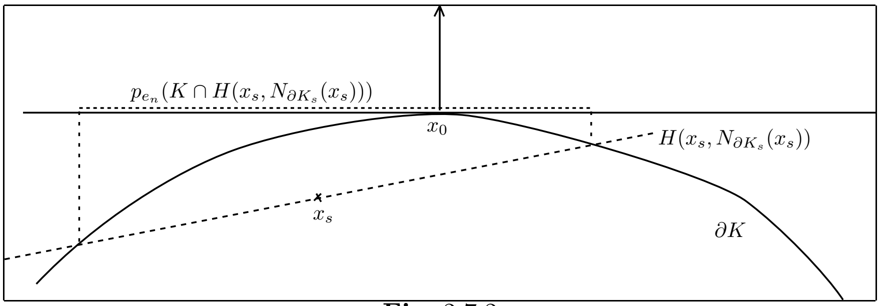

Fig. 2.7.2

Since $T \circ p_{e_n} = p_{e_n} \circ T$ and since T is volume preserving in hyperplanes that are orthogonal to e_n we get

$$\begin{aligned} &\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \\ &\quad \geq \mathrm{vol}_{n-1}(p_{e_n}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s)))) \\ &\quad = \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle \mathrm{vol}_{n-1}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s))). \end{aligned}$$

The last equality follows from (13) and (23). By (18) we then get that the latter is greater than or equal to

$$(1-c^2\delta) \ \mathrm{vol}_{n-1}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s)))),$$

which, in turn, by (20) and (21) is greater than or equal to

$$(1-c^2\delta)\mathrm{vol}_{n-1}(B_2^{n-1})\ \left(2(r-\eta)h_s-h_s^2\right)^{\frac{n-1}{2}}.$$

By (22) and as the function $\left(2(r-\eta)\Delta-\Delta^2\right)^{\frac{n-1}{2}}$ is increasing in Δ for $\Delta \leq r-\eta$, the latter is greater or equal

$$(1-c^2\delta)\mathrm{vol}_{n-1}(B_2^{n-1})\ \left(1-\frac{(1+\tan^2\theta)\Delta_s}{2(r-\eta)}\right)^{\frac{n-1}{2}}\ \left(2(r-\eta)\Delta_s-\Delta_s^2\right)^{\frac{n-1}{2}}. \tag{25}$$

In the last inequality we have also used that $(1-\frac{(1+\tan^2\theta)\Delta_s}{2(r-\eta)})^{\frac{n-1}{2}} \leq 1$.
$\Delta_s \leq \frac{4c^2\delta(r-\eta)}{(n-1)(1+\tan^2\theta)}$ implies that

$$\left(1-\frac{(1+\tan^2\theta)\Delta_s}{2(r-\eta)}\right)^{\frac{n-1}{2}} \geq \left(1-\frac{2c^2}{n-1}\delta\right)^{\frac{n-1}{2}} \geq 1-c^2\delta.$$

$\Delta_s \leq 2\left(r-\eta\frac{1+(1-c^2\delta)^{\frac{2}{n-1}}}{1-(1-c^2\delta)^{\frac{2}{n-1}}}\right)$ implies that

$$2(r-\eta)-2(r+\eta)(1-c^2\delta)^{\frac{2}{n-1}} \geq \Delta_s(1-(1-c^2\delta)^{\frac{2}{n-1}})$$

which is equivalent to

$$(2(r-\eta)-\Delta_s) \geq (1-c^2\delta)^{\frac{2}{n-1}}\ (2(r+\eta)-\Delta_s)$$

and

$$\left(2(r-\eta)\Delta_s-\Delta_s^2\right)^{\frac{n-1}{2}} \geq (1-c^2\delta)\left(2(r+\eta)\Delta_s-\Delta_s^2\right)^{\frac{n-1}{2}}.$$

Hence we get for all $s \leq s_\delta$ that (25) is greater than

$$\begin{aligned}
&(1-c^2\delta)^3\mathrm{vol}_{n-1}(B_2^{n-1})\ \left(2(r+\eta)\Delta_s-\Delta_s^2\right)^{\frac{n-1}{2}}\\
&\quad=(1-c^2\delta)^3\mathrm{vol}_{n-1}(B_2^n(x_0-(r+\eta)N_{\partial K}(x_0),r+\eta)\\
&\qquad\qquad\cap H(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)))\\
&\quad=(1-c^2\delta)^3\mathrm{vol}_{n-1}(B_2^n(x_0-(r+\eta)N_{\partial K}(x_0),r+\eta)\\
&\qquad\qquad\cap\, T(H(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)))),
\end{aligned}$$

as T does not change the last coordinate. By (17) the latter is greater than

$$\begin{aligned}
&(1-c^2\delta)^3\mathrm{vol}_{n-1}(T(K)\cap T(H(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)))\\
&\quad=(1-c^2\delta)^3\mathrm{vol}_{n-1}(K\cap H(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)))\\
&\quad\geq\frac{(1-c^2\delta)^3}{1+\delta}\ \frac{\mathrm{vol}_{n-1}(\partial K\cap H^-(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)))}{\left(1+\frac{2\Delta_s a_n^3}{(a_n-\Delta_s)^2\min_{1\leq i\leq(n-1)}a_i^2}\right)^{\frac{1}{2}}}\\
&\quad\geq\frac{(1-c^2\delta)^3}{(1+\delta)^{\frac{3}{2}}}\mathrm{vol}_{n-1}(\partial K\cap H^-(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0))).
\end{aligned}$$

The second last inequality follows with (15) and the last inequality follows as $\Delta_s \leq \frac{a_n^2 \delta}{8 \ \min_{1\leq i\leq (n-1)} b_i}$.
Therefore we get altogether that

$$\begin{aligned} &\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \qquad (26)\\ &\quad \geq \frac{(1-c^2\delta)^3}{(1+\delta)^{\frac{3}{2}}} \mathrm{vol}_{n-1}(\partial K \cap H^-(x_0 - \Delta_s N_{\partial K}(x_0), N_{\partial K}(x_0))). \end{aligned}$$

Hence, by (12)

$$s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) = \int_{\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))} f(x) \mathrm{d}\mu.$$

By (8)

$$s \geq (1-\delta) f(x_0) \mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))).$$

By (26)

$$s \geq \frac{(1-\delta)(1-c^2\delta)^3}{(1+\delta)^{\frac{3}{2}}} f(x_0) \mathrm{vol}_{n-1}(\partial K \cap H^-(x_0 - \Delta_s N_{\partial K}(x_0), N_{\partial K}(x_0)))).$$

By (8) and (10)

$$\begin{aligned} s &\geq \frac{(1-\delta)(1-c^2\delta)^3}{(1+\delta)^{\frac{5}{2}}} \int_{\partial K \cap H^-(x_0 - \Delta_s N_{\partial K}(x_0), N_{\partial K}(x_0)))} f(x) \mathrm{d}\mu \\ &= \frac{(1-\delta)(1-c^2\delta)^3}{(1+\delta)^{\frac{5}{2}}} \mathbb{P}_f(\partial K \cap H^-(x_0 - \Delta_s N_{\partial K}(x_0), N_{\partial K}(x_0)))). \end{aligned}$$

For ϵ given, we choose now $s_\epsilon = s_\delta$. By our choice of δ, this finishes (i).

(ii) We assume that the assertion is not true. Then

$$\exists \epsilon > 0 \forall s_\epsilon > 0 \exists s, 0 < s < s_\epsilon \exists N_{\partial K_s}(x_s) : \mathbb{P}_f(\partial K \cap H(x_s, N_{\partial K_s}(x_s))) \geq (1+\epsilon) s.$$

We consider $y_s \in H(x_s, N_{\partial K_s}(x_s))$ such that $T(y_s)$ is the center of the $n-1$-dimensional Euclidean ball

$$B_2^n(x_0 - (r-\eta)N(x_0), r-\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))).$$

Since $y_s \in H(x_s, N_{\partial K_s}(x_s))$ we have $y_s \notin \overset{\circ}{K}_s$. Consequently, by the definition of K_s there is a hyperplane H such that $y_s \in H$ and $\mathbb{P}_f(\partial K \cap H^-) \leq s$.

On the other hand, we shall show that for all hyperplanes H with $y_s \in H$ we have $\mathbb{P}_f(\partial K \cap H^-) > s$ which gives a contradiction.

We choose δ as in the proof of (i) and moreover so small that $\epsilon > 10\delta$ and s_δ small enough so that the two following estimates hold.

$$\begin{aligned}(1+\epsilon)s &\leq \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \\ &\leq (1+\delta)f(x_0)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))\end{aligned}$$

We verify this. As f is continuous at x_0, for all $\delta > 0$ there exists α such that for all $x \in B_2^n(x_0, \alpha) \cap \partial K$

$$(1-\delta)f(x_0) \leq f(x) \leq (1+\delta)f(x_0).$$

By Lemma 2.5, for all $\rho > 0$ there is s_ρ such that for all s with $0 < s \leq s_\rho$

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \rho.$$

Moreover, the indicatrix at x_0 exists and is an ellipsoid. Therefore we can choose s_ρ sufficiently small so that for all s with $0 < s \leq s_\rho$

$$\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \subseteq B_2^n(x_0, \alpha).$$

Thus there is s_δ such that for all s with $0 < s \leq s_\delta$

$$\begin{aligned}\mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) &= \int_{\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))} f(x) \mathrm{d}\mu(x) \\ &\leq (1+\delta)f(x_0)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))).\end{aligned}$$

Thus

$$(1+\epsilon)s \leq (1+\delta)f(x_0)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))).$$

Since the indicatrix at x_0 exists and is an ellipsoid for all ρ there is s_ρ such that for all $x \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s))$

$$< N_{\partial K}(x), N_{\partial K_s}(x_s) > \geq 1 - \rho.$$

Therefore

$$(1+\epsilon)s \leq (1+2\delta)f(x_0)\mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K_s}(x_s)))$$

which by (23) equals

$$(1+2\delta)f(x_0)\frac{\mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s))))}{< N_{\partial K}(x_0), N_{\partial K_s}(x_s) >}.$$

By Lemma 2.5 for all s with $0 < s \leq s_\delta$

$$(1+\epsilon)s \leq (1+3\delta)f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s)))).$$

Since $T \circ p_{e_n} = p_{e_n} \circ T$ and since T is volume preserving in hyperplanes that are orthogonal to e_n we get

$$(1+\epsilon)s \leq (1+3\delta)f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s))))).$$

Since

$$T(K) \cap T(H(x_s, N_{\partial K_s}(x_s)))) \subseteq B_2^n(x_0 - (r+\eta)N_{\partial K}(x_0), r+\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))))$$

we get

$$(1+\epsilon)s \le (1+3\delta)f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(B_2^n(x_0 - (r+\eta)N_{\partial K}(x_0), r+\eta)) \cap T(H(x_s, N_{\partial K_s}(x_s))))$$

and thus

$$(1+\epsilon)s \le (1+4\delta)f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))))).$$

Since $T(y_s)$ is the center of

$$B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H(x_s, N_{\partial K_s}(x_s))))$$

we have for all hyperplanes H with $y_s \in H$

$$(1+\epsilon)s \le (1+4\delta)f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H)).$$

Thus we get for all hyperplanes H with $y_s \in H$ and

$$B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H) \subseteq T(K) \cap T(H)$$

that

$$(1+\epsilon)s \le (1+5\delta)\mathbb{P}_f(\partial K \cap H^-).$$

Please note that $\epsilon > 10\delta$. We can choose s_δ so small that we have for all s with $0 < s \le s_\delta$ and all hyperplanes H with $y_s \in H$ and

$$B_2^n(x_0 - (r-\eta)N_{\partial K}(x_0), r-\eta) \cap T(H) \not\subseteq T(K) \cap T(H)$$

that

$$s < \mathbb{P}_f(\partial K \cap H^-).$$

Thus we have $s < \mathbb{P}_f(\partial K \cap H^-)$ for all H which is a contradiction. □

Lemma 2.8. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin at x_0 exists and is an ellipsoid. Let $f : \partial K \to \mathbb{R}$ be a a.e. positive, integrable function with $\int f \mathrm{d}\mu = 1$ that is continuous at x_0. Let $\mathcal{E}$ be the standard approximating ellipsoid at x_0. For $0 \le s \le T$ let x_s be given by*

$$\{x_s\} = [x_T, x_0] \cap \partial K_s$$

and $\bar{x}_s$ *by*

$$\{\bar{x}_s\} = H(x_s, N_{\partial K_s}(x_s)) \cap \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\}.$$

The map $\Phi : \partial K \cap H(x_s, N_{\partial K_s}(x_s)) \to \partial \mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))$ *is defined by*

$$\{\Phi(y)\} = \partial \mathcal{E} \cap \{\bar{x}_s + t(y - x_s) | t \geq 0\}.$$

Then, for every $\epsilon > 0$ *there is* s_ϵ *such that we have for all* s *with* $0 < s < s_\epsilon$ *and all* $z \in \partial \mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))$

$$\left| \frac{1}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} - \frac{1}{\sqrt{1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2}} \right| \leq \frac{\epsilon}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}}.$$

Proof. During this proof several times we choose the number s_ϵ sufficiently small in order to assure certain properties. Overall, we take the minimum of all these numbers.

Note that $\bar{x}_s \in K$ and by Lemma 2.7.(i) x_s is an interior point of K for s with $0 < s \leq s_\epsilon$. Therefore the angles between any of the normals are strictly larger than 0 and the expressions are well-defined.

Let z_s be given by

$$\{z_s\} = \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\} \cap H(x_s, N_{\partial K}(x_0)).$$

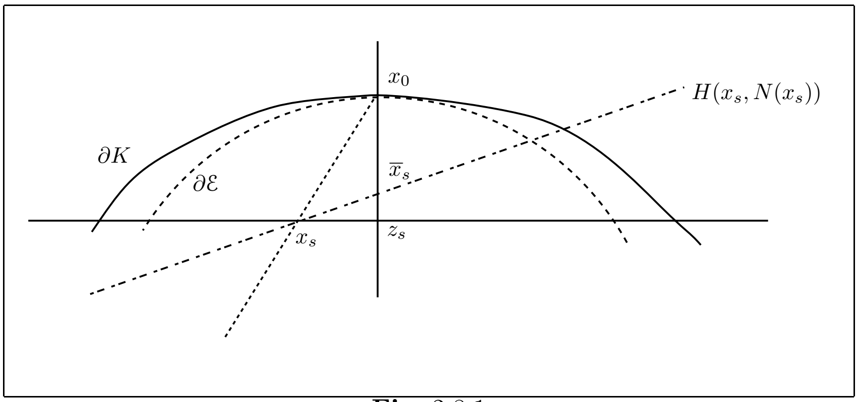

Fig. 2.8.1

In Figure 2.8.1 we see the plane through x_0 spanned by $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$. The point x_s is not necessarily in this plane, but z_s is. The point x_s is contained in the intersection of the planes $H(x_s, N_{\partial K_s}(x_s))$ and $H(x_s, N_{\partial K}(x_0))$.

As in the proof of Lemma 2.7 let b_i, $i = 1, \dots, n-1$ be the lenghts of the principal axes of the indicatrix of Dupin. Then, by Lemma 1.2 and by (3) in the standard approximating ellipsoid $\mathcal{E}$ at x_0 the lengths of the principal axes are given by

$$a_i = b_i \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{1}{n-1}} \qquad i = 1, \dots, n-1 \qquad \text{and} \qquad a_n = \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}.$$

We can assume that $x_0 = 0$ and $N_{\partial K}(x_0) = e_n$. The standard approximating ellipsoid $\mathcal{E}$ is centered at $x_0 - a_n N_{\partial K}(x_0)$ and given by

$$\sum_{i=1}^{n-1} \left| \frac{x_i}{a_i} \right|^2 + \left| \frac{x_n}{a_n} + 1 \right|^2 \leq 1.$$

We consider the transform $T : \mathbb{R}^n \to \mathbb{R}^n$

$$T(x) = \left(\frac{x_1}{a_1} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, \dots, \frac{x_{n-1}}{a_{n-1}} \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}, x_n \right).$$

See (5) and (7). This transforms the ellipsoid into a Euclidean sphere with radius $\rho = \left(\prod_{i=1}^{n-1} b_i \right)^{\frac{2}{n-1}}$, i.e.

$$T(\mathcal{E}) = B_2^n \left((0, \dots, 0, -\rho), \rho \right).$$

Let $\delta > 0$ be given. Then there exists s_δ such that for all s with $0 < s \leq s_\delta$ and all normals $N_{\partial K_s}(x_s)$ at x_s (the normal may not be unique)

$$f(x_0)\ \mathrm{vol}_{n-1}(T(\mathcal{E}) \cap T(H(x_s, N_{\partial K_s}(x_s)))) \leq (1+\delta)s. \tag{27}$$

Indeed, by Lemma 2.7.(ii) we have

$$\mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \leq (1+\delta)s.$$

Now

$$\begin{aligned} (1+\delta)s &\geq \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \\ &= \int_{\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))} f(x) \mathrm{d}\mu_{\partial K}(x). \end{aligned}$$

By continuity of f at x_0

$$\begin{aligned} (1+\delta)^2 s &\geq f(x_0) \mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \\ &\geq f(x_0) \mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K_s}(x_s))). \end{aligned}$$

We have $N_{\partial K}(x_0) = e_n$. By (23) we see that the latter equals

$$f(x_0)\frac{\mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s))))}{< N_{\partial K}(x_0), N_{\partial K_s}(x_s) >}.$$

Since $< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \leq 1$

$$(1+\delta)^2 s \geq f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s)))).$$

Since T is volume preserving in all hyperplanes orthogonal to $N_{\partial K}(x_0)$

$$(1+\delta)^2 s \geq f(x_0)\mathrm{vol}_{n-1}(T(p_{e_n}(K \cap H(x_s, N_{\partial K_s}(x_s))))).$$

Since $T \circ p_{e_n} = p_{e_n} \circ T$

$$\begin{aligned}(1+\delta)^2 s &\geq f(x_0)\mathrm{vol}_{n-1}(p_{e_n}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s))))) \\ &= f(x_0)\left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|}\right\rangle \\ &\qquad \times \mathrm{vol}_{n-1}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s)))).\end{aligned}$$

The latter equality follows since $e_n = N_{\partial K}(x_0)$. As in the proof of Lemma 2.7. (i) we get

$$(1+\delta)^3 s \geq f(x_0)\mathrm{vol}_{n-1}(T(K) \cap T(H(x_s, N_{\partial K_s}(x_s)))).$$

$T(\mathcal{E})$ approximates $T(K)$ well as $\mathcal{E}$ approximates K well. By Lemma 2.5 we have $< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \delta$. This and Lemma 1.2 give

$$(1+\delta)^4 s \geq f(x_0)\mathrm{vol}_{n-1}(T(\mathcal{E}) \cap T(H(x_s, N_{\partial K_s}(x_s)))).$$

Now we pass to a new δ and establish (27).

$\bar{x}_s$ is the point where the plane $H(x_s, N_{\partial K_s}(x_s))$ and the line through x_0 with direction $N_{\partial K}(x_0)$ intersect.

$$\{\bar{x}_s\} = H(x_s, N_{\partial K_s}(x_s)) \cap \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\}$$

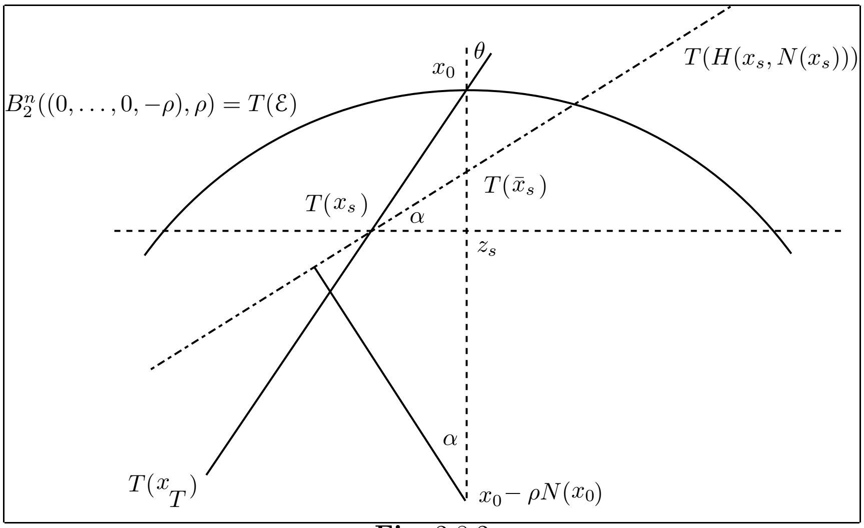

Fig. 2.8.2

In Figure 2.8.2 we see the plane through x_0 spanned by the vectors $N_{\partial K}(x_0)$ and $T^{-1t}(N_{\partial K_s}(x_s))$. The point z_s is also contained in this plane. The line through x_0, $T(x_s)$, and $T(x_T)$ is not necessarily in this plane. We see only its projection onto this plane. Also the angle θ is not necessarily measured in this plane. θ is measured in the plane spanned by $N_{\partial K}(x_0)$ and $x_0 - T(x_T)$.

α is the angle between the hyperplanes

$$T(H(x_s, N_{\partial K}(x_s))) \quad \text{and} \quad H(z_s, N_{\partial K}(x_0)).$$

Please observe that $\bar{x}_s = T(\bar{x}_s)$, $z_s = T(z_s)$ and that the plane

$$T(H(x_s, N_{\partial K_s}(x_s)))$$

is orthogonal to $T^{-1t}(N_{\partial K_s}(x_s))$.

We observe that for small enough s_δ we have for s with $0 < s \le s_\delta$

$$\|x_0 - \bar{x}_s\| \ge (1-\delta)\|x_0 - z_s\| \tag{28}$$

which is the same as

$$\|x_0 - T(\bar{x}_s)\| \ge (1-\delta)\|x_0 - z_s\|.$$

We check the inequality. Figure 2.8.2 gives us that

$$\|\bar{x}_s - z_s\| \le \tan\theta \tan\alpha \|x_0 - z_s\|.$$

We would have equality here if the angle θ would be contained in the plane that is seen in Figure 2.8.2. The angle θ is fixed, but we can make sure that the angle α is arbitrarily small. By Lemma 2.5 it is enough to choose s_δ sufficiently small. Thus (28) is established.

By Figure 2.8.2 the radius of the $n-1$-dimensional ball

$$B_2^n(x_0 - \rho N_{\partial K}(x_0), \rho) \cap T(H(x_s, N_{\partial K_s}(x_s)))$$

with $\rho = \left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}$ equals

$$\sqrt{\rho^2 - (\rho - \|x_0 - \bar{x}_s\|)^2 \cos^2 \alpha}$$

which by (28) is greater than or equal to

$$\begin{aligned} &\sqrt{\rho^2 - (\rho - (1-\delta)\|x_0 - z_s\|)^2 \cos^2 \alpha} \\ &\qquad = \sqrt{\rho^2 - (\rho - (1-\delta)\|x_0 - z_s\|)^2 \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle^2}. \end{aligned}$$

By (27) we get with a new δ

$$\begin{aligned} &\left[\rho^2 - (\rho - (1-\delta)\|x_0 - z_s\|)^2 \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle^2\right]^{\frac{n-1}{2}} \\ &\qquad\qquad\qquad \times \mathrm{vol}_{n-1}(B_2^{n-1}) \\ &\qquad \le \mathrm{vol}_{n-1}(T(\mathcal{E}) \cap H(T(x_s), T^{-1t}(N_{\partial K_s}(x_s)))) \le \frac{(1+\delta)s}{f(x_0)}. \end{aligned} \tag{29}$$

On the other hand,

$$\begin{aligned} s \le \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K}(x_0))) &= \mathbb{P}_f(\partial K \cap H^-(z_s, N_{\partial K}(x_0))) \\ &= \int_{\partial K \cap H^-(z_s, N_{\partial K}(x_0))} f(x) \mathrm{d}\mu(x). \end{aligned}$$

Now we use the continuity of f at x_0 and Lemma 1.4 to estimate the latter.

$$s \le (1+\delta) f(x_0) \mathrm{vol}_{n-1}(K \cap H(z_s, N_{\partial K}(x_0)))$$

As above we use that T is volume-preserving in hyperplanes orthogonal to $N_{\partial K}(x_0)$. Note that $T(H(z_s, N_{\partial K}(x_0))) = H(z_s, N_{\partial K}(x_0))$.

$$s \le (1+\delta) f(x_0) \mathrm{vol}_{n-1}(T(K) \cap H(z_s, N_{\partial K}(x_0)))$$

Since $T(\mathcal{E})$ approximates $T(K)$ well (Lemma 1.2)

$$s \le (1+\delta)^2 f(x_0) \mathrm{vol}_{n-1}(T(\mathcal{E}) \cap H(z_s, N_{\partial K}(x_0))).$$

Therefore (29) is less than

$$\begin{aligned}
&(1+\delta)^3 \mathrm{vol}_{n-1}(T(\mathcal{E}) \cap H(z_s, N_{\partial K}(x_0))) \\
&\quad = (1+\delta)^3 (\rho^2 - (\rho - \|x_0 - z_s\|)^2)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}(B_2^{n-1}) \\
&\quad = (1+\delta)^3 (2\rho\|x_0 - z_s\| - \|x_0 - z_s\|^2)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}(B_2^{n-1}).
\end{aligned}$$

From this we get

$$\begin{aligned}
\rho^2 - (\rho - (1-\delta)\|x_0 - z_s\|)^2 \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle^2 \\
\leq (1+\delta)^{\frac{6}{n-1}} (2\rho\|x_0 - z_s\| - \|x_0 - z_s\|^2)
\end{aligned}$$

which gives us

$$\begin{aligned}
&(\rho - (1-\delta)\|x_0 - z_s\|)^2 \left(1 - \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle^2 \right) \\
&\quad \leq (1+\delta)^{\frac{6}{n-1}} (2\rho\|x_0 - z_s\| - \|x_0 - z_s\|^2) \\
&\qquad\qquad -2(1-\delta)\rho\|x_0 - z_s\| + (1-\delta)^2\|x_0 - z_s\|^2.
\end{aligned}$$

This is less than $c\delta\rho\|x_0 - z_s\|$ where c is a numerical constant.

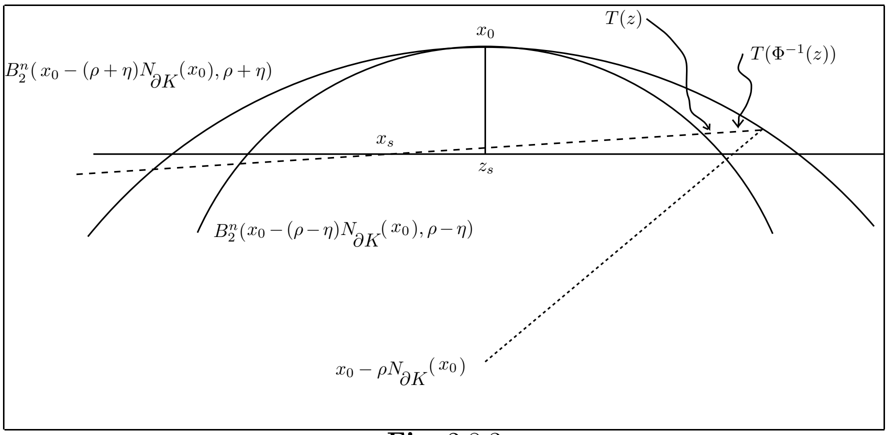

Fig. 2.8.3

Thus we have

$$1 - \left\langle N_{\partial K}(x_0), \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|} \right\rangle^2 \leq c\delta \frac{\rho\|x_0 - z_s\|}{(\rho - \|x_0 - z_s\|)^2}.$$

If we choose s_δ sufficiently small we get for all s with $0 < s \leq s_\delta$

$$1-\left\langle N_{\partial K}(x_0),\frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|}\right\rangle^2\le\delta\|x_0-z_s\|. \tag{30}$$

This is equivalent to

$$1-\left\langle N_{\partial K}(x_0),\frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|}\right\rangle\le\delta\|x_0-z_s\| \tag{31}$$

which is the same as

$$\left\|N_{\partial K}(x_0)-\frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|}\right\|\le\sqrt{2\delta\|x_0-z_s\|}. \tag{32}$$

Now we show that for every $\epsilon>0$ there is s_ϵ such that we have for all s with $0<s\le s_\epsilon$

$$\|N_{\partial K}(\Phi^{-1}(z))-N_{\partial\mathcal{E}}(z)\|\le\epsilon\sqrt{\|x_0-z_s\|}. \tag{33}$$

By Lemma 2.6 it is enough to show

$$\left\|\frac{T^{-1t}(N_{\partial K}(\Phi^{-1}(z)))}{\|T^{-1t}(N_{\partial K}(\Phi^{-1}(z)))\|}-\frac{T^{-1t}(N_{\partial\mathcal{E}}(z))}{\|T^{-1t}(N_{\partial\mathcal{E}}(z))\|}\right\|\le\epsilon\sqrt{\|x_0-z_s\|}.$$

T transforms the approximating ellipsoid $\mathcal{E}$ into the Euclidean ball $T(\mathcal{E})=B_2^n(x_0-\rho N_{\partial K}(x_0),\rho)$. We have

$$N_{\partial TK}(T(\Phi^{-1}(z)))=\frac{T^{-1t}(N_{\partial K}(\Phi^{-1}(z)))}{\|T^{-1t}(N_{\partial K}(\Phi^{-1}(z)))\|}$$

and

$$N_{\partial T\mathcal{E}}(T(z))=\frac{T^{-1t}(N_{\partial\mathcal{E}}(z))}{\|T^{-1t}(N_{\partial\mathcal{E}}(z))\|}.$$

Therefore, the above inequality is equivalent to

$$\|N_{\partial TK}(T(\Phi^{-1}(z)))-N_{\partial T\mathcal{E}}(T(z))\|\le\epsilon\sqrt{\|x_0-z_s\|}.$$

$T(z)$ and $T(\Phi^{-1}(z)))$ are elements of the hyperplane $T(H(x_s,N_{\partial K_s}(x_s)))$ that is orthogonal to $T^{-1t}(N_{\partial K_s}(x_s))$. We want to verify now this inequality.

It follows from Lemma 1.2 that for every η there is a δ so that

$$\begin{aligned}&B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0))\\&\subseteq T(K)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0))\\&\subseteq B_2^n(x_0-(\rho+\eta)N_{\partial K}(x_0),\rho+\eta)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0)).\end{aligned} \tag{34}$$

For s_η sufficiently small we get for all s with $0<s\le s_\eta$

$$\begin{aligned}T(H^-(x_s,&N_{\partial K_s}(x_s)))\cap B_2^n(x_0-(\rho+\eta)N_{\partial K}(x_0),\rho+\eta)\\&\subseteq H^-(x_0-2\|x_0-z_s\|N_{\partial K}(x_0),N_{\partial K}(x_0))\\&\qquad\cap B_2^n(x_0-(\rho+\eta)N_{\partial K}(x_0),\rho+\eta).\end{aligned} \tag{35}$$

We verify this. By (30) the angle β between the vectors

$$N_{\partial K}(x_0) \qquad \text{and} \qquad \frac{T^{-1t}(N_{\partial K_s}(x_s))}{\|T^{-1t}(N_{\partial K_s}(x_s))\|}$$

satisfies $\sin^2\beta \le \delta\|x_0 - z_s\|$. In case (35) does not hold we have

$$\tan\beta \ge \frac{1}{4}\sqrt{\frac{\|x_0 - z_s\|}{\rho+\eta}}.$$

This is true since $T(H(x_s, N_{\partial K_s}(x_s)))$ intersects the two hyperplanes $H(x_0 - \|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0))$ and $H(x_0 - 2\|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0))$. Compare Figure 2.8.4.

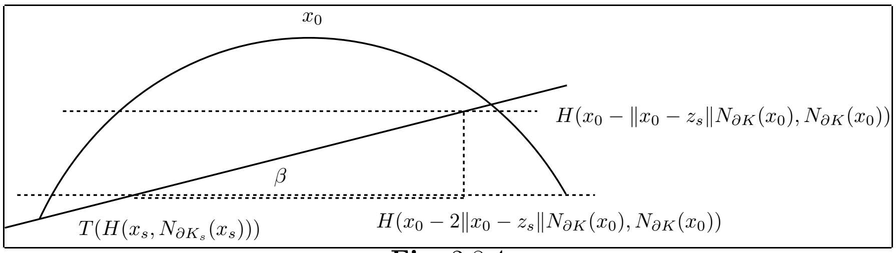

Fig. 2.8.4

This is impossible if we choose δ sufficiently small.

Let s_η be such that (35) holds. The distance of $T(\Phi^{-1}(z)))$ to the boundary of $B_2^n(x_0 - (\rho - \eta)N_{\partial K}(x_0), \rho - \eta)$ is less than $\frac{4\eta}{\rho-\eta}\|x_0 - z_s\|$. We check this. $T(\Phi^{-1}(z)))$ is contained in $B_2^n(x_0 - (\rho + \eta)N_{\partial K}(x_0), \rho + \eta)$ but not in $B_2^n(x_0 - (\rho - \eta)N_{\partial K}(x_0), \rho - \eta)$. See Figure 2.8.5.

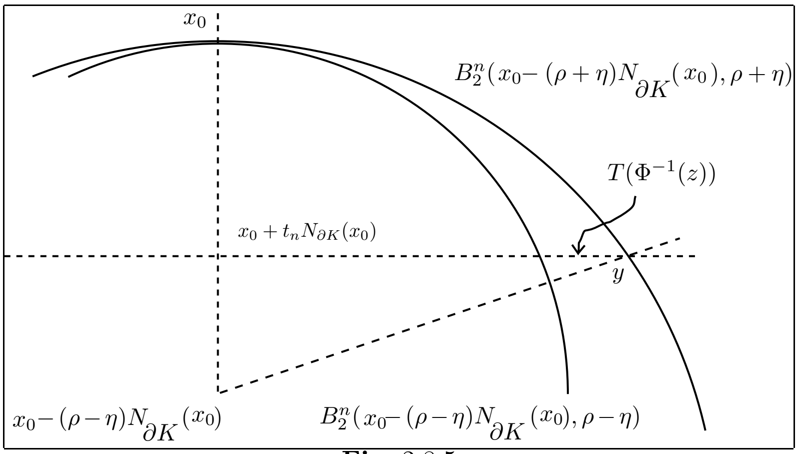

Fig. 2.8.5

Let t_n denote the n-th coordinate of $T(\Phi^{-1}(z)))$. By Figure 2.8.5 we get

$$\begin{aligned}
&\|(x_0-(\rho-\eta)N_{\partial K}(x_0))-y\|^2\\
&\qquad=(\rho-\eta-|t_n|)^2+(2|t_n|(\rho+\eta)-t_n^2)\\
&\qquad=(\rho-\eta)^2+4\eta|t_n|.
\end{aligned}$$

Thus the distance of $T(\Phi^{-1}(z)))$ to the boundary of

$$B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$$

is less than

$$\begin{aligned}
&\|(x_0-(\rho-\eta)N_{\partial K}(x_0))-y\|-(\rho-\eta)\\
&\qquad=\sqrt{(\rho-\eta)^2+4\eta|t_n|}-(\rho-\eta)\\
&\qquad=(\rho-\eta)\left\{\sqrt{1+\frac{4\eta|t_n|}{(\rho-\eta)^2}}-1\right\}\\
&\qquad\le(\rho-\eta)\frac{2\eta|t_n|}{(\rho-\eta)^2}=\frac{2\eta|t_n|}{\rho-\eta}.
\end{aligned}$$

By (35) we have $|t_n|\le 2\|x_0-z_s\|$. Thus we get

$$\|(x_0-(\rho-\eta)N_{\partial K}(x_0))-y\|-(\rho-\eta)\le\frac{4\eta\|x_0-z_s\|}{\rho-\eta}.$$

Thus the distance of $T(\Phi^{-1}(z)))$ to the boundary of

$$B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$$

is less than

$$\frac{4\eta}{\rho-\eta}\|x_0-z_s\|. \tag{36}$$

By (34)

$$\begin{aligned}
&B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0))\\
&\qquad\subseteq T(K)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0)).
\end{aligned}$$

Therefore a supporting hyperplane of $\partial T(K)$ at $T(\Phi^{-1}(z)))$ cannot intersect

$$B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)\cap H^-(x_0-\delta N_{\partial K}(x_0),N_{\partial K}(x_0)).$$

Therefore, if we choose s_ϵ small enough a supporting hyperplane of $\partial T(K)$ at $T(\Phi^{-1}(z)))$ cannot intersect

$$B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta).$$

We consider now a supporting hyperplane of $B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$ that is parallel to $T(H(\Phi^{-1}(z),N_{\partial K}(\Phi^{-1}(z))))$. Let w be the contact point of this supporting hyperplane and $B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$. Thus the hyperplane is $H(w,N_{\partial K}(\Phi^{-1}(z))))$ and

$$N_{\partial B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)}(w)=N_{\partial TK}(T(\Phi^{-1}(z))). \tag{37}$$

We introduce two points $v\in\partial B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$ and u.

$$v=x_0-(\rho-\eta)N_{\partial K}(x_0)+(\rho-\eta)\frac{T(\Phi^{-1}(z)))-(x_0-(\rho-\eta)N_{\partial K}(x_0))}{\|T(\Phi^{-1}(z)))-(x_0-(\rho-\eta)N_{\partial K}(x_0))\|}$$

$$\{u\}=[x_0-(\rho-\eta)N_{\partial K}(x_0),T(\Phi^{-1}(z))]\cap H(w,T^{-1t}(N_{\partial K}(\Phi^{-1}(z))))$$

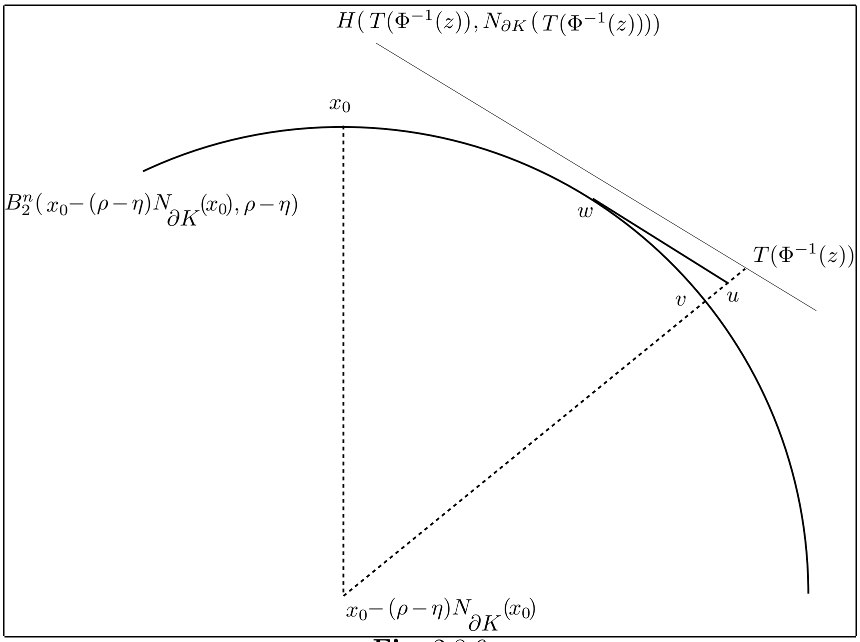

Fig. 2.8.6

We claim that

$$\|w-u\|\le\epsilon\sqrt{\|x_0-z_s\|}.$$

We check this inequality. By the Pythagorean theorem (see Figure 2.8.6)

$$\|w-u\|=\sqrt{\|u-(x_0-(\rho-\eta)N_{\partial K}(x_0))\|^2-(\rho-\eta)^2}.$$

By (36) the distance $\|T(\Phi^{-1}(z)))-v\|$ of $T(\Phi^{-1}(z)))$ to the boundary of $B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$ is less than $\frac{4\eta}{\rho-\eta}\|x-z_s\|$. Since $\|v-u\|\le\|v-T(\Phi^{-1}(z)))\|$ we get with $\epsilon=\frac{4\eta}{\rho-\eta}$

$$\begin{aligned}\|w-u\| &\leq \sqrt{(\rho-\eta+\epsilon\|x_0-z_s\|)^2-(\rho-\eta)^2}\\ &\leq \sqrt{2\epsilon\rho\|x_0-z_s\|)+(\epsilon\|x_0-z_s\|)^2}.\end{aligned}$$

This implies

$$\|w-u\| \leq \epsilon\sqrt{\|x_0-z_s\|}$$

and also

$$\|w-v\| \leq \epsilon\sqrt{\|x_0-z_s\|}.$$

Since

$$\begin{aligned}N(w) &= N_{\partial B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)}(w)\\ N(v) &= N_{\partial B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)}(v)\end{aligned}$$

we get

$$\|N(w)-N(v)\| = \frac{\|w-v\|}{\rho-\eta} \leq \epsilon\frac{\sqrt{\|x_0-z_s\|}}{\rho-\eta}.$$

Since $N(w)=N_{\partial K}(T(\Phi^{-1}(z))))$ we get

$$\|N_{\partial T(K)}(T(\Phi^{-1}(z))))-N(v)\| \leq \epsilon\frac{\sqrt{\|x_0-z_s\|}}{\rho-\eta}.$$

We observe that

$$\|v-T(z)\| \leq \frac{\epsilon}{\rho}\sqrt{\|x_0-z_s\|}.$$

This is done as above. Both points are located between the two Euclidean balls $B_2^n(x_0-(\rho-\eta)N_{\partial K}(x_0),\rho-\eta)$ and $B_2^n(x_0-(\rho+\eta)N_{\partial K}(x_0),\rho+\eta)$. The line passing through both points also intersects both balls and thus the distance between both points must be smaller than $\frac{\epsilon}{\rho}\sqrt{\|x_0-z_s\|}$.

From this we conclude in the same way as we have done for $N(v)$ and $N_{\partial K}(T(\Phi^{-1}(z))))$ that we have with a new ϵ

$$\|N(v)-N_{\partial T\mathcal{E}}(T(z))\| \leq \frac{\epsilon}{\rho}\sqrt{\|x_0-z_s\|}.$$

Therefore we get by triangle inequality

$$\|N_{\partial TK}(T(\Phi^{-1}(z))))-N_{\partial T\mathcal{E}}(T(z))\| \leq \frac{\epsilon}{\rho}\sqrt{\|x_0-z_s\|}$$

and thus finally the claimed inequality (33) with a new ϵ

$$\|N_{\partial K}(\Phi^{-1}(z))-N_{\partial\mathcal{E}}(z)\| \leq \epsilon\sqrt{\|x_0-z_s\|}.$$

Now we show

$$1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2 \geq c\|x_0-z_s\|. \tag{38}$$

For all s with $0 < s \le s_\epsilon$ the distance of $T(x_s)$ to the boundary of $T\mathcal{E} = B_2^n(x_0 - \rho N_{\partial K}(x_0), \rho)$ is larger than $c\|x_0 - z_s\|$. Thus the height of the cap

$$T\mathcal{E} \cap H^-(x_s, N_{\partial K_s}(x_s))$$

is larger than $c\|x_0 - z_s\|$. The radius of the cap is greater than $\sqrt{2c\rho\|x_0 - z_s\|}$. By Figure 2.8.2 there is a c such that we have for all s with $0 < s \le s_\eta$

$$\|T(x_s) - x_0\| \le c\|x_0 - z_s\|.$$

By triangle inequality we get with a new c

$$\|x_0 - T(z)\| \ge c\sqrt{\rho\|x_0 - z_s\|}.$$

We have

$$N_{\partial T\mathcal{E}}(T(z)) = \tfrac{1}{\rho}(T(z)) - (x_0 - \rho N_{\partial K}(x_0))).$$

We get

$$\begin{aligned} c\sqrt{\rho\|x_0 - z_s\|} &\le \|x_0 - T(z)\| \\ &= \|\rho N_{\partial K}(x_0) - (T(z) - (x_0 - \rho N_{\partial K}(x_0))))\| \\ &= \rho\|N_{\partial K}(x_0) - N_{\partial T\mathcal{E}}(T(z))\|. \end{aligned}$$

Since $T(N_{\partial K}(x_0))) = N_{\partial K}(x_0)$ we get by Lemma 2.6 with a new c

$$c\sqrt{\|x_0 - z_s\|} \le \|N_{\partial K}(x_0) - N_{\partial \mathcal{E}}(z))\|.$$

We have by (32) and Lemma 2.6

$$\|N_{\partial K}(x_0) - N_{\partial K_s}(x_s)\| \le \delta\sqrt{\|x_0 - z_s\|}. \tag{39}$$

Now we get by triangle inequality

$$c\sqrt{\|x_0 - z_s\|} \le \|N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(z))\|.$$

By (33) and triangle inequality we get

$$c\sqrt{\|x_0 - z_s\|} \le \|N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(\Phi^{-1}(z))\|.$$

Therefore we get with a new constant c

$$\begin{aligned} c\|x_0 - z_s\| &\le 1- < N_{\partial K_s}(x_s), N_{\partial K}(\Phi^{-1}(z)) > \\ &\le 1- < N_{\partial K_s}(x_s), N_{\partial K}(\Phi^{-1}(z)) >^2 . \end{aligned}$$

We have

$$
\begin{aligned}
&| < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2 - < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2 | \\
&\quad = | < N_{\partial K}(\Phi^{-1}(z)) + N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) > \times \\
&\qquad\qquad\qquad < N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) > | \\
&\quad \le 2| < N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) > | \\
&\quad \le 2| < N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(z) > | \\
&\qquad\qquad\qquad +2| < N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z), N_{\partial \mathcal{E}}(z) > | \\
&\quad \le 2\|N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z)\| \, \|N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(z)\| \\
&\qquad\qquad\qquad +2|1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial \mathcal{E}}(z) > |.
\end{aligned}
$$

By (33)

$$\|N_{\partial K}(\Phi^{-1}(z)) - N_{\partial \mathcal{E}}(z)\| \le \epsilon \sqrt{\|x_0 - z_s\|}$$

which is the same as

$$1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial \mathcal{E}}(z) > \le \tfrac{1}{2}\epsilon^2 \|x - z_s\|.$$

We get

$$
\begin{aligned}
&| < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2 - < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2 | \\
&\qquad \le 2\epsilon \sqrt{\|x_0 - z_s\|} \, \|N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(z)\| + \epsilon^2 \|x_0 - z_s\|.
\end{aligned} \tag{40}
$$

We show

$$\|N_{\partial K_s}(x_s) - N_{\partial \mathcal{E}}(z)\| \le c\sqrt{\|x_0 - z_s\|}. \tag{41}$$

By (35) we have

$$\|N_{\partial T K_s}(T x_s) - N_{\partial T \mathcal{E}}(Tz)\| \le c\sqrt{\|x_0 - z_s\|}.$$

(41) follows now from this and Lemma 2.6. (40) and (41) give now

$$
\begin{aligned}
&| < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2 - < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2 | \\
&\qquad \le 2\epsilon \sqrt{\|x_0 - z_s\|}\sqrt{\|x_0 - z_s\|} + \epsilon^2 \|x_0 - z_s\| \le 3\epsilon \|x_0 - z_s\|.
\end{aligned}
$$

With this we get

$$
\begin{aligned}
&\left| \frac{1}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} - \frac{1}{\sqrt{1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2}} \right| \\
&= \frac{\left| \sqrt{1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2} - \sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2} \right|}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}\sqrt{1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2}} \\
&\le \frac{\left| < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2 - < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2 \right|}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}\,(1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2)} \\
&\le \frac{1}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \frac{3\epsilon \|x_0 - z_s\|}{(1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2)}.
\end{aligned}
$$

By (38) we have that $1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2 \geq c\|x_0 - z_s\|$. Therefore we get

$$\left| \frac{1}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} - \frac{1}{\sqrt{1- < N_{\partial K}(\Phi^{-1}(z)), N_{\partial K_s}(x_s) >^2}} \right|$$
$$\leq \frac{3\epsilon}{c\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}}.$$

□

Lemma 2.9. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin at x_0 exists and is an ellipsoid. Let $f : \partial K \to \mathbb{R}$ be a integrable, a.e. positive function with $\int f \mathrm{d}\mu = 1$ that is continuous at x_0. Let $\bar{x}_s$ and Φ be as given in Lemma 2.8 and z_s as given in the proof of Lemma 2.8.*
(i) For every ϵ there is s_ϵ so that we have for all s with $0 < s \leq s_\epsilon$

$$\begin{aligned}(1-\epsilon) &\sup_{y \in \partial K \cap H(x_s, N_{\partial K_s}(x_s))} | < N_{\partial K}(x_0), y - x_0 > | \\ &\leq \|x_0 - z_s\| \\ &\leq (1+\epsilon) \inf_{y \in \partial K \cap H(x_s, N_{\partial K_s}(x_s))} | < N_{\partial K}(x_0), y - x_0 > |.\end{aligned}$$

(ii) For every ϵ there is s_ϵ so that we have for all s with $0 < s \leq s_\epsilon$ and all $z \in \partial \mathcal{E} \cap H(x_s, N_{\partial K_s}(x_0))$

$$\begin{aligned}(1-\epsilon) &< N_{\partial K \cap H}(\Phi^{-1}(z)), z - x_s > \\ &\leq < N_{\partial \mathcal{E} \cap H}(z), z - x_s > \\ &\leq (1+\epsilon) < N_{\partial K \cap H}(\Phi^{-1}(z))), z - x_s >\end{aligned}$$

where $H = H(x_s, N_{\partial K_s}(x_s))$ and the normals are taken in the plane H.
(iii) Let $\phi : \partial K \cap H \to \mathbb{R}$ be the real valued, positive function such that

$$\Phi(y) = \bar{x}_s + \phi(y)(y - \bar{x}_s).$$

For every ϵ there is s_ϵ such that we have for all s with $0 < s \leq s_\epsilon$ and all $y \in \partial K \cap H(x_s, N_{\partial K_s}(x_s))$

$$1 - \epsilon \leq \phi(y) \leq 1 + \epsilon.$$

Proof. We may suppose that $x_0 = 0$ and $N_{\partial K}(x_0) = e_n$.

(i) We put

$$m_s = \inf_{y \in \partial K \cap H(x_s, N_{\partial K_s}(x_s))} | < N_{\partial K}(x_0), y - x_0 > |.$$

We show now the right hand inequality. Let ρ be strictly greater than all the lengths of the principal axes of the standard approximating ellipsoid $\mathcal{E}$. Then there is $\eta > 0$

$$\mathcal{E} \cap H(x_0 - \eta N_{\partial K}(x_0), N_{\partial K}(x_0)) \subseteq B_2^n(x_0 - \rho N_{\partial K}(x_0), \rho) \cap H(x_0 - \eta N_{\partial K}(x_0), N_{\partial K}(x_0)).$$

Let α_s denote the angle between $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$. Recall that in the proof of Lemma 2.8 we put

$$\{z_s\} = \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\} \cap H(x_s, N_{\partial K}(x_0)).$$

Then we have

$$\begin{aligned} \tan \alpha_s &\geq \frac{\|x_0 - z_s\| - m_s}{c\|x_0 - z_s\| + \sqrt{\rho^2 - (\rho - \|x_0 - z_s\|)^2}} \\ &\geq \frac{\|x_0 - z_s\| - m_s}{c\|x_0 - z_s\| + \sqrt{2\rho\|x_0 - z_s\| - \|x_0 - z_s\|^2}} \\ &\geq \frac{\|x_0 - z_s\| - m_s}{c\|x_0 - z_s\| + \sqrt{2\rho\|x_0 - z_s\|}}. \end{aligned}$$

To see this consult Figure 2.9.1.

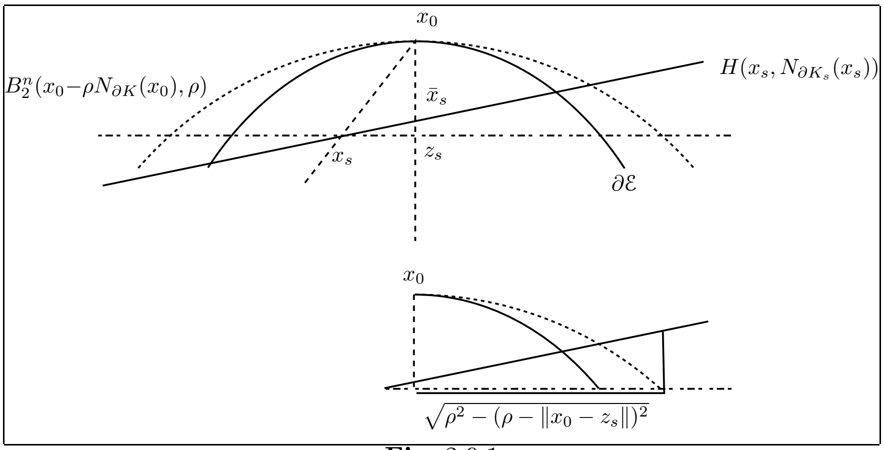

Fig. 2.9.1

In Figure 2.9.1 we see the plane through x_0 that is spanned by $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$. The point x_s is not necessarily in this plane.

On the other hand, by (39)

$$\sin^2 \alpha_s = 1 - < N_{\partial K}(x_0), N_{\partial K_s}(x_s) >^2 \leq \epsilon \|x_0 - z_s\|$$

which implies for sufficiently small ϵ

$$\tan \alpha_s \le \sqrt{2\epsilon \|x_0 - z_s\|}.$$

Altogether we get

$$\sqrt{2\epsilon \|x_0 - z_s\|} \ge \frac{\|x_0 - z_s\| - m_s}{c\|x_0 - z_s\| + \sqrt{2\rho \|x_0 - z_s\|}}$$

and thus

$$(c\sqrt{2\epsilon} + 4\sqrt{\epsilon\rho})\|x_0 - z_s\| \ge \|x_0 - z_s\| - m_s.$$

Finally we get with a new constant c

$$(1 - 2c\sqrt{\epsilon})\|x_0 - z_s\| \le m_s.$$

The left hand inequality is proved similarly.

(ii) By (i) we have for all s with $0 < s \le s_\epsilon$

$$\begin{aligned} \partial K \cap H^-(x_s + \epsilon \|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ \subseteq \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \\ \subseteq \partial K \cap H^-(x_s - \epsilon \|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0)). \end{aligned}$$

$p_{N_{\partial K}(x_0)}$ is the orthogonal projection onto the subspace orthogonal to $N_{\partial K}(x_0)$. From this we get

$$\begin{aligned} p_{N_{\partial K}(x_0)}(K \cap H(x_s + \epsilon \|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ \subseteq p_{N_{\partial K}(x_0)}(K \cap H(x_s, N_{\partial K_s}(x_s))) \\ \subseteq p_{N_{\partial K}(x_0)}(K \cap H(x_s - \epsilon \|x_0 - z_s\| N_{\partial K}(x_0), N_{\partial K}(x_0))). \end{aligned}$$

Let $\mathcal{D}$ be the indicatrix of Dupin at x_0. By Lemma 1.1 for every ϵ there is t_ϵ so that for all t with $0 < t \le t_\epsilon$

$$(1 - \epsilon)\mathcal{D} \subseteq \tfrac{1}{\sqrt{2t}} p_{N_{\partial K}(x_0)}(K \cap H(x_0 - tN_{\partial K}(x_0), N_{\partial K}(x_0))) \subseteq (1 + \epsilon)\mathcal{D}.$$

By choosing a proper s_ϵ we get for all s with $0 < s \le s_\epsilon$

$$(1 - \epsilon)\mathcal{D} \subseteq \frac{1}{\sqrt{2\|x_0 - z_s\|}} p_{N_{\partial K}(x_0)}(K \cap H(x_s, N_{\partial K_s}(x_s))) \subseteq (1 + \epsilon)\mathcal{D}. \quad (42)$$

We get the same inclusions for $\mathcal{E}$ instead of K.

$$(1 - \epsilon)\mathcal{D} \subseteq \frac{1}{\sqrt{2\|x_0 - z_s\|}} p_{N_{\partial \mathcal{E}}(x_0)}(\mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))) \subseteq (1 + \epsilon)\mathcal{D} \quad (43)$$

Consider now $y \in \partial K \cap H(x_s, N_{\partial K_s}(x_s))$ and $\Phi(y)$. Since

$$p_{N_{\partial K}(x_0)}(\bar{x}_s) = x_0 = 0$$

there is $\lambda > 0$ so that

$$p_{N_{\partial K}(x_0)}(y) = \lambda p_{N_{\partial K}(x_0)}(\Phi(y)).$$

By (42) and (43) we get with a new s_ϵ

$$\|N_{p_{N_{\partial K}(x_0)}(\partial K \cap H)}(p_{N_{\partial K}(x_0)}(y)) - N_{p_{N_{\partial K}(x_0)}(\partial \mathcal{E} \cap H)}(p_{N_{\partial K}(x_0)}(\Phi(y)))\| < \epsilon$$

where $H = H(x_s, N_{\partial K_s}(x_s))$ and the normals are taken in the subspace of the first $n-1$ coordinates. The projection $p_{N_{\partial K}(x_0)}$ is an isomorphism between $\mathbb{R}^{n-1}$ and $H(x_s, N_{\partial K_s}(x_s))$. The norm of this isomorphism equals 1 and the norm of its inverse is less than $1+\epsilon$ if we choose s_ϵ sufficiently small. Therefore, if we choose a new s_ϵ we get for all s with $0 < s \le s_\epsilon$

$$\|N_{\partial K \cap H}(y) - N_{\partial \mathcal{E} \cap H}(\Phi(y))\| < \epsilon.$$

(iii) follows from (42) and (43) and from the fact that the projection $p_{N_{\partial K}(x_0)}$ is an isomorphism between $\mathbb{R}^{n-1}$ and $H(x_s, N_{\partial K_s}(x_s))$ whose norm equals 1 and the norm of its inverse is less than $1+\epsilon$. Indeed, the norm of the inverse depends only on the angle between $\mathbb{R}^n$ and $H(x_s, N_{\partial K_s}(x_s))$. The angle between these two planes will be as small as we wish if we choose s_ϵ small enough. $\square$

Lemma 2.10. *(i) Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin at x_0 exists and is an ellipsoid. Let $f : \partial K \to \mathbb{R}$ be a integrable, a.e. positive function with $\int f \mathrm{d}\mu = 1$. Suppose that f is continuous at x_0 and $f(x_0) > 0$. Let x_s and Φ as given by Lemma 2.8 and let z_s be given as in the proof of Lemma 2.8 by*

$$\{z_s\} = \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\} \cap H(x_s, N_{\partial K}(x_0)).$$

For every $x_0 \in \partial K$ and every $\epsilon > 0$ there is s_ϵ so that we have for all s with $0 < s < s_\epsilon$

$$\begin{aligned} \Bigg| \int_{\partial K \cap H(x_s, N_{\partial K_s}(x_s))} & \frac{f(y)}{\sqrt{1- < N_{\partial K}(y), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial K \cap H(x_s, N(x_s))}(y) \\ & - \int_{\partial \mathcal{E} \cap H(z_s, N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K}(x_0) >^2}} \mathrm{d}\mu_{\partial \mathcal{E} \cap H(x_s, N(x_0))}(z) \Bigg| \\ \le \epsilon & \int_{\partial \mathcal{E} \cap H(z_s, N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))}{\sqrt{1- < N_{\partial \mathcal{E}}(z), N_{\partial K}(x_0) >^2}} \mathrm{d}\mu_{\partial \mathcal{E} \cap H(x_s, N_{\partial K}(x_0))}(z). \end{aligned}$$

(ii) Let B_2^n denote the Euclidean ball and $(B_2^n)_s$ its surface body with respect to the constant density $(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}$. Let $\{x_s\} = \partial (B_2^n)_s \cap [0, e_n]$ and H_s the tangent hyperplane to $(B_2^n)_s$ at x_s. For every $\epsilon > 0$ there is s_ϵ so that we have for all s with $0 < s < s_\epsilon$

$$
\begin{aligned}
(1-\epsilon)&\left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{n-3}{n-1}}\mathrm{vol}_{n-2}(\partial B_2^{n-1})\\
&\le \int_{\partial B_2^n\cap H_s}\frac{1}{\sqrt{1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2}}\mathrm{d}\mu_{\partial B_2^n\cap H_s}(y)\\
&\le \left(s\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{n-3}{n-1}}\mathrm{vol}_{n-2}(\partial B_2^{n-1}).
\end{aligned}
$$

(iii) Let $a_1,\dots,a_n>0$ and

$$
\mathcal{E}=\left\{x\left|\sum_{i=1}^n\left|\frac{x(i)}{a_i}\right|^2\le 1\right.\right\}.
$$

Let $\mathcal{E}_s$, $0<s\le\frac12$, be the surface bodies with respect to the constant density $(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$. Moreover, let $\lambda_{\mathcal{E}}:\mathbb{R}^+\to[0,a_n]$ be such that $\lambda_{\mathcal{E}}(s)e_n\in\partial\mathcal{E}_s$ and H_s the tangent hyperplane to $\mathcal{E}_s$ at $\lambda_{\mathcal{E}}(s)e_n$. Then, for all $\epsilon>0$ there is s_ϵ such that for all s and t with $0\le s,t\le\frac12$

$$
\begin{aligned}
&\int_{\partial\mathcal{E}\cap H_s}\frac{1}{\sqrt{1-<N_{\partial\mathcal{E}_s}(x_s),N_{\partial\mathcal{E}}(y)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H_s}(y)\\
&\qquad\le(1+\epsilon)(\tfrac{s}{t})^{\frac{n-3}{n-1}}\int_{\partial\mathcal{E}\cap H_t}\frac{1}{\sqrt{1-<N_{\partial\mathcal{E}_t}(x_t),N_{\partial\mathcal{E}}(y)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H_t}(y).
\end{aligned}
$$

Please note that $N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n)=N_{\partial\mathcal{E}}(a_ne_n)=e_n$.

Proof. (i) In the first part of the proof H denotes $H(x_s,N_{\partial K_s}(x_s))$. We prove first that for every ϵ there is s_ϵ so that we have for all s with $0<s\le s_\epsilon$

$$
\begin{aligned}
&\left|\int_{\partial K\cap H}\frac{f(y)}{\sqrt{1-<N_{\partial K}(y),N_{\partial K_s}(x_s)>^2}}\ \mathrm{d}\mu_{\partial K\cap H}(y)\right.\\
&\qquad\left.-\int_{\partial\mathcal{E}\cap H}\frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z)\right| \qquad (44)\\
&\le\epsilon\int_{\partial\mathcal{E}\cap H}\frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z).
\end{aligned}
$$

$\bar{x}_s$ and Φ are as given in Lemma 2.8. There is a real valued, positive function $\phi:\partial K\cap H\to\mathbb{R}$ such that

$$
\Phi(y)=\bar{x}_s+\phi(y)(y-\bar{x}_s).
$$

By Lemma 1.8 we have with $y=\Phi^{-1}(z)$

$$\begin{aligned}
&\int_{\partial K\cap H} \frac{f(y)}{\sqrt{1-<N_{\partial K}(y),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial K\cap H}(y)\\
&=\int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))\phi^{-n+2}(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\\
&\qquad\qquad\times\frac{<N_{\partial\mathcal{E}\cap H}(z),\frac{z}{\|z\|}>}{<N_{\partial K\cap H}(\Phi^{-1}(z)),\frac{z}{\|z\|}>}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z)\\
&=\int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))\phi^{-n+2}(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\\
&\qquad\qquad\times\frac{<N_{\partial\mathcal{E}\cap H}(z),z>}{<N_{\partial K\cap H}(\Phi^{-1}(z)),z>}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z).
\end{aligned}$$

With this we get

$$\begin{aligned}
&\left|\int_{\partial K\cap H} \frac{f(y)}{\sqrt{1-<N_{\partial K}(y),N_{\partial K_s}(x_s)>^2}}\,\mathrm{d}\mu_{\partial K\cap H}(y)\right.\\
&\qquad\qquad\left.-\int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z)\right|\\
&\le\left|\int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}\right.\\
&\qquad\left.-\frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z)\right|\\
&+\left|\int_{\partial\mathcal{E}\cap H}\frac{f(\Phi^{-1}(z))\left(1-\phi^{-n+2}(\Phi^{-1}(z))\frac{<N_{\partial\mathcal{E}\cap H}(z),z>}{<N_{\partial K\cap H}(\Phi^{-1}(z)),z>}\right)}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z)\right|.
\end{aligned}$$

By Lemma 2.8 we have

$$\begin{aligned}
&\left|\frac{1}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}-\frac{1}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\right|\\
&\qquad\le\frac{\epsilon}{\sqrt{1-<N_{\partial\mathcal{E}}(z),N_{\partial K_s}(x_s)>^2}}
\end{aligned}$$

which gives the right estimate of the first summand.

We apply Lemma 2.9.(ii) and (iii) to the second summand. The second summand is less than

$$\epsilon\int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))}{\sqrt{1-<N_{\partial K}(\Phi^{-1}(z)),N_{\partial K_s}(x_s)>^2}}\mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z).$$

Now we apply Lemma 2.8 and get that this is less than or equal to

$$3\epsilon \int_{\partial\mathcal{E}\cap H} \frac{f(\Phi^{-1}(z))}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H}(z).$$

This establishes (44). Now we show

$$\left| \int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))} \frac{f(\Phi^{-1}(y))}{\sqrt{1- < N_{\partial\mathcal{E}}(y), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))}(y) - \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z) \right|$$
$$\le \epsilon \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z). \tag{45}$$

Since f is continuous at x_0 and $f(x_0) > 0$ it is equivalent to show

$$\left| \int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))} \frac{f(x_0)}{\sqrt{1- < N_{\partial\mathcal{E}}(y), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))}(y) - \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{f(x_0)}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z) \right|$$
$$\le \epsilon \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{f(x_0)}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z)$$

which is of course the same as

$$\left| \int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))} \frac{1}{\sqrt{1- < N_{\partial\mathcal{E}}(y), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))}(y) - \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{1}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z) \right|$$
$$\le \epsilon \int_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))} \frac{1}{\sqrt{1- < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H(z_s,N_{\partial K}(x_0))}(z). \tag{46}$$

We put $\mathcal{E}$ in such a position that $N_{\partial K}(x_0) = e_n$, $x_0 = r_n e_n$, and such that $\mathcal{E}$ is given by the equation

$$\sum_{i=1}^{n} \left| \frac{y_i}{r_i} \right|^2 = 1.$$

Let $\xi \in \partial B_2^n$ and $y = (r(\xi, y_n)\xi, y_n) \in \partial\mathcal{E}$. Then

$$N_{\partial\mathcal{E}}(y) = \frac{\left(\frac{y_1}{r_1^2}, \dots, \frac{y_n}{r_n^2} \right)}{\sqrt{\sum_{i=1}^n \frac{y_i^2}{r_i^4}}} = \frac{\left(\frac{r(\xi,y_n)\xi_1}{r_1^2}, \dots, \frac{r(\xi,y_n)\xi_{n-1}}{r_{n-1}^2}, \frac{y_n}{r_n^2} \right)}{\sqrt{\frac{y_n^2}{r_n^4} + r(\xi, y_n)^2 \sum_{i=1}^{n-1} \frac{\xi_i^2}{r_i^4}}}$$

with

$$r(\xi, y_n) = \frac{\sqrt{r_n^2 - y_n^2}}{r_n \sqrt{\sum_{i=1}^{n-1} \frac{\xi_i^2}{r_i^2}}}. \tag{47}$$

As $N_{\partial K}(x_0) = e_n$ we get

$$< N_{\partial \mathcal{E}}(y), N_{\partial K}(x_0) >= \frac{y_n}{r_n^2 \sqrt{\sum_{i=1}^{n} \frac{y_i^2}{r_i^4}}}.$$

Therefore

$$\frac{1}{1- < N_{\partial \mathcal{E}}(y), N_{\partial K}(x_0) >^2} = \frac{\sum_{i=1}^{n} \frac{y_i^2}{r_i^4}}{\sum_{i=1}^{n-1} \frac{y_i^2}{r_i^4}}.$$

For $y, z \in \partial \mathcal{E}$ we get

$$\frac{1- < N_{\partial \mathcal{E}}(z), N_{\partial K}(x_0) >^2}{1- < N_{\partial \mathcal{E}}(y), N_{\partial K}(x_0) >^2} = \frac{\sum_{i=1}^{n} \frac{y_i^2}{r_i^4}}{\sum_{i=1}^{n} \frac{z_i^2}{r_i^4}} \frac{\sum_{i=1}^{n-1} \frac{z_i^2}{r_i^4}}{\sum_{i=1}^{n-1} \frac{y_i^2}{r_i^4}}.$$

For $y, z \in \partial \mathcal{E}$ with the same direction ξ we get by (47)

$$\frac{1- < N_{\partial \mathcal{E}}(z), N_{\partial K}(x_0) >^2}{1- < N_{\partial \mathcal{E}}(y), N_{\partial K}(x_0) >^2} = \frac{\sum_{i=1}^{n} \frac{y_i^2}{r_i^4}}{\sum_{i=1}^{n} \frac{z_i^2}{r_i^4}} \left(\frac{r_n^2 - z_n^2}{r_n^2 - y_n^2} \right).$$

We can choose s_ϵ sufficiently small so that we have for all s with $0 < s \leq s_\epsilon$, and all $y \in \partial \mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))$, $z \in \partial \mathcal{E} \cap H(x_s, N_{\partial K}(x_0))$

$$|y_n - r_n| < \epsilon \qquad\qquad |z_n - r_n| < \epsilon$$

and by Lemma 2.9.(i)

$$1 - \epsilon \leq \frac{r_n - z_n}{r_n - y_n} \leq 1 + \epsilon.$$

We pass to a new ϵ and obtain: We can choose s_ϵ sufficiently small so that we have for all s with $0 < s \leq s_\epsilon$, and all $y \in \partial \mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))$, $z \in \partial \mathcal{E} \cap H(x_s, N_{\partial K}(x_0))$ such that $p_{e_n}(y)$ and $p_{e_n}(z)$ are colinear

$$1 - \epsilon \leq \frac{1- < N_{\partial \mathcal{E}}(z), N_{\partial K}(x_0) >^2}{1- < N_{\partial \mathcal{E}}(y), N_{\partial K}(x_0) >^2} \leq 1 + \epsilon. \tag{48}$$

By Lemma 2.5 we have

$$< N_{\partial K}(x_0), N_{\partial K_s}(x_s) > \geq 1 - \epsilon.$$

Therefore, the orthogonal projection p_{e_n} restricted to the hyperplane

$$H(x_s, N_{\partial K_s}(x_s))$$

is a linear isomorphism between this hyperplane and $\mathbb{R}^{n-1}$ and moreover, $\|p_{e_n}\| = 1$ and $\|p_{e_n}^{-1}\| \leq \frac{1}{1-\epsilon}$. By this, there is s_ϵ such that for all s with $0 < s \leq s_\epsilon$

$$\begin{aligned}(1-\epsilon)&\int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(y),N_{\partial K_s}(x_s)>^2}} \\ &\leq \int_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))} \frac{\mathrm{d}\mu_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}(z)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(z)),N_{\partial K_s}(x_s)>^2}} \\ &\leq \int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(y),N_{\partial K_s}(x_s)>^2}}\end{aligned}$$

where $z = p_{e_n}(y)$. Let q_{e_n} denote the orthogonal projection from

$$H(x_s, N_{\partial K}(x_0))$$

to $\mathbb{R}^{n-1}$. q_{e_n} is an isometry. Therefore

$$\begin{aligned}&\int_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_0))} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_0))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(y),N_{\partial K_s}(x_s)>^2}} \\ &\quad= \int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))} \frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(q_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}.\end{aligned}$$

Thus, in order to show (46) it suffices to show

$$\begin{aligned}&\left|\int_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))} \frac{\mathrm{d}\mu_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}\right. \\ &\qquad \left.-\int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))} \frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(q_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}\right| \\ &\leq \epsilon \int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))} \frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(q_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}.\end{aligned}$$

Let $\rho : q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0))) \to p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))$ be the radial map defined by

$$\{\rho(y)\} = \{ty \mid t \geq 0\} \cap p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0))).$$

We have

$$
\begin{aligned}
&(1-\epsilon)\int_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}\frac{\mathrm{d}\mu_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}\\
&\le\int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}\frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(\rho(y))),N_{\partial K_s}(x_s)>^2}}\\
&\le(1+\epsilon)\int_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}\frac{\mathrm{d}\mu_{p_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}.
\end{aligned}
$$

To see this, consider the indicatrix of Dupin $\mathcal{D}$ of K at x_0. We have by (43)

$$
\begin{aligned}
(1-\epsilon)\mathcal{D}&\subseteq\frac{1}{\sqrt{2\|x_0-z_s\|}}q_{e_n}(\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))\subseteq(1+\epsilon)\mathcal{D}\\
(1-\epsilon)\mathcal{D}&\subseteq\frac{1}{\sqrt{2\|x_0-z_s\|}}p_{e_n}(\mathcal{E}\cap H(x_s,N_{\partial K}(x_s)))\subseteq(1+\epsilon)\mathcal{D}.
\end{aligned}
$$

They imply that with a new s_ϵ the surface element changes at most by a factor $(1+\epsilon)$. Thus, in order to verify (46), it is enough to show

$$
\begin{aligned}
&\left|\int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}\frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(p_{e_n}^{-1}(\rho(y))),N_{\partial K_s}(x_s)>^2}}\right.\\
&\qquad\left.-\int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}\frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(q_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}\right| \qquad (49)\\
&\le\epsilon\int_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}\frac{\mathrm{d}\mu_{q_{e_n}(\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0)))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}}(q_{e_n}^{-1}(y)),N_{\partial K_s}(x_s)>^2}}.
\end{aligned}
$$

We verify this. By (48) there is s_ϵ so that we have for all s with $0<s\le s_\epsilon$, and all $y\in\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))$, $z\in\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0))$ such that $p_{e_n}(y)$ and $p_{e_n}(z)$ are colinear

$$
1-\epsilon\le\frac{\|N_{\partial\mathcal{E}}(z)-N_{\partial K}(x_0)\|}{\|N_{\partial\mathcal{E}}(y)-N_{\partial K}(x_0)\|}\le 1+\epsilon.
$$

By (39) for every ϵ there is s_ϵ such that for all s with $0<s\le s_\epsilon$

$$
\|N_{\partial K}(x_0)-N_{\partial K_s}(x_s)\|\le\epsilon\sqrt{\|x_0-z_s\|}
$$

and by the formula following (2.8.13) for all $y\in\partial\mathcal{E}\cap H(x_s,N_{\partial K_s}(x_s))$ and $z\in\partial\mathcal{E}\cap H(x_s,N_{\partial K}(x_0))$

$$
\begin{aligned}
\|N_{\partial\mathcal{E}}(y)-N_{\partial K_s}(x_s)\|&\ge c\sqrt{\|x_0-z_s\|}\\
\|N_{\partial\mathcal{E}}(z)-N_{\partial K_s}(x_s)\|&\ge c\sqrt{\|x_0-z_s\|}.
\end{aligned}
$$

Therefore,

$$\|N_{\partial K}(x_0) - N_{\partial K_s}(x_s)\| \le \epsilon\sqrt{\|x_0 - z_s\|} \le \tfrac{\epsilon}{c}\|N_{\partial\mathcal{E}}(z) - N_{\partial K_s}(x_s)\|.$$

By triangle inequality

$$\|N_{\partial\mathcal{E}}(z) - N_{\partial K_s}(x_s)\| \le (1 + \tfrac{\epsilon}{c})\|N_{\partial\mathcal{E}}(z) - N_{\partial K}(x_0)\| \tag{50}$$

and the same inequality for y. In the same way we get the estimates from below. Thus there is s_ϵ so that we have for all s with $0 < s \le s_\epsilon$, and all $y \in \partial\mathcal{E} \cap H(x_s, N_{\partial K_s}(x_s))$, $z \in \partial\mathcal{E} \cap H(x_s, N_{\partial K}(x_0))$ such that $p_{e_n}(y)$ and $p_{e_n}(z)$ are colinear

$$1 - \epsilon \le \frac{\|N_{\partial\mathcal{E}}(z) - N_{\partial K_s}(x_s)\|}{\|N_{\partial\mathcal{E}}(y) - N_{\partial K_s}(x_s)\|} \le 1 + \epsilon$$

which is the same as

$$1 - \epsilon \le \frac{1 - < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}{1 - < N_{\partial\mathcal{E}}(y), N_{\partial K_s}(x_s) >^2} \le 1 + \epsilon.$$

This establishes (49) and consequently (45). Combining the formulas (44) and (45) gives

$$\begin{aligned}
&\left|\int_{\partial K \cap H(x_s, N_{\partial K_s}(x_s))} \frac{f(y)\mathrm{d}\mu_{\partial K \cap H(x_s, N(x_s))}(y)}{\sqrt{1 - < N_{\partial K}(y), N_{\partial K_s}(x_s) >^2}}\right. \\
&\qquad \left. - \int_{\partial\mathcal{E} \cap H(z_s, N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))\mathrm{d}\mu_{\partial\mathcal{E} \cap H(x_s, N(x_0))}(z)}{\sqrt{1 - < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}}\right| \\
&\qquad\qquad \le \epsilon \int_{\partial\mathcal{E} \cap H(z_s, N_{\partial K}(x_0))} \frac{f(\Phi^{-1}(z))\mathrm{d}\mu_{\partial\mathcal{E} \cap H(x_s, N_{\partial K}(x_0))}(z)}{\sqrt{1 - < N_{\partial\mathcal{E}}(z), N_{\partial K_s}(x_s) >^2}}.
\end{aligned}$$

It is left to replace $N_{\partial K_s}(x_s)$ by $N_{\partial K}(x_0)$. This is done by using the formula (50) relating the two normals.

(ii) For every $\epsilon > 0$ there is s_ϵ such that for all s with $0 < s \le s_\epsilon$

$$(1 - \epsilon)s \le \frac{\mathrm{vol}_{n-1}(B_2^n \cap H_s)}{\mathrm{vol}_{n-1}(\partial B_2^n)} \le \frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H_s^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)} = s.$$

$B_2^n \cap H_s$ is the boundary of a $n-1$-dimensional Euclidean ball with radius

$$r = \left(\frac{\mathrm{vol}_{n-1}(B_2^n \cap H_s)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.$$

Therefore

$$\left((1 - \epsilon)s \frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \le r \le \left(s \frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.$$

We have $N(x_s) = e_n$ and $\sqrt{1- < e_n, N_{\partial B_2^n}(y) >^2}$ is the sine of the angle between e_n and $N_{\partial B_2^n}(y)$. This equals the radius r of $B_2^n \cap H_s$. Altogether we get

$$\int_{\partial B_2^n \cap H_s} \frac{\mathrm{d}\mu_{\partial B_2^n \cap H_s}(y)}{\sqrt{1- < N(x_s), N_{\partial B_2^n}(y) >^2}}$$
$$= r^{n-3}\mathrm{vol}_{n-2}(\partial B_2^{n-1}) \leq \left(s \frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{\frac{n-3}{n-1}} \mathrm{vol}_{n-2}(\partial B_2^{n-1}).$$

(iii) $\mathcal{E} \cap H_s$ and $\mathcal{E} \cap H_t$ are homothetic, $n-1$-dimensional ellipsoids. The factor ϕ_0 by which we have to multiply $\mathcal{E} \cap H_s$ in order to recover $\mathcal{E} \cap H_t$ is

$$\phi_0 = \left(\frac{\mathrm{vol}_{n-1}(\mathcal{E} \cap H_t)}{\mathrm{vol}_{n-1}(\mathcal{E} \cap H_s)} \right)^{\frac{1}{n-1}}.$$

On the other hand, for all $\epsilon > 0$ there is s_ϵ such that for all s with $0 < s \leq s_\epsilon$

$$(1-\epsilon)s \leq \frac{\mathrm{vol}_{n-1}(\mathcal{E} \cap H_s)}{\mathrm{vol}_{n-1}(\partial \mathcal{E})} \leq \frac{\mathrm{vol}_{n-1}(\partial \mathcal{E} \cap H_s^-)}{\mathrm{vol}_{n-1}(\partial \mathcal{E})} = s.$$

Therefore

$$\left(\frac{(1-\epsilon)t}{s} \right)^{\frac{1}{n-1}} \leq \phi_0 \leq \left(\frac{t}{(1-\epsilon)s} \right)^{\frac{1}{n-1}}.$$

The volume of a volume element of $\partial \mathcal{E} \cap H_s$ that is mapped by the homothety onto one in $\partial \mathcal{E} \cap H_t$ increases by ϕ_0^{n-2}.

Now we estimate how much the angle between $N_{\partial \mathcal{E}}(y)$ and $N_{\partial \mathcal{E}_s}(x_s) = e_n$ changes. The normal to $\mathcal{E}$ at y is

$$\left(\frac{y_i}{a_i^2 \sqrt{\sum_{k=1}^n \frac{y_k^2}{a_k^4}}} \right)_{i=1}^n.$$

Thus

$$< N_{\partial \mathcal{E}}(y), e_n >= \frac{y_n}{a_n^2 \sqrt{\sum_{k=1}^n \frac{y_k^2}{a_k^4}}}$$

and

$$1- < N_{\partial \mathcal{E}}(y), e_n >^2 = \frac{\sum_{k=1}^{n-1} \frac{y_k^2}{a_k^4}}{\sum_{k=1}^n \frac{y_k^2}{a_k^4}}.$$

Let $y(s) \in \mathcal{E} \cap H_s$ and $y(t) \in \mathcal{E} \cap H_t$ be vectors such that $(y_1(s), \dots, y_{n-1}(s))$ and $(y_1(t), \dots, y_{n-1}(t))$ are colinear. Then

$$(y_1(t),\dots,y_{n-1}(t))=\phi_0(y_1(s),\dots,y_{n-1}(s))$$

Thus

$$\frac{1-<N_{\partial\mathcal{E}}(y(t)),e_n>^2}{1-<N_{\partial\mathcal{E}}(y(s)),e_n>^2}=\frac{\sum_{k=1}^{n-1}\frac{y_k^2(t)}{a_k^4}\sum_{k=1}^{n}\frac{y_k^2(s)}{a_k^4}}{\sum_{k=1}^{n}\frac{y_k^2(t)}{a_k^4}\sum_{k=1}^{n-1}\frac{y_k^2(s)}{a_k^4}}=\phi_0^2\frac{\sum_{k=1}^{n}\frac{y_k^2(s)}{a_k^4}}{\sum_{k=1}^{n}\frac{y_k^2(t)}{a_k^4}}.$$

For every $\epsilon>0$ there is s_ϵ such that for all s with $0<s\le s_\epsilon$ we have $a_n-\epsilon\le y_n(s)\le a_n$. Therefore there is an appropriate s_ϵ such that for all s with $0<s\le s_\epsilon$

$$1-\epsilon\le\frac{\sum_{k=1}^{n}\frac{y_k^2(t)}{a_k^4}}{\sum_{k=1}^{n}\frac{y_k^2(s)}{a_k^4}}\le 1+\epsilon.$$

Thus

$$(1-\epsilon)\phi_0\le\frac{\sqrt{1-<N_{\partial\mathcal{E}}(y(t)),e_n>^2}}{\sqrt{1-<N_{\partial\mathcal{E}}(y(s)),e_n>^2}}\le(1+\epsilon)\phi_0.$$

Consequently, with a new s_ϵ

$$\begin{aligned}&\int_{\partial\mathcal{E}\cap H_s}\frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H_s}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_s}(x_s),N_{\partial\mathcal{E}}(y)>^2}}\\&\le(1+\epsilon)\phi_0^{-(n-3)}\int_{\partial\mathcal{E}\cap H_t}\frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H_t}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_t}(x_t),N_{\partial\mathcal{E}}(y)>^2}}\\&\le(1+\epsilon)\left(\frac{s}{t}\right)^{\frac{n-3}{n-1}}\int_{\partial\mathcal{E}\cap H_t}\frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H_t}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_t}(x_t),N_{\partial\mathcal{E}}(y)>^2}}.\end{aligned}$$

□

Lemma 2.11. *Let K be a convex body in $\mathbb{R}^n$ such that for all $t>0$ the inclusion $K_t\subseteq\overset{\circ}{K}$ holds and that K has everywhere a unique normal. Let $f:\partial K\to\mathbb{R}$ a continuous, positive function with $\int_{\partial K}f(x)\mathrm{d}\mu_{\partial K}(x)=1$.*
(i) Let $t<T$ and $\epsilon>0$ such that $t+\epsilon<T$. Let $x\in\partial K_t$ and let $H(x,N_{\partial K_t}(x))$ be a hyperplane such that

$$\mathbb{P}_f(\partial K\cap H^-(x,N_{\partial K_t}(x)))=t.$$

Let $h(x,\epsilon)$ be defined by

$$\mathbb{P}_f(\partial K\cap H^-(x-h(x,\epsilon)N_{\partial K_t}(x),N_{\partial K_t}(x)))=t+\epsilon.$$

Then we have for sufficiently small ϵ

$$\epsilon-o(\epsilon)=\int_{\partial K\cap H(x,N_{\partial K_t}(x))}\frac{f(y)h(x,\epsilon)\mathrm{d}\mu_{\partial K\cap H(x,N_{\partial K_t}(x)))(y)}}{\sqrt{1-<N_{\partial K_t}(x),N_{\partial K}(y)>^2}}.$$

(ii) Let $t+\epsilon<T$, $x\in\partial K_{t+\epsilon}$, and $H(x,N_{\partial K_{t+\epsilon}}(x))$ a hyperplane such that

$$\mathbb{P}_f(\partial K\cap H^-(x,N_{\partial K_{t+\epsilon}}(x)))=t+\epsilon.$$

Let $k(x,\epsilon)$ be defined

$$\mathbb{P}_f(\partial K\cap H(x+k(x,\epsilon)N_{\partial K_{t+\epsilon}}(x),N_{\partial K_{t+\epsilon}}(x)))=t.$$

Then we have

$$\epsilon+o(\epsilon)=\int_{\partial K\cap H(x,N_{\partial K_{t+\epsilon}}(x))}\frac{f(y)k(x,\epsilon)\mathrm{d}\mu_{\partial K\cap H(x,N_{\partial K_{t+\epsilon}}(x))(y)}}{\sqrt{1-<N_{\partial K_{t+\epsilon}}(x),N_{\partial K}(y)>^2}}.$$

(iii) Let $\mathcal{E}$ be an ellipsoid

$$\mathcal{E}=\left\{x\left|\sum_{i=1}^n\left|\frac{x_i}{a_i}\right|^2\le 1\right.\right\}$$

and $\mathcal{E}_s$, $0<s\le\frac12$ surface bodies with respect to the constant density. $\{x_s\}=[0,a_ne_n]\cap\partial\mathcal{E}_s$. Let $\Delta:(0,T)\to[0,\infty)$ be such that $\Delta(s)$ is the height of the cap $\mathcal{E}\cap H^-(x_s,N_{\partial\mathcal{E}_s}(x_s))$. Then Δ is a differentiable, increasing function and

$$\frac{\mathrm{d}\Delta}{\mathrm{d}s}(s)=\left(\int_{\partial\mathcal{E}\cap H_s}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{\sqrt{1-<N_{\partial\mathcal{E}_s}(x_s),N_{\partial\mathcal{E}}(y)>^2}}\mathrm{d}\mu(y)\right)^{-1}$$

where $H_s=H(x_s,N_{\partial\mathcal{E}_s}(x_s))$.

Proof. (i) As $K_t\subset\overset{\circ}{K}$ we can apply Lemma 2.2 and assure that for all $0<t<T$ and all $x\in\partial K_t$ there is a normal $N_{\partial K_t}(x)$ with

$$t=\int_{\partial K\cap H^-(x,N_{\partial K_t}(x))}f(z)\mathrm{d}\mu_{\partial K}(z).$$

We have

$$\begin{aligned}\epsilon&=\int_{\partial K\cap H^-(x-h(x,\epsilon)N_{\partial K_t}(x)),N_{\partial K_t}(x))}f(z)\mathrm{d}\mu_{\partial K}(z)\\&\qquad\qquad-\int_{\partial K\cap H^-(x,N_{\partial K_t}(x))}f(z)\mathrm{d}\mu_{\partial K}(z)\\&=\int_{\partial K\cap H^-(x-h(x,\epsilon)N_{\partial K_t}(x)),N_{\partial K_t}(x))\cap H^+(x,N_{\partial K_t}(x))}f(z)\mathrm{d}\mu_{\partial K}(z).\end{aligned}$$

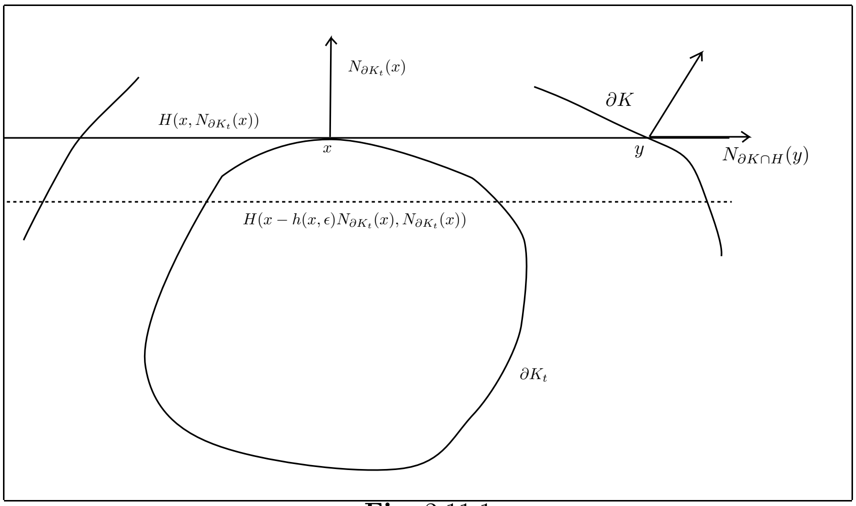

Fig. 2.11.1

Consider now small ϵ. Since K has everywhere a unique normal a surface element of

$$\partial K \cap H^-(x - h(x,\epsilon)N_{\partial K_t}(x)), N_{\partial K_t}(x)) \cap H^+(x, N_{\partial K_t}(x))$$

at y has approximately the area

$$h(x,\epsilon)d\mu_{\partial K \cap H(x,N_{\partial K_t}(x))}(y)$$

divided by the cosine of the angle between $N_{\partial K}(y)$ and $N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y)$. The latter normal is taken in the plane $H(x, N_{\partial K_t}(x))$. The vector $N_{\partial K}(y)$ is contained in the plane spanned by $N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y)$ and $N_{\partial K_t}(x)$. Thus we have

$$\begin{aligned} N_{\partial K}(y) = \ & < N_{\partial K}(y), N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y) > N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y) \\ & + < N_{\partial K}(y), N_{\partial K_t}(x) > N_{\partial K_t}(x) \end{aligned}$$

which implies

$$1 =< N_{\partial K}(y), N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y) >^2 + < N_{\partial K}(y), N_{\partial K_t}(x) >^2 .$$

We get for the approximate area of the surface element

$$\frac{h(x,\epsilon)d\mu_{\partial K \cap H(x,N_{\partial K_t}(x))}(y)}{< N_{\partial K}(y), N_{\partial K \cap H(x,N_{\partial K_t}(x))}(y) >} = \frac{h(x,\epsilon)d\mu_{\partial K \cap H(x,N_{\partial K_t}(x))}(y)}{\sqrt{1- < N_{\partial K}(y), N_{\partial K_t}(x) >^2}}.$$

Since f is a continuous function

$$\epsilon + o(\epsilon) = \int_{\partial(K \cap H(x,N_{\partial K_t}(x)))} \frac{f(y)h(x,\epsilon)\mathrm{d}\mu_{\partial K \cap H(x,N_{\partial K_t}(x)))(y)}}{\sqrt{1- < N_{\partial K_t}(x), N_{\partial K}(y) >^2}}.$$

(iii) By the symmetries of the ellipsoids e_n is a normal to the surface body $\mathcal{E}_s$. In fact we have

$$\mathbb{P}\{\partial\mathcal{E} \cap H^-(x_s, e_n)\} = s.$$

This follows from Lemma 2.4. Moreover,

$$h(x_s, \epsilon) \le \Delta(s+\epsilon) - \Delta(s) \le k(x_s, \epsilon).$$

□

Lemma 2.12. *Let K be a convex body in $\mathbb{R}^n$ that has everywhere a unique normal and let $f : \partial K \to \mathbb{R}$ be a continuous, positive function with $\int_{\partial K} f(x) \mathrm{d}\mu_{\partial K}(x) = 1$. K_s, $0 \le s \le T$, are the surface bodies of K with respect to the density f. Suppose that for all t with $0 < t \le T$ we have $K_t \subseteq \overset{\circ}{K}$. Let $G : K \to \mathbb{R}$ be a continuous function. Then*

$$\int_{K_s} G(x) \mathrm{d}x$$

is a continuous, decreasing function of s on the interval $[0, T]$ and a differentiable function on $(0, T)$. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x) \mathrm{d}x = -\int_{\partial K_s} \frac{G(x_s) \mathrm{d}\mu_{\partial K_s}(x_s)}{\int_{\partial K \cap H_s} \frac{f(y)}{\sqrt{1 - <N_{\partial K_s}(x_s), N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K \cap H_s}(y)}.$$

where $H_s = H(x_s, N_{\partial K_s}(x_s))$. The derivative is bounded on all intervals $[a, T)$ with $[a, T) \subset (0, T)$ and

$$\int_K G(x) \mathrm{d}x = \int_0^T \int_{\partial K_s} \frac{G(x_s) \mathrm{d}\mu_{\partial K_s}(x_s) \mathrm{d}s}{\int_{\partial K \cap H_s} \frac{f(y)}{\sqrt{1 - <N_{\partial K_s}(x_s), N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K \cap H_s}(y)}.$$

Proof. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x) \mathrm{d}x &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{K_{s+\epsilon}} G(x) \mathrm{d}x - \int_{K_s} G(x) \mathrm{d}x \right) \\ &= -\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{K_s \setminus K_{s+\epsilon}} G(x) \mathrm{d}x \end{aligned}$$

provided that the right hand side limit exists.

Let $\Delta(x_s, \epsilon)$ be the distance of x_s to $\partial K_{s+\epsilon}$. By Lemma 2.4.(iv), for all s and $\delta > 0$ there is $\epsilon > 0$ such that $d_H(K_s, K_{s+\epsilon}) < \delta$. By this and the continuity of G we get

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x) \mathrm{d}x = -\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\partial K_s} G(x_s) \Delta(x_s, \epsilon) \mathrm{d}\mu_{\partial K_s}(x_s).$$

We have to show that the right hand side limit exists. By Lemma 2.11.(i) we have

$$\epsilon - o(\epsilon) = \int_{\partial(K\cap H(x,N_{\partial K_t}(x)))} \frac{f(y)h(x,\epsilon)\mathrm{d}\mu_{\partial K\cap H(x,N_{\partial K_t}(x))(y)}}{\sqrt{1- < N_{\partial K_t}(x), N_{\partial K}(y) >^2}}.$$

Since $h(x_s,\epsilon) \le \Delta(x_s,\epsilon)$ we get

$$\begin{aligned}
&\mathrm{liminf}_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\partial K_s} G(x_s)\Delta(x_s,\epsilon)\mathrm{d}\mu_{\partial K_s}(x_s)\\
&\qquad\ge \int_{\partial K_s} \frac{G(x_s)}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\mathrm{d}\mu_{\partial K_s}(x_s)
\end{aligned}$$

where $H_s = H(x_s, N_{\partial K_s}(x_s))$. We show the inverse inequality for the Limes Superior. This is done by using Lemma 2.11.(ii).

We show now that the function satisfies the fundamental theorem of calculus.

$$\begin{aligned}
&\int_{\partial K\cap H_s} \frac{f(y)\mathrm{d}\mu_{\partial K\cap H_s}(y)}{\sqrt{1- < N_{\partial K_s}(x_s), N_{\partial K}(y) >^2}}\\
&\qquad\ge \int_{\partial K\cap H_s} f(y)\mathrm{d}\mu_{\partial K\cap H_s}(y) \ge \min_{y\in\partial K} f(y)\mathrm{vol}_{n-2}(\partial K\cap H_s).
\end{aligned}$$

By the isoperimetric inequality there is a constant $c>0$ such that

$$\int_{\partial K\cap H_s} \frac{f(y)\mathrm{d}\mu_{\partial K\cap H_s}(y)}{\sqrt{1- < N_{\partial K_s}(x_s), N_{\partial K}(y) >^2}} \ge c\min_{y\in\partial K} f(y)\mathrm{vol}_{n-1}(K\cap H_s).$$

By our assumption $K_s \subseteq \overset{\circ}{K}$ the distance between ∂K and ∂K_s is strictly larger than 0. From this we conclude that there is a constant $c>0$ such that for all $x_s\in\partial K_s$

$$\mathrm{vol}_{n-1}(K\cap H_s) \ge c.$$

This implies that for all s with $0<s<T$ there is a constant $c_s>0$

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\int_{K_s} G(x)\mathrm{d}x\right| \le c_s.$$

Thus, on all intervals $[a,T)\subset(0,T)$ the derivative is bounded and therefore the function is absolutely continuous. We get for all t_0, t with $0<t_0\le t<T$

$$\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}s}\int_{K_s} G(x)\mathrm{d}x = \int_{K_t} G(x)\mathrm{d}x - \int_{K_{t_0}} G(x)\mathrm{d}x.$$

We take the limit of $t_0\to 0$. By Lemma 2.3.(iii) we have $\bigcup_{t>0} K_t \supseteq \overset{\circ}{K}$. The monotone convergence theorem implies

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x)\mathrm{d}x = \int_{K_t} G(x)\mathrm{d}x - \int_K G(x)\mathrm{d}x.$$

Now we take the limit $t \to T$. By Lemma 2.3 we have $K_T = \bigcap_{t<T} K_t$. The monotone convergence theorem implies

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x)\mathrm{d}x = \int_{K_T} G(x)\mathrm{d}x - \int_K G(x)\mathrm{d}x.$$

Since the volume of K_T equals 0 we get

$$\int_0^T \frac{\mathrm{d}}{\mathrm{d}s} \int_{K_s} G(x)\mathrm{d}x = -\int_K G(x)\mathrm{d}x.$$

□

3 The Case of the Euclidean Ball

We present here a proof of the main theorem in case that the convex body is the Euclidean ball. This result was proven by J. Müller [Mü]. We include the results of chapter 3 for the sake of completeness. Most of them are known.

Proposition 3.1. *(*Müller*) We have*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{N^{-\frac{2}{n-1}}} \\
&\quad = \frac{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}{2(n+1)!} \left(\frac{(n-1)\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})} \right)^{\frac{n+1}{n-1}} \Gamma\left(n+1+\tfrac{2}{n-1}\right) \\
&\quad = \frac{(n-1)^{\frac{n+1}{n-1}} (\mathrm{vol}_{n-1}(\partial B_2^n))^{\frac{n+1}{n-1}}}{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.
\end{aligned}$$

We want to show first that almost all random polytopes are simplicial.

Lemma 3.1. *The n^2-dimensional Hausdorff measure of the real $n \times n$-matrices with determinant* 0 *equals* 0.

Proof. We use induction. For $n = 1$ the only matrix with determinant 0 is the zeromatrix. Let A_{11} be the submatrix of the matrix A that is obtained by deleting the first row and column. We have

$$\{A|\det(A) = 0\} \subseteq \{A|\det(A_{11}) = 0\} \cup \{A|\det(A) = 0 \text{ and } \det(A_{11}) \neq 0\}.$$

Since

$$\{A|\det(A_{11})=0\}=\mathbb{R}^{n^2-(n-1)^2}\times\{B\in M_{n-1}|\det(B)=0\}$$

we get by the induction assumption that $\{A|\det(A_{11})=0\}$ is a nullset. We have

$$\begin{aligned}&\{A|\det(A)=0 \text{ and } \det(A_{11})\neq 0\}\\&=\left\{A\,\middle|\,a_{11}=\frac{1}{\det(A_{11})}\sum_{i=1}^{n}a_{1i}(-1)^{1+i}\det(A_{1i}))\right\}.\end{aligned}$$

Since this is the graph of a function it is a nullset. □

Lemma 3.2. *The $n(n-1)$-dimensional Hausdorff measure of the real $n\times n$-matrices whose determinant equal* 0 *and whose columns have Euclidean norm equal to* 1 *is* 0*.*

Proof. Let $A_{i,j}$ be the submatrix of the matrix A that is obtained by deleting the i-th row and j-th column. We have

$$\{A|\det(A)=0\}\subseteq\{A|\det(A_{11})=0\}\cup\{A|\det(A)=0 \text{ and } \det(A_{11})\neq 0\}.$$

By Lemma 3.1 the set of all $(n-1)\times(n-1)$ matrices with determinant equal to 0 has $(n-1)^2$-dimensional Hausdorff measure 0. Therefore, the set

$$\{(a_1,\dots,a_{n-1})|\det(\bar{a}_1,\dots,\bar{a}_{n-1})=0\}$$

has $(n-1)^2$-dimensional Hausdorff measure 0 where $\bar{a}_i$ is the vector a_i with the first coordinate deleted. From this we conclude that $\{A|\det(A_{11})=0\}$ has $n(n-1)$-dimensional Hausdorff measure 0.

As in Lemma 3.1 we have

$$\begin{aligned}&\{A|\det(A)=0 \text{ and } \det(A_{11})\neq 0\}\\&=\left\{A\,\middle|\,a_{11}=\frac{1}{\det(A_{11})}\sum_{i=1}^{n}a_{1i}(-1)^{1+i}\det(A_{1i}))\right\}.\end{aligned}$$

By this and since the columns of the matrix have Euclidean length 1 the above set is the graph of a differentiable function of $n(n-1)-1$ variables. Thus the $n(n-1)$-dimensional Hausdorff measure is 0. □

The next lemma says that almost all random polytopes of points chosen from a convex body are simplicial. Intuitively this is obvious. Suppose that we have chosen $x_1,\dots,x_n$ and we want to choose x_{n+1} so that it is an element of the hyperplane spanned by $x_1,\dots,x_n$, then we are choosing it from a nullset.

Lemma 3.3. *Let K be a convex body in $\mathbb{R}^n$ and $\mathbb{P}$ the normalized Lebesgue measure on K. Let $\mathbb{P}_K^N$ the N-fold probability measure of $\mathbb{P}$. Then*
(i)

$$\mathbb{P}_K^N\{(x_1,\dots,x_N)|\exists i_1,\dots,i_{n+1}\exists H : x_{i_1},\dots,x_{i_{n+1}} \in H\} = 0$$

where H denotes a hyperplane in $\mathbb{R}^n$.
(ii)

$$\mathbb{P}_K^N\{(x_1,\dots,x_N)|\ \exists i_1,\dots,i_n : x_{i_1},\dots,x_{i_n} \text{ are linearly dependent}\} = 0$$

Proof. (i) It suffices to show that

$$\mathbb{P}_K^N\{(x_1,\dots,x_N)|\exists H : x_1,\dots,x_{n+1} \in H\} = 0.$$

Let $X = (x_1,\dots,x_n)$. We have that

$$\begin{aligned}\{(x_1,\dots,x_N)|\exists H : x_1,\dots,x_{n+1} \in H\} &= \{(x_1,\dots,x_N)|\det(X) = 0\}\\ \cup\Big\{(x_1,\dots,x_N)|\det(X)\neq 0 \text{ and } &\exists t_1,\dots,t_{n-1} :\\ &x_{n+1} = x_n + \sum_{i=1}^{n-1} t_i(x_i - x_n)\Big\}.\end{aligned}$$

The set with $\det(X) = 0$ has measure 0 by Lemma 3.1. Now we consider the second set. $\det(X) \neq 0$ and $x_{n+1} = x_n + \sum_{i=1}^{n-1} t_i(x_i - x_n)$ imply that

$$X^{-1}(x_{n+1}) = X^{-1}\left(x_n + \sum_{i=1}^{n-1} t_i(x_i - x_n)\right) = e_n + \sum_{i=1}^{n-1} t_i(e_i - e_n).$$

We get

$$t_i =< X^{-1}(x_{n+1}), e_i > \qquad i = 1,\dots,n-1.$$

Therefore we get

$$\begin{aligned}&\left\{(x_1,\dots,x_N)\,\middle|\,\det(X)\neq 0 \text{ and } \exists t_1,\dots,t_{n-1} : x_{n+1} = x_n + \sum_{i=1}^{n-1} t_i(x_i - x_n)\right\}\\ &\subseteq \Big\{(x_1,\dots,x_n,z,x_{n+2},\dots,x_N)\Big|\det(X)\neq 0 \text{ and}\\ &\qquad z = x_n + \sum_{i=1}^{n-1} < X^{-1}(x_{n+1}), e_i > (x_i - x_n)\Big\}.\end{aligned}$$

We have that

$$\frac{\partial z}{\partial x_{n+1}(j)} = \sum_{i=1}^{n-1} < X^{-1}(e_j), e_i > (x_i - x_n).$$

Since all the vectors $\frac{\partial z}{\partial x_{n+1}(j)}$, $j=1,\dots,n$ are linear combinations of the vectors $x_i - x_n$, $i=1,\dots,n-1$, the rank of the matrix

$$\left(\frac{\partial z}{\partial x_{n+1}(j)}\right)_{j=1}^{n}$$

is at most $n-1$. Therefore, the determinant of the Jacobian of the function mapping $(x_1,\dots,x_N)$ onto $(x_1,\dots,x_n,z,x_{n+2},\dots,x_N)$ is 0. Thus the set

$$\left\{(x_1,\dots,x_N)\,\middle|\,\det(X)\neq 0 \text{ and } \exists t_1,\dots,t_{n-1}: x_{n+1}=x_n+\sum_{i=1}^{n-1} t_i(x_i-x_n)\right\}$$

has measure 0. □

Lemma 3.4. *Let $\mathbb{P}_{\partial B_2^n}$ be the normalized surface measure on ∂B_2^n. Let $\mathbb{P}_{\partial B_2^n}^N$ the N-fold probability measure of $\mathbb{P}_{\partial B_2^n}$. Then we have*
(i)

$$\mathbb{P}_{\partial B_2^n}^N\{(x_1,\dots,x_N)|\exists i_1,\dots,i_{n+1}\exists H: x_{i_1},\dots,x_{i_{n+1}}\in H\}=0$$

where H denotes a hyperplane in $\mathbb{R}^n$.
(ii)

$$\mathbb{P}_{\partial B_2^n}^N\{(x_1,\dots,x_N)|\ \exists i_1,\dots,i_n: x_{i_1},\dots,x_{i_n} \text{ are linearly dependent}\}=0$$

Proof. Lemma 3.4 is shown in the same way as Lemma 3.3. We use in addition the Cauchy-Binet formula ([EvG], p. 89). □

Lemma 3.5. *Almost all random polytopes of points chosen from the boundary of the Euclidean ball with respect to the normalized surface measure are simplicial.*

Lemma 3.5 follows from Lemma 3.4.(i).

Let F be a $n-1$-dimensional face of a polytope. Then dist(F) is the distance of the hyperplane containing F to the origin 0. We define

$$\Phi_{j_1,\dots,j_k}(x)=\frac{1}{n}\mathrm{vol}_{n-1}([x_{j_1},\dots,x_{j_k}])\mathrm{dist}(x_{j_1},\dots,x_{j_k})$$

if $[x_{j_1},\dots,x_{j_k}]$ is a $n-1$-dimensional face of the polytope $[x_1,\dots,x_N]$ and if $0\in H^+$ where H denotes the hyperplane containing the face $[x_{j_1},\dots,x_{j_k}]$ and H^+ the halfspace containing $[x_1,\dots,x_N]$. We define

$$\Phi_{j_1,\dots,j_k}(x)=-\frac{1}{n}\mathrm{vol}_{n-1}([x_{j_1},\dots,x_{j_k}])\mathrm{dist}(x_{j_1},\dots,x_{j_k})$$

if $[x_{j_1}, \dots, x_{j_k}]$ is a $n-1$-dimensional face of the polytope $[x_1, \dots, x_N]$ and if $0 \in H^-$. We put

$$\Phi_{j_1,\dots,j_k}(x) = 0$$

if $[x_{j_1}, \dots, x_{j_k}]$ is not a $n-1$-dimensional face of the polytope $[x_1, \dots, x_N]$.

Lemma 3.6. *Let $x_1, \dots, x_N \in \mathbb{R}^n$ such that $[x_1, \dots, x_N]$ is a simplicial polytope. Then we have*

$$\mathrm{vol}_n([x_1, \dots, x_N]) = \sum_{\{j_1,\dots,j_n\} \subseteq \{1,\dots,N\}} \Phi_{j_1,\dots,j_n}(x).$$

Note that the above formula holds if $0 \in [x_1, \dots, x_N]$ and if $0 \notin [x_1, \dots, x_N]$.

$\mathrm{d}L_k^n$ is the measure on all k-dimensional affine subspaces of $\mathbb{R}^n$ and $\mathrm{d}L_k^n(0)$ is the measure on all k-dimensional subspaces of $\mathbb{R}^n$ [San].

Lemma 3.7. [Bla1, San]

$$\bigwedge_{i=0}^{k} \mathrm{d}x_i^n = (k! vol_k([x_0, \dots, x_k]))^{n-k} \bigwedge_{i=0}^{k} \mathrm{d}x_i^k \mathrm{d}L_k^n$$

where $\mathrm{d}x_i^n$ is the volume element in $\mathbb{R}^n$ and $\mathrm{d}x_i^k$ is the volume element in L_k^n.

The above formula can be found as formula (12.22) on page 201 in [San]. We need this formula here only in the case $k = n-1$. It can be found as formula (12.24) on page 201 in [San]. The general formula can also be found in [Mil]. See also [Ki] and [Pe].

Lemma 3.8.

$$\mathrm{d}L_{n-1}^n = \mathrm{d}p \mathrm{d}\mu_{\partial B_2^n}(\xi)$$

where p is the distance of the hyperplane from the origin and ξ is the normal of the hyperplane.

This lemma is formula (12.40) in [San].

Let X be a metric space. Then a sequence of probability measures $\mathbb{P}_n$ converges weakly to a probability measure $\mathbb{P}$ if we have for all $\phi \in C(X)$ that

$$\lim_{n \to \infty} \int_X \phi \mathrm{d}\mathbb{P}_n = \int_X \phi \mathrm{d}\mathbb{P}_n.$$

See ([Bil], p.7). In fact, we have that two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ coincide on the underlying Borel σ-algebra if we have for all continuous functions ϕ that

$$\int_X \phi \mathrm{d}\mathbb{P}_1 = \int_X \phi \mathrm{d}\mathbb{P}_2.$$

Lemma 3.9. *We put*

$$A_\epsilon = B_2^n(0, r+\epsilon) \setminus B_2^n(0,r)$$

and as probability measure $\mathbb{P}_\epsilon$ *on* $A_\epsilon \times A_\epsilon \times \cdots \times A_\epsilon$

$$\mathbb{P}_\epsilon = \frac{\chi_{A_\epsilon} \times \cdots \times \chi_{A_\epsilon}(x_1)\mathrm{d}x_1 \dots \mathrm{d}x_k}{((r+\epsilon)^n - r^n)^k (\mathrm{vol}_n(B_2^n))^k}.$$

Then $\mathbb{P}_\epsilon$ *converges weakly for* ϵ *to* 0 *to the* k*-fold product of the normalized surface measure on* $\partial B_2^n(0,r)$

$$\frac{\mu_{\partial B_2^n(0,r)}(x_1) \dots \mu_{\partial B_2^n(0,r)}(x_k)}{r^{k(n-1)}(\mathrm{vol}_{n-1}(\partial B_2^n))^k}.$$

Proof. All the measures are being viewed as measures on $\mathbb{R}^n$, otherwise it would not make sense to talk about convergence. For the proof we consider a continuous function ϕ on $\mathbb{R}^n$ and Riemann sums for the Euclidean sphere. □

Lemma 3.10. *[Mil]*

$$\begin{aligned}&\mathrm{d}\mu_{\partial B_2^n}(x_1) \cdots \mathrm{d}\mu_{\partial B_2^n}(x_n) \\ &\quad = (n-1)! \frac{\mathrm{vol}_{n-1}([x_1, \dots, x_n])}{(1-p^2)^{\frac{n}{2}}} \mathrm{d}\mu_{\partial B_2^n \cap H}(x_1) \cdots \mathrm{d}\mu_{\partial B_2^n \cap H}(x_n) \mathrm{d}p \mathrm{d}\mu_{\partial B_2^n}(\xi)\end{aligned}$$

where ξ *is the normal to the plane* H *through* $x_1, \dots, x_n$ *and* p *is the distance of the plane* H *to the origin.*

Proof. We put

$$A_\epsilon = B_2^n(0, 1+\epsilon) \setminus B_2^n(0,1)$$

and as probability measure $\mathbb{P}_\epsilon$ on $A_\epsilon \times A_\epsilon \times \cdots \times A_\epsilon$

$$\mathbb{P}_\epsilon = \frac{\chi_{A_\epsilon} \times \cdots \times \chi_{A_\epsilon}(x_1)\mathrm{d}x_1 \dots \mathrm{d}x_n}{((1+\epsilon)^n - 1)^n (\mathrm{vol}_n(B_2^n))^n}.$$

Then, by Lemma 3.9, $\mathbb{P}_\epsilon$ converges for ϵ to 0 to the n-fold product of the normalized surface measure on ∂B_2^n

$$\frac{\mu_{\partial B_2^n}(x_1) \dots \mu_{\partial B_2^n}(x_n)}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}.$$

By Lemma 3.7 we have

$$\bigwedge_{i=1}^{n} \mathrm{d}x_i^n = (n-1)!\mathrm{vol}_{n-1}([x_1,\dots,x_n])\mathrm{d}L_{n-1}^n \bigwedge_{i=1}^{n} \mathrm{d}x_i^{n-1}$$

and by Lemma 3.8

$$\mathrm{d}L_{n-1}^n = \mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi).$$

We get

$$\bigwedge_{i=1}^{n} \mathrm{d}x_i^n = (n-1)!\mathrm{vol}_{n-1}([x_1,\dots,x_n]) \bigwedge_{i=1}^{n} \mathrm{d}x_i^{n-1}\mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi).$$

Thus we get

$$\mathbb{P}_\epsilon = \chi_{A_\epsilon} \times \cdots \times \chi_{A_\epsilon}(n-1)!\mathrm{vol}_{n-1}([x_1,\dots,x_n]) \\ \times \frac{\mathrm{d}x_1^{n-1}\dots\mathrm{d}x_n^{n-1}\mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi)}{((1+\epsilon)^n-1)^n(\mathrm{vol}_n(B_2^n))^n}.$$

This can also be written as

$$\mathbb{P}_\epsilon = (n-1)!\mathrm{vol}_{n-1}([x_1,\dots,x_n]) \\ \times \frac{\chi_{A_\epsilon\cap H} \times \cdots \times \chi_{A_\epsilon\cap H}\mathrm{d}x_1^{n-1}\dots\mathrm{d}x_n^{n-1}\mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi)}{((1+\epsilon)^n-1)^n(\mathrm{vol}_n(B_2^n))^n}$$

where H is the hyperplane with normal ξ that contains the points $x_1,\dots,x_n$. p is the distance of H to 0. $A_\epsilon \cap H$ is the set-theoretic difference of a Euclidean ball of dimension $n-1$ with radius $(1-p^2+2\epsilon+\epsilon^2)^{\frac{1}{2}}$ and a ball with radius $(1-p^2)^{\frac{1}{2}}$. By Lemma 3.9 we have that

$$\frac{\chi_{A_\epsilon\cap H} \times \cdots \times \chi_{A_\epsilon\cap H}\mathrm{d}x_1^{n-1}\dots\mathrm{d}x_n^{n-1}}{((1-p^2+2\epsilon+\epsilon^2)^{\frac{n-1}{2}}-(1-p^2)^{\frac{n-1}{2}})^n(\mathrm{vol}_{n-1}(B_2^{n-1}))^n}$$

converges weakly to the n-fold product of the normalized surface measure on $\partial B_2^n \cap H$

$$\frac{\mathrm{d}\mu_{\partial B_2^n\cap H}\dots\mathrm{d}\mu_{\partial B_2^n\cap H}}{(1-p^2)^{n\frac{n-2}{2}}(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}.$$

Therefore we get that

$$\frac{\chi_{A_\epsilon\cap H} \times \cdots \times \chi_{A_\epsilon\cap H}\mathrm{d}x_1^{n-1}\dots\mathrm{d}x_n^{n-1}}{((1+\epsilon)^n-1)^n(\mathrm{vol}_n(B_2^n))^n}$$

converges to

$$\left(\frac{(n-1)\mathrm{vol}_{n-1}(B_2^{n-1})}{n\ \mathrm{vol}_n(B_2^n)}\right)^n (1-p^2)^{n\frac{n-1}{2}-n}\frac{\mathrm{d}\mu_{\partial B_2^n\cap H}\dots\mathrm{d}\mu_{\partial B_2^n\cap H}}{(1-p^2)^{n\frac{n-2}{2}}(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n} \\ = \frac{\mathrm{d}\mu_{\partial B_2^n\cap H}\dots\mathrm{d}\mu_{\partial B_2^n\cap H}}{(1-p^2)^{\frac{n}{2}}(\mathrm{vol}_{n-1}(\partial B_2^n))^n}.$$

□

Lemma 3.11. [Mil]

$$\int_{\partial B_2^n(0,r)} \cdots \int_{\partial B_2^n(0,r)} (\mathrm{vol}_n([x_1,\dots,x_{n+1}]))^2 \times \mathrm{d}\mu_{\partial B_2^n(0,r)}(x_1)\cdots \mathrm{d}\mu_{\partial B_2^n(0,r)}(x_{n+1})$$
$$= \frac{(n+1)r^{2n}}{n!n^n}(\mathrm{vol}_{n-1}(\partial B_2^n(r)))^{n+1} = \frac{(n+1)r^{n^2+2n-1}}{n!n^n}(\mathrm{vol}_{n-1}(\partial B_2^n))^{n+1}$$

We just want to refer to [Mil] for the proof. But we want to indicate an alternative proof here. One can use

$$\lim_{N\to\infty} \mathbb{E}(\partial B_2^n, N) = \mathrm{vol}_n(B_2^n)$$

and the computation in the proof of Proposition 3.1.

Lemma 3.12. *Let C be a cap of a Euclidean ball with radius* 1*. Let s be the surface area of this cap and r its radius. Then we have*

$$\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)}\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} - c\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}} \le r(s) \le \left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} - \frac{1}{2(n+1)}\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} + c\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}$$

where c is a numerical constant.

Proof. The surface area s of a cap of the Euclidean ball of radius 1 is

$$s = \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \int_0^\alpha \sin^{n-2} t \mathrm{d}t$$

where α is the angle of the cap. Then $\alpha = \arcsin r$ where r is the radius of the cap. For all t with $t \ge 0$

$$t - \tfrac{1}{3!}t^3 \le \sin t \le t - \tfrac{1}{3!}t^3 + \tfrac{1}{5!}t^5.$$

Therefore we get for all t with $t \ge 0$

$$\sin^{n-2} t \ge (t - \tfrac{1}{3!}t^3)^{n-2} = t^{n-2}(1-\tfrac{1}{3!}t^2)^{n-2} \ge t^{n-2}(1-\tfrac{n-2}{3!}t^2) = t^{n-2} - \tfrac{n-2}{3!}t^n.$$

Now we use $(1-u)^k \le 1 - ku + \frac{1}{2}k(k-1)u^2$ and get for all $t \ge 0$

$$\sin^{n-2} t \le t^{n-2} - \tfrac{n-2}{3!} t^n + c t^{n+2}.$$

Thus

$$\begin{aligned} s &\ge \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \int_0^\alpha t^{n-2} - \tfrac{n-2}{3!} t^n \mathrm{d}t \\ &= \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \left(\tfrac{1}{n-1} \alpha^{n-1} - \tfrac{n-2}{6(n+1)} \alpha^{n+1} \right) \\ &= \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \left(\tfrac{1}{n-1} (\arcsin r)^{n-1} - \tfrac{n-2}{6(n+1)} (\arcsin r)^{n+1} \right) \end{aligned}$$

and

$$\begin{aligned} s \le{} & \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \times \\ & \left(\tfrac{1}{n-1} (\arcsin r)^{n-1} - \tfrac{n-2}{6(n+1)} (\arcsin r)^{n+1} + \tfrac{c}{n+3} (\arcsin r)^{n+3} \right). \end{aligned}$$

We have

$$\arcsin r = r + \frac{1}{2} \frac{r^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{r^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{r^7}{7} + \cdots$$

Thus we have for all sufficiently small r that

$$r + \tfrac{1}{3!} r^3 \le \arcsin r \le r + \tfrac{1}{3!} r^3 + r^5.$$

We get with a new constant c

$$\begin{aligned} s &\ge \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \left(\tfrac{1}{n-1} (r + \tfrac{1}{3!} r^3)^{n-1} - \tfrac{n-2}{6(n+1)} (r + \tfrac{1}{3!} r^3 + r^5)^{n+1} \right) \\ &\ge \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \left(\tfrac{1}{n-1} r^{n-1} + \tfrac{1}{3!} r^{n+1} - \tfrac{n-2}{6(n+1)} r^{n+1} - c r^{n+3} \right) \\ &= \mathrm{vol}_{n-2}(\partial B_2^{n-1}) \left(\tfrac{1}{n-1} r^{n-1} + \tfrac{1}{2(n+1)} r^{n+1} - c r^{n+3} \right) \\ &= \mathrm{vol}_{n-1}(B_2^{n-1}) \left(r^{n-1} + \tfrac{n-1}{2(n+1)} r^{n+1} - c(n-1) r^{n+3} \right). \end{aligned}$$

We get the inverse inequality

$$s \le \mathrm{vol}_{n-1}(B_2^{n-1}) \left(r^{n-1} + \tfrac{n-1}{2(n+1)} r^{n+1} + c(n-1) r^{n+3} \right)$$

in the same way. We put now

$$u = \frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}$$

and get

$$\begin{aligned} & u^{\frac{1}{n-1}} - \frac{1}{2(n+1)} u^{\frac{3}{n-1}} - c u^{\frac{5}{n-1}} \\ & \quad \le \left(r^{n-1} + \tfrac{n-1}{2(n+1)} r^{n+1} + c(n-1) r^{n+3} \right)^{\frac{1}{n-1}} \\ & \qquad - \frac{1}{2(n+1)} \left(r^{n-1} + \tfrac{n-1}{2(n+1)} r^{n+1} - c(n-1) r^{n+3} \right)^{\frac{3}{n-1}} \\ & \qquad - a \left(r^{n-1} + \tfrac{n-1}{2(n+1)} r^{n+1} - c(n-1) r^{n+3} \right)^{\frac{5}{n-1}}. \end{aligned}$$

If we choose a big enough then this can be estimated with a new constant c by

$$r - cr^5 \le r$$

provided r is small enough. The opposite inequality is shown in the same way. Altogether we have with an appropriate constant c

$$\begin{aligned}\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} &- \frac{1}{2(n+1)}\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} \\ -c\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}} &\le r(s) \le \left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \\ &-\frac{1}{2(n+1)}\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}} + c\left(\frac{s}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}.\end{aligned}$$

□

Proof. (Proof of Proposition 3.1) We have

$$\mathbb{P} = \frac{\mu_{\partial B_2^n}}{\mathrm{vol}_{n-1}(B_2^n)}$$

and

$$\mathbb{E}(\partial B_2^n, N) = \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \mathrm{vol}_n([x_1, \dots, x_N]) \mathrm{d}\mathbb{P}(x_1) \cdots \mathrm{d}\mathbb{P}(x_N).$$

By Lemma 3.5 almost all random polytopes are simplicial. Therefore we get with Lemma 3.6

$$\begin{aligned}&\mathbb{E}(\partial B_2^n, N) \\ &= \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \sum_{\{j_1,\dots,j_n\}\subseteq\{1,\dots,N\}} \Phi_{j_1,\dots,j_n}(x_1, \dots, x_N) \mathrm{d}\mathbb{P}(x_1) \cdots \mathrm{d}\mathbb{P}(x_N) \\ &= \binom{N}{n} \int_{\partial B_2^n} \cdots \int_{\partial B_2^n} \Phi_{1,\dots,n}(x_1, \dots, x_N) \mathrm{d}\mathbb{P}(x_1) \cdots \mathrm{d}\mathbb{P}(x_N).\end{aligned}$$

H is the hyperplane containing the points $x_1, \dots, x_n$. The set of points where H is not well defined has measure 0. H^+ is the halfspace containing the polytope $[x_1, \dots, x_N]$. We have

$$\begin{aligned}\mathbb{P}^{N-n}\{(x_{n+1}, \dots, x_N) | \Phi_{1,\dots,n}(x_1, \dots, x_N) \\ = \tfrac{1}{n}\mathrm{vol}_{n-1}([x_1, \dots, x_n])\mathrm{dist}(x_1, \dots, x_n)\Big\} \\ = \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\end{aligned}$$

and

$$\mathbb{P}^{N-n}\left\{(x_{n+1},\dots,x_N)|\Phi_{1,\dots,n}(x_1,\dots,x_N) = -\tfrac{1}{n}\mathrm{vol}_{n-1}([x_1,\dots,x_n])\mathrm{dist}(x_1,\dots,x_n)\right\} = \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}.$$

Therefore

$$\mathbb{E}(\partial B_2^n,N) = \binom{N}{n}\frac{1}{n}\int_{\partial B_2^n}\cdots\int_{\partial B_2^n}\mathrm{vol}_{n-1}([x_1,\dots,x_n])\mathrm{dist}(x_1,\dots,x_n) \times\left\{\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\right\} \times \mathrm{d}\mathbb{P}(x_1)\cdots\mathrm{d}\mathbb{P}(x_n).$$

By Lemma 3.10 we get

$$\mathbb{E}(\partial B_2^n,N) = \frac{1}{n}\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int_{\partial B_2^n}\int_0^1 p(1-p^2)^{-\frac{n}{2}} \times\left\{\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\right\} \times\int_{\partial B_2^n\cap H}\cdots\int_{\partial B_2^n\cap H}(\mathrm{vol}_{n-1}([x_1,\dots,x_n]))^2 \times\mathrm{d}\mu_{\partial B_2^n\cap H}(x_1)\cdots\mathrm{d}\mu_{\partial B_2^n\cap H}(x_n)\mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi).$$

We apply Lemma 3.11 for the dimension $n-1$

$$\mathbb{E}(\partial B_2^n,N) = \frac{1}{n}\binom{N}{n}\frac{(n-1)!}{(\mathrm{vol}_{n-1}(\partial B_2^n))^n}\int_{\partial B_2^n}\int_0^1 p(1-p^2)^{-\frac{n}{2}} \times\left\{\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\right\} \times\frac{nr^{n^2-2}}{(n-1)!(n-1)^{n-1}}(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n\mathrm{d}p\mathrm{d}\mu_{\partial B_2^n}(\xi).$$

Since $r(p)=\sqrt{1-p^2}$ we get

$$\mathbb{E}(\partial B_2^n,N) = \binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^n}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}}\frac{1}{(n-1)^{n-1}}\int_0^1 r^{n^2-n-2}\sqrt{1-r^2} \left\{\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^+)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n} - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n\cap H^-)}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\right\}\mathrm{d}p.$$

Now we introduce the surface area s of a cap with height $1-p$ as a new variable. By Lemma 1.5 we have

$$\frac{\mathrm{d}p}{\mathrm{d}s} = -\left(r^{n-3}\mathrm{vol}_{n-2}(\partial B_2^{n-1})\right)^{-1}.$$

Thus we get

$$\begin{aligned}\mathbb{E}(\partial B_2^n, N) = &\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-1}}\frac{1}{(n-1)^{n-1}}\\ &\times\int_0^{\frac{1}{2}\mathrm{vol}_{n-1}(\partial B_2^n)} r^{(n-1)^2}\sqrt{1-r^2}\\ &\times\left\{\left(1-\frac{s}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}-\left(\frac{s}{\mathrm{vol}_{n-1}(\partial B_2^n)}\right)^{N-n}\right\}\mathrm{d}s.\end{aligned}$$

Now we introduce the variable

$$u = \frac{s}{\mathrm{vol}_{n-1}(\partial B_2^n)}$$

and obtain

$$\begin{aligned}\mathbb{E}(\partial B_2^n, N) = &\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}}\frac{1}{(n-1)^{n-1}}\\ &\times\int_0^{\frac{1}{2}} r^{(n-1)^2}\sqrt{1-r^2}\left\{(1-u)^{N-n}-u^{N-n}\right\}\mathrm{d}u.\end{aligned}$$

By Lemma 3.12 we get

$$\begin{aligned}&\mathbb{E}(\partial B_2^n, N)\\ &\le\binom{N}{n}\frac{(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{n-1}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{n-2}}\frac{1}{(n-1)^{n-1}}\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}\\ &\quad\times\left\{\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}-\frac{1}{2(n+1)}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}}\right.\\ &\qquad\qquad\left.+c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}\right\}^{(n-1)^2}\\ &\quad\times\left\{1-\left[\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}-\frac{1}{2(n+1)}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}}\right.\right.\\ &\qquad\qquad\left.\left.-c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}\right]^2\right\}^{\frac{1}{2}}\mathrm{d}u.\end{aligned}$$

From this we get

$$
\begin{aligned}
&\mathbb{E}(\partial B_2^n, N)\\
&\le \binom{N}{n}\mathrm{vol}_{n-1}(\partial B_2^n)\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}u^{n-1}\times\\
&\left\{1-\frac{1}{2(n+1)}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}+c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{4}{n-1}}\right\}^{(n-1)^2}\\
&\times\left\{1-\left[\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}-\frac{1}{2(n+1)}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{3}{n-1}}\right.\right.\\
&\qquad\qquad\left.\left.-c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{5}{n-1}}\right]^2\right\}^{\frac{1}{2}}\mathrm{d}u.
\end{aligned}
$$

This implies that we get for a new constant c

$$
\begin{aligned}
&\mathbb{E}(\partial B_2^n, N)\\
&\le \binom{N}{n}\mathrm{vol}_{n-1}(\partial B_2^n)\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}u^{n-1}\\
&\times\left\{1-\frac{(n-1)^2}{2(n+1)}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}+c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{4}{n-1}}\right\}\\
&\times\left(1-\left\{\frac{1}{2}\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}-c\left(\frac{u\ \mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{4}{n-1}}\right\}\right)\mathrm{d}u.
\end{aligned}
$$

This gives, again with a new constant c

$$
\begin{aligned}
\mathbb{E}(\partial B_2^n, N)\le{}&\binom{N}{n}\mathrm{vol}_{n-1}(\partial B_2^n)\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}u^{n-1}\mathrm{d}u\\
&-\binom{N}{n}\frac{n^2-n+2}{2(n+1)}\frac{\mathrm{vol}_{n-1}(\partial B_2^n)^{\frac{n+1}{n-1}}}{\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n-1}}}\\
&\qquad\times\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}u^{n-1+\frac{2}{n-1}}\mathrm{d}u\\
&+c\binom{N}{n}\int_0^{\frac{1}{2}}\left\{(1-u)^{N-n}-u^{N-n}\right\}u^{n-1+\frac{4}{n-1}}\mathrm{d}u.
\end{aligned}
$$

From this we get

$$
\begin{aligned}
\mathbb{E}(\partial B_2^n, N)\le{}&\binom{N}{n}\mathrm{vol}_{n-1}(\partial B_2^n)B(N-n+1,n)\\
&-\binom{N}{n}\frac{n^2-n+2}{2(n+1)}\frac{\mathrm{vol}_{n-1}(\partial B_2^n)^{\frac{n+1}{n-1}}}{\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n-1}}}B(N-n+1,n+\tfrac{2}{n-1})\\
&+c\binom{N}{n}B(N-n+1,n+\tfrac{4}{n-1})+c\left(\frac{1}{2}\right)^{-N+\frac{2}{n-1}}.
\end{aligned}
$$

This implies

$$\begin{aligned}\mathbb{E}(\partial B_2^n, N) &\le \mathrm{vol}_n(B_2^n)\\ &-\binom{N}{n}\frac{n^2-n+2}{2(n+1)}\frac{\mathrm{vol}_{n-1}(\partial B_2^n)^{\frac{n+1}{n-1}}}{\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n-1}}}\frac{\Gamma(N-n+1)\Gamma(n+\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})}\\ &+c\binom{N}{n}\frac{\Gamma(N-n+1)\Gamma(n+\frac{4}{n-1})}{\Gamma(N+1+\frac{4}{n-1})}+c\left(\frac{1}{2}\right)^{-N+\frac{2}{n-1}}.\end{aligned}$$

We have the asymptotic formula

$$\lim_{k\to\infty}\frac{\Gamma(k+\beta)}{\Gamma(k)k^\beta}=1.$$

Therefore we get that $\mathbb{E}(\partial B_2^n, N)$ is asymptotically less than

$$\begin{aligned}\mathrm{vol}_n(B_2^n)-\frac{n^2-n+2}{2(n+1)}\frac{\mathrm{vol}_{n-1}(\partial B_2^n)^{\frac{n+1}{n-1}}}{\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n-1}}}\frac{\Gamma(n+\frac{2}{n-1})}{n!N^{\frac{2}{n-1}}}\\ +c\frac{\Gamma(n+\frac{4}{n-1})}{n!N^{\frac{4}{n-1}}}+c\left(\frac{1}{2}\right)^{-N+\frac{2}{n-1}}.\end{aligned}$$

We apply now $x\Gamma(x)=\Gamma(x+1)$ to $x=n+\frac{2}{n-1}$.

$$\begin{aligned}\mathbb{E}(\partial B_2^n, N)\le \mathrm{vol}_n(B_2^n)-\frac{n-1}{2(n+1)!}\frac{\mathrm{vol}_{n-1}(\partial B_2^n)^{\frac{n+1}{n-1}}}{\mathrm{vol}_{n-1}(B_2^{n-1})^{\frac{2}{n-1}}}\frac{\Gamma(n+1+\frac{2}{n-1})}{N^{\frac{2}{n-1}}}\\ +c\frac{\Gamma(n+\frac{4}{n-1})}{n!N^{\frac{4}{n-1}}}+c\left(\frac{1}{2}\right)^{-N+\frac{2}{n-1}}.\end{aligned}$$

The other inequality is proved similarly. □

4 Probabilistic Estimates

4.1 Probabilistic Estimates for General Convex Bodies

Lemma 4.1. *Let K be a convex body in $\mathbb{R}^n$ with 0 as an interior point. The $n(n-1)$-dimensional Hausdorff measure of the real $n\times n$-matrices whose determinant equal 0 and whose columns are elements of ∂K is 0.*

Proof. We deduce this lemma from Lemma 3.2. We consider the map $rp:\partial B_2^n\to\partial K$

$$rp^{-1}(x)=\frac{x}{\|x\|}$$

and $Rp : \partial B_2^n \times \cdots \times \partial B_2^n \to \partial K \times \cdots \times \partial K$ with

$$Rp(x_1, \dots, x_n) = (rp(x_1), \dots, rp(x_n)).$$

Rp is a Lipschitz-map and the image of a nullset is a nullset. □

Lemma 4.2. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a continuous, positive function with $\int f d\mu = 1$. Then we have for all $x \in \overset{\circ}{K}$*

$$\mathbb{P}_f^N\{(x_1, \dots, x_N) | x \in \partial [x_1, \dots, x_N]\} = 0.$$

Let $\epsilon = (\epsilon(i))_{1 \le i \le n}$ be a sequence of signs, that is $\epsilon(i) = \pm 1$, $1 \le i \le n$. We denote, for a given sequence ϵ of signs, by K^ϵ the following subset of K

$$K^\epsilon = \{x = (x(1), x(2), \dots, x(n)) \in K | \ \forall i = 1, \dots, n : \operatorname{sgn}(x(i)) = \epsilon(i)\}.$$

Lemma 4.3. *(i) Let K be a convex body in $\mathbb{R}^n$, a, b positive constants and $\mathcal{E}$ an ellipsoid with center 0 such that $a\mathcal{E} \subseteq K \subseteq b\mathcal{E}$. Then we have*

$$\mathbb{P}_{\partial K}^N\{(x_1, \dots, x_N) | 0 \notin [x_1, \dots, x_N]\} \le 2^n \left(1 - \frac{1}{2^n}\left(\frac{a}{b}\right)^{n-1}\right)^N.$$

(ii) Let K be a convex body in $\mathbb{R}^n$, 0 an interior point of K, and let $f : \partial K \to \mathbb{R}$ be a continuous, nonnegative function with $\int_{\partial K} f(x) \mathrm{d}\mu = 1$. Then we have

$$\mathbb{P}_f^N\{(x_1, \dots, x_N) | 0 \notin [x_1, \dots, x_N]\} \le 2^n \left(1 - \min_{\epsilon} \int_{\partial K^\epsilon} f(x) \mathrm{d}\mu\right)^N.$$

(Here we do not assume that the function f is strictly positive.)

Proof. (i) A rotation puts K into such a position that

$$\mathcal{E} = \left\{ x \left| \sum_{i=1}^n \left| \frac{x(i)}{a_i} \right|^2 \le 1 \right. \right\}.$$

We have for all ϵ

$$\frac{a^{n-1}}{2^n} \mathrm{vol}_{n-1}(\partial \mathcal{E}) \le \mathrm{vol}_{n-1}(\partial K^\epsilon).$$

We show this. Let $p_{K,a\mathcal{E}}$ be the metric projection from ∂K onto $\partial a\mathcal{E}$. We have $p_{K,a\mathcal{E}}(\partial K^\epsilon) = \partial a\mathcal{E}^\epsilon$. Thus we get

$$\frac{a^{n-1}}{2^n} \mathrm{vol}_{n-1}(\partial \mathcal{E}) = a^{n-1} \mathrm{vol}_{n-1}(\partial \mathcal{E}^\epsilon) \le \mathrm{vol}_{n-1}(\partial K^\epsilon).$$

We have

$$\{(x_1,\dots,x_N)|\ \forall\epsilon\ \exists i: x_i\in\partial K^{\epsilon}\}\subseteq\{(x_1,\dots,x_N)|0\in[x_1,\dots,x_N]\}$$

and therefore

$$\{(x_1,\dots,x_N)|\ \exists\epsilon\ \forall i: x_i\notin\partial K^{\epsilon}\}\supseteq\{(x_1,\dots,x_N)|0\notin[x_1,\dots,x_N]\}.$$

Consequently

$$\bigcup_{\epsilon}\{(x_1,\dots,x_N)|\ \ \forall i: x_i\notin\partial K^{\epsilon}\}\supseteq\{(x_1,\dots,x_N)|0\notin[x_1,\dots,x_N]\}.$$

Therefore we get

$$\begin{aligned}\mathbb{P}_f^N\{(x_1,\dots,x_N)|0\notin[x_1,\dots,x_N]\}\ &\le\sum_{\epsilon}\left(1-\frac{\mathrm{vol}_{n-1}(\partial K^{\epsilon})}{\mathrm{vol}_{n-1}(\partial K)}\right)^N\\ &\le 2^n\left(1-\frac{\min_{\epsilon}\mathrm{vol}_{n-1}(\partial K^{\epsilon})}{\mathrm{vol}_{n-1}(\partial K)}\right)^N\\ &\le 2^n\left(1-\frac{a^{n-1}}{2^n}\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-1}(\partial K)}\right)^N\\ &\le 2^n\left(1-\frac{1}{2^n}\left(\frac{a}{b}\right)^{n-1}\right)^N.\end{aligned}$$

(ii) As in (i)

$$\begin{aligned}\mathbb{P}_f^N\{(x_1,\dots,x_N)|0\notin[x_1,\dots,x_N]\}\ &\le\mathbb{P}_f^N\{(x_1,\dots,x_N)|\exists\epsilon\forall i: x_i\notin\partial K^{\epsilon}\}\\ &\le 2^n\left(1-\min_{\epsilon}\int_{\partial K^{\epsilon}}f(x)\mathrm{d}\mu(x)\right)^N.\end{aligned}$$

□

Lemma 4.4. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K}f\mathrm{d}\mu=1$. Suppose that for all $0<t\le T$ we have $K_t\subseteq\overset{\circ}{K}$ and that there are $r,R>0$ with*

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R)$$

and let $N_{\partial K_s}(x_s)$ be a normal such that $s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))$. Then there is s_0 that depends only on r, R, and f such that we have for all s with $0<s\le s_0$ and for all sequences of signs ϵ,δ

$$\begin{aligned}&\mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta})\\ &\qquad\le C(r,R,f,\theta,n)\mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\epsilon})\end{aligned}$$

where the signed sets are taken in the plane $H(x_s,N_{\partial K_s}(x_s))$ with x_s as the origin and any orthogonal coordinate system. θ is the angle between $N_{\partial K}(x_0)$ and x_0-x_T.

The important point in Lemma 4.4 is that s_0 and the constant in the inequality depend only on r, R, and f.

Another approach is to use that x_s is the center of gravity of $K \cap H(x_s, N_{\partial K_s}(x_s))$ with respect to the weight

$$\frac{f(y)}{< N_{\partial K \cap H}(y), N_{\partial K}(y) >}$$

where $H = H(x_s, N_{\partial K_s}(x_s))$. See Lemma 2.4.

Proof. We choose s_0 so small that $x_0 - rN_{\partial K}(x_0) \in K_{s_0}$. We show first that there is s_0 that depends only on r and R such that we have for all s with $0 \leq s \leq s_0$

$$\sqrt{1 - \frac{2R\Delta}{r^2}\left(\frac{\max_{x \in \partial K} f(x)}{\min_{x \in \partial K} f(x)}\right)^{\frac{2}{n-1}}} \leq \langle N_{\partial K}(x_0), N_{\partial K_s}(x_s) \rangle \tag{51}$$

where Δ is the distance of x_0 to the hyperplane $H(x_s, N_{\partial K}(x_0))$

$$\Delta = < N_{\partial K}(x_0), x_0 - x_s > .$$

Let α denote the angle between $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$. From Figure 4.4.1 and 4.4.2 we deduce that the height of the cap

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_s, N_{\partial K_s}(x_s))$$

is greater than

$$r(1 - \cos\alpha) = r(1 - \langle N_{\partial K}(x_0), N_{\partial K_s}(x_s) \rangle).$$

Here we use that $x_T \in K_{s_0}$ and $x_0 - rN_{\partial K}(x_0) \in K_{s_0}$.

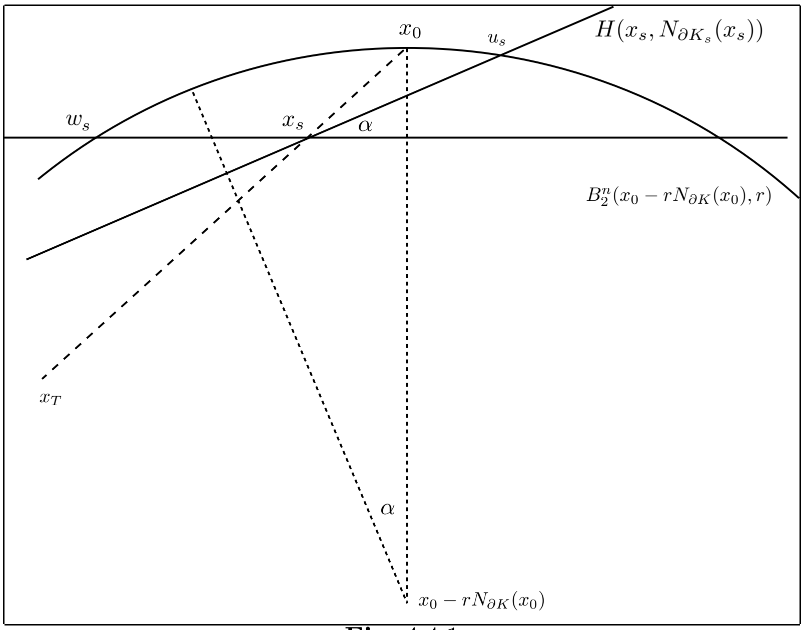

Fig. 4.4.1

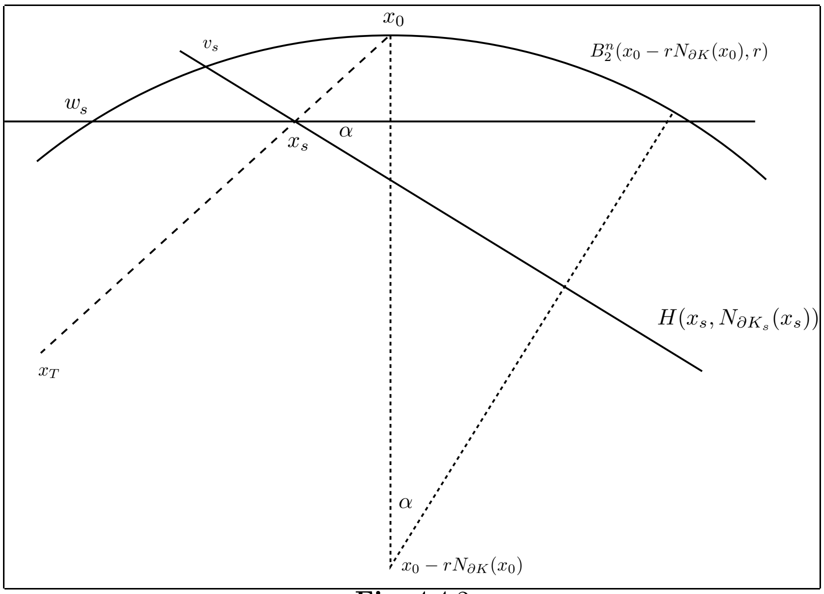

Fig. 4.4.2

In both graphics we see the plane through x_0 that is spanned by $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$. The points x_s and x_T are not necessarily in this plane.

We have

$$\begin{aligned}\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))) &= \int_{\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))} f(x)\mathrm{d}\mu_{\partial K}(x)\\ &\geq \min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))).\end{aligned}$$

Since $B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K$ we get

$$\begin{aligned}&\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &\qquad\geq \min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(\partial B_2^n(x_0-rN_{\partial K}(x_0),r)\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &\qquad\geq \min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^n(x_0-rN_{\partial K}(x_0),r)\cap H(x_s,N_{\partial K_s}(x_s))).\end{aligned}$$

Since the height of the cap is greater than $r(1-\cos\alpha)$ we get

$$\begin{aligned}&\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &\qquad\geq \min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})\left(2r^2(1-\cos\alpha)-r^2(1-\cos\alpha)^2\right)^{\frac{n-1}{2}}\\ &\qquad= \min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})\left(r^2(1-\cos^2\alpha)\right)^{\frac{n-1}{2}}.\end{aligned}\tag{52}$$

On the other hand

$$\begin{aligned}s=\mathbb{P}_f&(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &=\int_{\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))} f(x)\mathrm{d}\mu_{\partial K}(x)\\ &=\int_{\partial K\cap H^-(x_s,N_{\partial K}(x_0))} f(x)\mathrm{d}\mu_{\partial K}(x)\\ &\leq \max_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K}(x_0))).\end{aligned}$$

Since $B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R)$ we get for sufficiently small s_0

$$\begin{aligned}&\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &\qquad\leq \max_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(\partial B_2^n(x_0-RN_{\partial K}(x_0),R)\cap H^-(x_s,N_{\partial K}(x_0)))\\ &\qquad\leq \max_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})(2R\Delta)^{\frac{n-1}{2}}.\end{aligned}\tag{53}$$

Since

$$s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\leq \mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K}(x_0)))$$

we get by (52) and (53)

$$\begin{aligned}&\min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})\left(r^2(1-\cos^2\alpha)\right)^{\frac{n-1}{2}}\\ &\qquad\qquad\leq \max_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})(2R\Delta)^{\frac{n-1}{2}}.\end{aligned}$$

This implies

$$\cos\alpha \geq \sqrt{1-\frac{2R\Delta}{r^2}\left(\frac{\max_{x\in\partial K} f(x)}{\min_{x\in\partial K} f(x)}\right)^{\frac{2}{n-1}}}.$$

Thus we have established (51).

The distance of x_s to $\partial K \cap H(x_s, N_{\partial K_s}(x_s))$ is greater than the distance of x_s to $\partial B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H(x_s, N_{\partial K_s}(x_s))$. We have $\|x_s - (x_0 - \Delta N_{\partial K}(x_0))\| = \Delta \tan\theta$. Let $\bar{x}_s$ be the image of x_s under the orthogonal projection onto the 2-dimensional plane seen in Figures 4.4.1 and 4.4.2. Then $\|\bar{x}_s - x_s\| \leq \Delta \tan\theta$. There is a $n-1$-dimensional ball with center $\bar{x}_s$ and radius $\min\{\|\bar{x}_s - u_s\|, \|\bar{x}_s - v_s\|\}$ that is contained in $K \cap H(x_s, N_{\partial K_s}(x_s))$.

We can choose s_0 small enough so that for all s with $0 < s \leq s_0$ we have $\cos\alpha \geq \frac{1}{2}$.

$$\tan\alpha = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} \leq 2\frac{\sqrt{2R\Delta}}{r}\left(\frac{\max_{x\in\partial K} f(x)}{\min_{x\in\partial K} f(x)}\right)^{\frac{1}{n-1}} \tag{54}$$

We compute the point of intersection of the line through v_s and $\bar{x}_s$ and the line through x_0 and w_s. Formula (54) and the fact that the height of the cap $B_2^n(x_0 - rN_{\partial K}(x_0), N_{\partial K}(x_0)) \cap H^-(x_s, N_{\partial K}(x_0))$ is Δ and its radius $2r\Delta - \Delta^2$ give further

$$c\sqrt{\Delta} \leq \min\{\|\bar{x}_s - u_s\|, \|\bar{x}_s - v_s\|\}$$

where c is a constant depending only on r, R, f, n. Thus $K \cap H(x_s, N_{\partial K_s}(x_s))$ contains a Euclidean ball with center $\bar{x}_s$ and radius greater $c\sqrt{\Delta}$. Therefore, $K \cap H(x_s, N_{\partial K_s}(x_s))$ contains a Euclidean ball with center x_s and radius greater $c\sqrt{\Delta} - \Delta\tan\theta$. On the other hand,

$$K \cap H(x_s, N_{\partial K_s}(x_s)) \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H(x_s, N_{\partial K_s}(x_s)).$$

Following arguments as above we find that $K \cap H(x_s, N_{\partial K_s}(x_s))$ is contained in a Euclidean ball with center x_s and radius $C\sqrt{\Delta}$ where C is a constant that depends only on r, R, f, n. Therefore, with new constants c, C we get for all sequences of signs δ

$$c\Delta^{\frac{n-1}{2}} \leq \mathrm{vol}_{n-1}((K \cap H(x_s, N_{\partial K_s}(x_s)))^{\delta}) \leq C\Delta^{\frac{n-1}{2}}.$$

□

Lemma 4.5. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Let $f : \partial K \to \mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f d\mu = 1$. For all t with $0 < t \leq T$ we have $K_t \subseteq \overset{\circ}{K}$. Suppose that there are $r, R > 0$ with*

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \subseteq K \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R)$$

and let $N_{\partial K_s}(x_s)$ *be a normal such that* $s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))$*. Then there is* s_0 *that depends only on* r*,* R*, and* f *such that we have for all* s *with* $0<s\le s_0$

$$\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\le 3\ \mathrm{vol}_{n-1}(K\cap H(x_s,N_{\partial K_s}(x_s))).$$

Proof. Since

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R)$$

we can choose Δ sufficiently small so that we have for all $y\in\partial K\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))$

$$<N_{\partial K}(x_0),N_{\partial K}(y)>\ge 1-\tfrac{1}{8} \tag{55}$$

and Δ depends only on r and R. Since f is strictly positive we find s_0 that depends only on r, R, and f such that we have for all s with $0<s\le s_0$

$$K\cap H(x_s,N_{\partial K_s}(x_s))\subseteq K\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0)). \tag{56}$$

By (55) and (56)

$$<N_{\partial K}(x_0),N_{\partial K_s}(x_s)>\ge 1-\tfrac{1}{8}.$$

Thus

$$\begin{aligned}
&<N_{\partial K_s}(x_s),N_{\partial K}(y)>\\
&\qquad=<N_{\partial K}(x_0),N_{\partial K}(y)>+<N_{\partial K_s}(x_s)-N_{\partial K}(x_0),N_{\partial K}(y)>\\
&\qquad\ge 1-\tfrac{1}{8}-\|N_{\partial K_s}(x_s)-N_{\partial K}(x_0)\|\\
&\qquad=1-\tfrac{1}{8}-\sqrt{2-2<N_{\partial K_s}(x_s),N_{\partial K}(x_0)>}\ge 1-\tfrac{3}{8}.
\end{aligned}$$

Altogether

$$<N_{\partial K_s}(x_s),N_{\partial K}(y)>\ge 1-\tfrac{3}{8}.$$

Let $p_{N_{\partial K_s}(x_s)}$ be the metric projection from $\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))$ onto the plane $H(x_s,N_{\partial K_s}(x_s))$. With this we get now

$$\begin{aligned}
&\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\
&\qquad=\int_{K\cap H(x_s,N_{\partial K_s}(x_s))}\frac{1}{<N_{\partial K_s}(x_s),N_{\partial K}(p^{-1}_{N_{\partial K_s}(x_s)}(z))>}\mathrm{d}z\\
&\qquad\le 3\ \mathrm{vol}_{n-1}(K\cap H(x_s,N_{\partial K_s}(x_s))).
\end{aligned}$$

□

Lemma 4.6. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. x_s is defined by $\{x_s\} = [x_0, x_T] \cap K_s$. Let $f : \partial K \to \mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f \mathrm{d}\mu = 1$. For all t with $0 < t \leq T$ we have $K_t \subseteq \overset{\circ}{K}$. Suppose that there are $r, R > 0$ such that*

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \subseteq K \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R)$$

and let $N_{\partial K_s}(x_s)$ be a normal such that $s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))$. Then there is s_0 that depends only on r, R, and f such that for all s with $0 < s \leq s_0$ there are hyperplanes $H_1, \dots, H_{n-1}$ containing x_T and x_s such that the angle between two $n-2$-dimensional hyperplanes $H_i \cap H(x_s, N_{\partial K_s}(x_s))$ is $\frac{\pi}{2}$ and such that for

$$\partial K_{H,\epsilon} = \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \cap \bigcap_{i=1}^{n-1} H_i^{\epsilon_i}$$

and all sequences of signs ϵ and δ we have

$$\mathrm{vol}_{n-1}(\partial K_{H,\epsilon}) \leq c\ \mathrm{vol}_{n-1}(\partial K_{H,\delta})$$

where c depends on n, r, R, f and $d(x_T, \partial K)$ only.

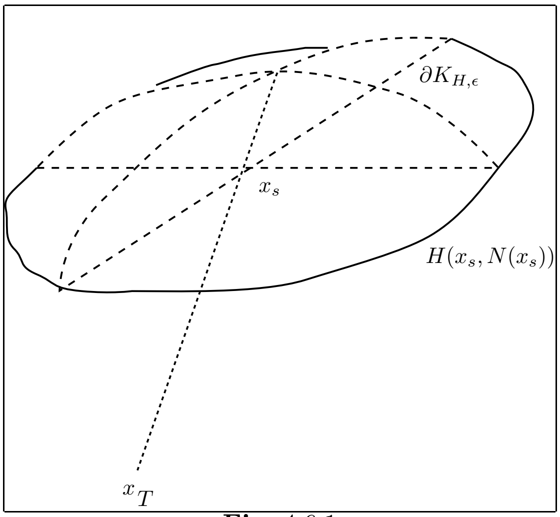

Fig. 4.6.1

Proof. Since x_T is an interior point of K we have $d(x_T, \partial K) > 0$. We choose s_0 so small that

$$B_2^n(x_T, \tfrac{1}{2} d(x_T, \partial K)) \subseteq K_{s_0}. \tag{57}$$

We choose hyperplanes H_i, $i=1,...,n-1$, such that they contain x_T and x_s and such that the angles between the hyperplanes $H_i \cap H(x_s, N_{\partial K_s}(x_s))$, $i=1,...,n-1$ is $\frac{\pi}{2}$.

By Lemma 4.4 there is s_0 so that we have for all s with $0<s\leq s_0$ and for all sequences of signs ϵ and δ

$$\mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\epsilon}) \leq c\ \mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta})$$

where c depends only on r, R, and n. Then we have by Lemma 4.5

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial K_{H,\epsilon}) &\leq \mathrm{vol}_{n-1}(\partial K\cap H^{-}(x_s,N_{\partial K_s}(x_s)))\\ &\leq c\ \mathrm{vol}_{n-1}(K\cap H(x_s,N_{\partial K_s}(x_s))).\end{aligned}$$

Therefore we get with a new constant c that depends only on n, f, r and R

$$\mathrm{vol}_{n-1}(\partial K_{H,\epsilon}) \leq c\ \mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta}).$$

We consider the affine projections $q:\mathbb{R}^n\to H(x_s,N_{\partial K_s}(x_s))$ and $p:\mathbb{R}^n\to H(x_s,\frac{x_s-x_T}{\|x_s-x_T\|})$ given by $q(t(x_s-x_T)+y)=y$ where $y\in H(x_s,N_{\partial K_s}(x_s))$ and $p(t(x_s-x_T)+y)=y$ where $y\in H(x_s,\frac{x_s-x_T}{\|x_s-x_T\|})$. Please note that p is a metric projection and $q\circ p=q$. Since p is a metric projection we have

$$\mathrm{vol}_{n-1}(p(\partial K_{H,\delta})) \leq \mathrm{vol}_{n-1}(\partial K_{H,\delta}).$$

q is an affine, bijective map between the two hyperplanes and

$$q\circ p(\partial K_{H,\delta}) = q(\partial K_{H,\delta}) \supseteq (K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta}.$$

By this (compare the proof of Lemma 2.7)

$$\begin{aligned}\frac{\mathrm{vol}_{n-1}(\partial K_{H,\delta})}{< N_{\partial K_s}(x_s),\frac{x_s-x_T}{\|x_s-x_T\|}>} &\geq \mathrm{vol}_{n-1}(q(\partial K_{H,\delta}))\\ &\geq \mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta}).\end{aligned}$$

By (57) the cosine of the angle between the plane $H(x_s,N_{\partial K_s}(x_s))$ and the plane orthogonal to x_s-x_T is greater than $\frac{1}{2}\frac{d(x_T,\partial K)}{\|x_s-x_T\|}$. Therefore we get

$$\mathrm{vol}_{n-1}(\partial K_{H,\delta}) \geq \frac{1}{2}\frac{d(x_T,\partial K)}{\|x_s-x_T\|}\mathrm{vol}_{n-1}((K\cap H(x_s,N_{\partial K_s}(x_s)))^{\delta}).$$

□

Lemma 4.7. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. x_s is defined by $\{x_s\}=[x_0,x_T]\cap K_s$. Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f d\mu=1$. Suppose that there are $r,R>0$ such that we have for all $x\in\partial K$*

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

and let $N_{\partial K_s}(x_s)$ be a normal such that $s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))$. Then there are constants s_0, a, and b with $0 \le a, b < 1$ that depend only on r, R, and f such that we have for all s with $0 < s \le s_0$ and for all $N \in \mathbb{N}$ and all $k = 1, \dots, N$

$$\begin{aligned}\mathbb{P}_f^N\{(x_1, \dots, x_N)|\ x_s \notin [x_1, \dots, x_N],\ x_1, \dots, x_k \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \\ \text{and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\} \\ \le (1-s)^{N-k}\, s^k 2^n (a^{N-k} + b^k).\end{aligned}$$

Proof. Let $H_1, \dots, H_{n-1}$ be hyperplanes and $\partial K_{H,\epsilon}$ as specified in Lemma 4.6:

$$\partial K_{H,\epsilon} = \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \cap \bigcap_{i=1}^{n-1} H_i^{\epsilon_i}.$$

We have by Lemma 4.6 that for all sequences of signs ϵ and δ

$$\mathrm{vol}_{n-1}(\partial K_{H,\epsilon}) \le c\ \mathrm{vol}_{n-1}(\partial K_{H,\delta})$$

where c depends on n, f, r, R and $d(x_T, \partial K)$. As

$$\begin{aligned}&\{(x_1, \dots, x_N)|x_s \in [x_1, \dots, x_N]\} \\ &\qquad \supseteq \{(x_1, \dots, x_N)|x_T \in [x_1, \dots, x_N] \text{ and } [x_s, x_0] \cap [x_1, \dots, x_N] \ne \emptyset\}\end{aligned}$$

we get

$$\begin{aligned}&\{(x_1, \dots, x_N)|x_s \notin [x_1, \dots, x_N]\} \\ &\qquad \subseteq \{(x_1, \dots, x_N)|x_T \notin [x_1, \dots, x_N] \text{ or } [x_s, x_0] \cap [x_1, \dots, x_N] = \emptyset\}.\end{aligned}$$

Therefore we get

$$\begin{aligned}&\{(x_1, \dots, x_N)|\ x_s \notin [x_1, \dots, x_N],\ x_1, \dots, x_k \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \\ &\qquad\qquad \text{and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\} \\ &\subseteq \{(x_1, \dots, x_N)|x_T \notin [x_1, \dots, x_N],\ x_1, \dots, x_k \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \\ &\qquad\qquad \text{and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\} \\ &\cup \{(x_1, \dots, x_N)|[x_s, x_0] \cap [x_1, \dots, x_N] = \emptyset,\ x_1, \dots, x_k \in \partial K \cap \\ &\qquad H^-(x_s, N_{\partial K_s}(x_s)) \text{ and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\}.\end{aligned}$$

With $H_s = H(x_s, N_{\partial K_s}(x_s))$

$$\begin{aligned}&\mathbb{P}_f^N\{(x_1, \dots, x_N)|\ x_s \notin [x_1, \dots, x_N],\ x_1, \dots, x_k \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)) \\ &\qquad\qquad \text{and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\} \\ &\quad \le (1-s)^{N-k}\, s^k\ \mathbb{P}_{f, \partial K \cap H_s^+}^{N-k}\{(x_{k+1}, \dots, x_N)|x_T \notin [x_{k+1}, \dots, x_N]\} \\ &\qquad + (1-s)^{N-k}\, s^k \mathbb{P}_{f, \partial K \cap H_s^-}^{k}\{(x_1, \dots, x_k)|[x_s, x_0] \cap [x_1, \dots, x_k] = \emptyset\}\end{aligned}$$

where we obtain $\mathbb{P}_{f,\partial K\cap H_s^+}$ from $\mathbb{P}_f$ by restricting it to the subset $\partial K\cap H_s^+$ and then normalizing it. The same for $\mathbb{P}_{f,\partial K\cap H_s^-}$. We have

$$\begin{aligned}\mathbb{P}^{N-k}_{f,\partial K\cap H_s^+}&\{(x_{k+1},\dots,x_N)|x_T\notin[x_{k+1},\dots,x_N]\}\\ &=\mathbb{P}^{N-k}_{\tilde f}\{(x_{k+1},\dots,x_N)|x_T\notin[x_{k+1},\dots,x_N]\}\end{aligned}\tag{58}$$

where $\tilde f:\partial(K\cap H^+(x_s,N_{\partial K_s}(x_s)))\to\mathbb{R}$ is given by

$$\tilde f(x)=\begin{cases}\dfrac{f(x)}{\mathbb{P}_f(\partial K\cap H_s^+)} & x\in\partial K\cap H^+(x_s,N_{\partial K_s}(x_s))\\ 0 & x\in \overset{\circ}{K}\cap H(x_s,N_{\partial K_s}(x_s)).\end{cases}$$

We apply Lemma 4.3.(ii) to $K\cap H^+(x_s,N_{\partial K_s}(x_s))$, $\tilde f$, and x_T as the origin. We get

$$\begin{aligned}\mathbb{P}^{N-k}_{\tilde f}&\{(x_{k+1},\dots,x_N)|x_T\notin[x_{k+1},\dots,x_N]\}\\ &\le 2^n\left(1-\min_{\epsilon}\int_{\partial(K\cap H_s^+)^\epsilon}\tilde f(x)\mathrm{d}\mu\right)^{N-k}.\end{aligned}\tag{59}$$

Since

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R)$$

we can choose s_0 sufficiently small so that for all s with $0<s\le s_0$

$$\min_{\epsilon}\int_{\partial(K\cap H_s^+)^\epsilon}\tilde f(x)d\mu\ge c>0$$

where c depends only on s_0 and s_0 can be chosen in such a way that it depends only on r, R, and f. Indeed, we just have to make sure that the surface area of the cap $K\cap H^-(x_s,N_{\partial K_s}(x_s))$ is sufficiently small. We verify the inequality. Since we have for all $x\in\partial K$

$$B_2^n(x-rN_{\partial K}(x),r)\subseteq K\subseteq B_2^n(x-RN_{\partial K}(x),R)$$

the point x_T is an interior point. We consider

$$B_2^n(x_T,\tfrac12 d(x_T,\partial K)).$$

Then, by considering the metric projection

$$\begin{aligned}&\tfrac{1}{2^n}\mathrm{vol}_{n-1}(\partial B_2^n(x_T,\tfrac12 d(x_T,\partial K)))\\ &\qquad=\mathrm{vol}_{n-1}(\partial B_2^n(x_T,\tfrac12 d(x_T,\partial K))^\epsilon)\le\mathrm{vol}_{n-1}(\partial K^\epsilon).\end{aligned}$$

We choose now

$$s_0 = \tfrac{1}{2^{n+1}}\mathrm{vol}_{n-1}(\partial B_2^n(x_T, \tfrac{1}{2}d(x_T,\partial K))) \min_{x\in\partial K} f(x).$$

Then we get

$$\begin{aligned}\mathbb{P}_f(\partial K\cap H_s^+)&\int_{\partial(K\cap H_s^+)^\epsilon} \tilde{f}(x)\mathrm{d}\mu(x)\\ &=\int_{\partial(K\cap H_s^+)^\epsilon} f(x)\mathrm{d}\mu(x)\\ &=\int_{\partial K^\epsilon} f(x)\mathrm{d}\mu(x)-\int_{\partial K^\epsilon\cap H_s^-} f(x)\mathrm{d}\mu.\end{aligned}$$

Since $\int_{\partial K^\epsilon\cap H_s^-} f(x)\mathrm{d}\mu = s\le s_0$

$$\begin{aligned}\mathbb{P}_f(\partial K\cap H_s^+)&\int_{\partial(K\cap H_s^+)^\epsilon} \tilde{f}(x)\mathrm{d}\mu(x)\\ &\ge\int_{\partial K^\epsilon} f(x)\mathrm{d}\mu(x)-s_0\\ &\ge \mathrm{vol}_{n-1}(\partial K^\epsilon)\min_{x\in\partial K} f(x)-s_0\\ &\ge \tfrac{1}{2^{n+1}}\mathrm{vol}_{n-1}(\partial B_2^n(x_T,\tfrac{1}{2}d(x_T,\partial K)))\min_{x\in\partial K} f(x).\end{aligned}$$

We put

$$a = 1-\min_{\epsilon}\int_{\partial(K\cap H_s^+)^\epsilon}\tilde{f}(x)\mathrm{d}\mu.$$

We get by (58) and (59)

$$\mathbb{P}^{N-k}_{f,\partial K\cap H_s^+}\{(x_{k+1},\dots,x_N)|x_T\notin[x_{k+1},\dots,x_N]\}\le 2^n a^{N-k}.$$

Moreover, since

$$\{(x_1,\dots,x_k)|\ [x_s,x_0]\cap[x_1,\dots,x_k]\neq\emptyset\}\supseteq\{(x_1,\dots,x_k)|\ \forall\epsilon\ \exists i: x_i\in\partial K_{H,\epsilon}\}$$

we get

$$\{(x_1,\dots,x_k)|\ [x_s,x_0]\cap[x_1,\dots,x_k]=\emptyset\}\subseteq\{(x_1,\dots,x_k)|\ \exists\epsilon\ \forall i: x_i\notin\partial K_{H,\epsilon}\}.$$

By Lemma 4.6 there is b with $0\le b<1$ so that

$$\mathbb{P}^k_{f,\partial K\cap H_s^-}\{(x_1,\dots,x_k)|[x_s,x_0]\cap[x_1,\dots,x_k]=\emptyset\}\le 2^{n-1}b^k.$$

Thus we get

$$\begin{aligned}\mathbb{P}^N_{\partial K}\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N],\ x_1,\dots,x_k\in\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))\\ \text{and } x_{k+1},\dots,x_N\in\partial K\cap H^+(x_s,N_{\partial K_s}(x_s))\}\\ \le(1-s)^{N-k}s^k2^n(a^{N-k}+b^k).\end{aligned}$$

□

Lemma 4.8. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. x_s is defined by $\{x_s\} = [x_0, x_T] \cap K_s$. Let $f : \partial K \to \mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f \mathrm{d}\mu = 1$. Suppose that there are $r, R > 0$ such that we have for all $x \in \partial K$*

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

and let $N_{\partial K_s}(x_s)$ be a normal such that $s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))$. Then there are constants s_0, a and b with $0 \le a, b < 1$ that depend only on r, R, and f such that we have for all s with $0 < s \le s_0$ and for all $N \in \mathbb{N}$ and all $k = 1, \dots, N$

$$\mathbb{P}_f^N\{(x_1, \dots, x_N)| \ x_s \notin [x_1, \dots, x_N]\} \le 2^n \left(a - as + s\right)^N + 2^n(1 - s + bs)^N.$$

s_0, a, and b are as given in Lemma 4.7.

Proof. We have

$$\begin{aligned}
&\mathbb{P}_f^N\{(x_1, \dots, x_N)| \ x_s \notin [x_1, \dots, x_N]\} \\
&\quad = \sum_{k=0}^{N} \binom{N}{k} \mathbb{P}_f^N\{(x_1, \dots, x_N)| \ x_s \notin [x_1, \dots, x_N], \ x_1, \dots, x_k \in \partial K \cap \\
&\qquad\quad H^-(x_s, N_{\partial K_s}(x_s)) \text{ and } x_{k+1}, \dots, x_N \in \partial K \cap H^+(x_s, N_{\partial K_s}(x_s))\}.
\end{aligned}$$

By Lemma 4.7 we get

$$\begin{aligned}
&\mathbb{P}_f^N\{(x_1, \dots, x_N)| \ x_s \notin [x_1, \dots, x_N]\} \\
&\quad \le 2^n \sum_{k=0}^{N} \binom{N}{k} (1 - s)^{N-k} s^k (a^{N-k} + b^k) \\
&\quad = 2^n \left(a - as + s\right)^N + 2^n(1 - s + bs)^N.
\end{aligned}$$

□

Lemma 4.9. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. x_s is defined by $\{x_s\} = [x_0, x_T] \cap K_s$. Let $f : \partial K \to \mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f \mathrm{d}\mu = 1$. Suppose that there are $r, R > 0$ such that we have for all $x \in \partial K$*

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

and let $N_{\partial K_s}(x_s)$ be a normal such that $s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))$. Then for all s_0 with $0 < s_0 \le T$

$$\lim_{N \to \infty} N^{\frac{2}{n-1}} \int_{s_0}^{T} \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1, \dots, x_N)| \ x_s \notin [x_1, \dots, x_N]\} \mathrm{d}\mu_{\partial K_s}(x_s) \mathrm{d}s}{\int_{\partial K \cap H_s} \frac{f(y)}{\sqrt{1 - <N_{\partial K_s}(x_s), N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K \cap H_s}(y)} = 0$$

where $H_s = H(x_s, N_{\partial K_s}(x_s))$.

Proof. Since $< N_{\partial K_s}(x_s), N_{\partial K}(y) > \leq 1$

$$N^{\frac{2}{n-1}} \int_{s_0}^{T} \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K\cap H_s}(y)} \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s$$

$$\leq \frac{N^{\frac{2}{n-1}}}{\min_{x\in\partial K} f(x)} \int_{s_0}^{T} \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\mathrm{vol}_{n-2}(\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))} \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s.$$

We observe that there is a constant $c_1 > 0$ such that

$$c_1 = d(\partial K, \partial K_{s_0}) = \inf\{\|x - x_{s_0}\| |x \in \partial K, x_{s_0} \in \partial K_{s_0}\}. \tag{60}$$

If not, there is $x_{s_0} \in \partial K \cap \partial K_{s_0}$. This cannot be because the condition

$$\forall x \in \partial K : B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R)$$

implies that K_{s_0} is contained in the interior of K. It follows that there is a constant $c_2 > 0$ that depends on K and f only such that for all $s \geq s_0$ and all $x_s \in \partial K_s$

$$\mathrm{vol}_{n-2}(\partial(K \cap H(x_s, N_{\partial K_s}(x_s)))) \geq c_2. \tag{61}$$

Therefore

$$N^{\frac{2}{n-1}} \int_{s_0}^{T} \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N(x_s),N(y)>^2}} \mathrm{d}\mu_{\partial K\cap H_s}(y)} \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s$$

$$\leq \frac{N^{\frac{2}{n-1}}}{c_2 \min_{x\in\partial K} f(x)} \times$$

$$\int_{s_0}^{T} \int_{\partial K_s} \mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\} \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s.$$

Now we apply Lemma 4.3.(ii) to K with x_s as the origin. Let

$$\partial K^{\epsilon}(x_s) = \{x \in \partial K | \forall i = 1,\dots,n : \mathrm{sgn}(x(i) - x_s(i)) = \epsilon_i\}.$$

With the notation of Lemma 4.3 we get that the latter expression is less than

$$\frac{2^n N^{\frac{2}{n-1}}}{c_2 \min_{x\in\partial K} f(x)} \int_{s_0}^{T} \int_{\partial K_s} \left(1 - \min_{\epsilon} \int_{\partial K^{\epsilon}(x_s)} f(x)\mathrm{d}\mu\right)^N \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s$$

$$\leq \frac{2^n N^{\frac{2}{n-1}}}{c_2 \min_{x\in\partial K} f(x)} \times$$

$$\int_{s_0}^{T} \int_{\partial K_s} \left(1 - \min_{x\in\partial K} f(x) \min_{\epsilon} \mathrm{vol}_{n-1}(\partial K^{\epsilon}(x_s))\right)^N \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s$$

$$\leq \frac{2^n N^{\frac{2}{n-1}} \mathrm{vol}_{n-1}(\partial K)(T - s_0)}{c_2 \min_{x\in\partial K} f(x)} \times$$

$$\left(1 - \min_{x\in\partial K} f(x) \inf_{s_0 \leq s \leq T} \min_{\epsilon} \mathrm{vol}_{n-1}(\partial K^{\epsilon}(x_s))\right)^N.$$

By (60) the ball with center x_s and radius c_1 is contained in K

$$c_1^{n-1}2^{-n}\mathrm{vol}_{n-1}(\partial B_2^n)=c_1^{n-1}\mathrm{vol}_{n-1}(\partial(B_2^n)^\epsilon)\le\mathrm{vol}_{n-1}(\partial K^\epsilon(x_s)).$$

Thus we obtain

$$N^{\frac{2}{n-1}}\int_{s_0}^T\int_{\partial K_s}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N(x_s),N(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s \tag{62}$$

$$\le\frac{2^nN^{\frac{2}{n-1}}\mathrm{vol}_{n-1}(\partial K)(T-s_0)}{c_2\min_{x\in\partial K}f(x)}\left(1-\min_{x\in\partial K}f(x)c_1^{n-1}2^{-n}\mathrm{vol}_{n-1}(\partial B_2^n)\right)^N.$$

Since f is strictly positive the latter expression tends to 0 for N to infinity. □

Lemma 4.10. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $x_s\in\partial K_s$ be given by the equation $\{x_s\}=[x_0,x_T]\cap\partial K_s$. Suppose that there are r,R with $0<r,R<\infty$ and*

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R).$$

Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K}f\mathrm{d}\mu=1$. Suppose that for all t with $0<t\le T$ we have $K_t\subseteq\overset{\circ}{K}$. Let the normals $N_{\partial K_s}(x_s)$ be such that

$$s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))).$$

Let Θ be the angle between $N_{\partial K}(x_0)$ and x_0-x_T and s_0 the minimum of

$$\frac{1}{2}\left(\frac{r}{8R}\right)^{\frac{n-1}{2}}\frac{(\min_{x\in\partial K}f(x))^2}{\max_{x\in\partial K}f(x)}\mathrm{vol}_{n-1}(B_2^{n-1})r^{n-1}\left(\tfrac14\cos^3\Theta\right)^{\frac{n-1}{2}}$$

and the constant $C(r,R,f,\Theta,n)$ of Lemma 4.4. Then we have for all s with $0<s<s_0$ and all $y\in\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))$

$$\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}\le\frac{30R}{r^2}\left(\frac{s\ \max_{x\in\partial K}f(x)}{(\min_{x\in\partial K}f(x))^2\,\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.$$

Proof. Θ is the angle between $N_{\partial K}(x_0)$ and x_0-x_T.

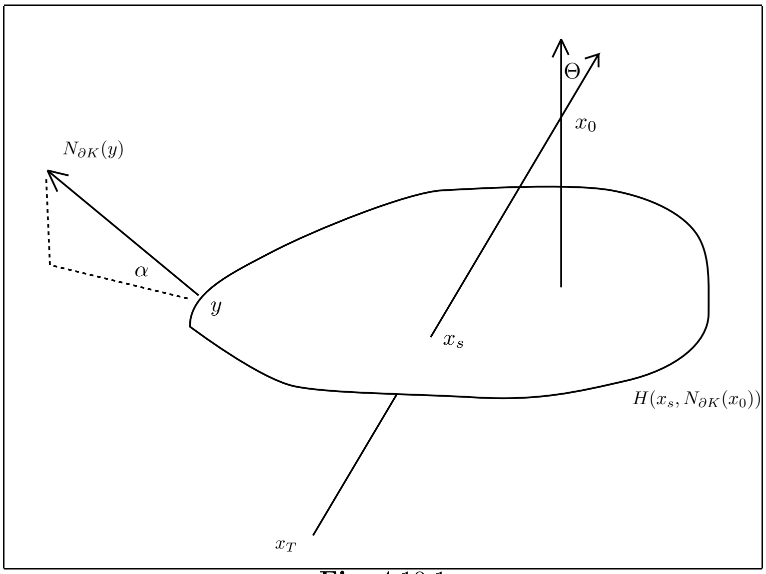

Fig. 4.10.1

Let $\Delta_r(s)$ be the height of the cap

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_s, N_{\partial K_s}(x_s))$$

and $\Delta_R(s)$ the one of

$$B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s)).$$

By assumption

$$s_0 \le \frac{1}{2}\left(\frac{r}{8R}\right)^{\frac{n-1}{2}} \frac{(\min_{x\in\partial K} f(x))^2}{\max_{x\in\partial K} f(x)} \mathrm{vol}_{n-1}(B_2^{n-1}) r^{n-1} \left(\tfrac{1}{4}\cos^3\Theta\right)^{\frac{n-1}{2}}. \quad (63)$$

First we want to make sure that for s with $0 < s < s_0$ the number $\Delta_r(s)$ is well-defined, i.e. the above cap is not the empty set. For this we have to show that $H(x_s, N_{\partial K_s}(x_s))$ intersects $B_2^n(x_0 - rN_{\partial K}(x_0), r)$. It is enough to show that for all s with $0 < s \le s_0$ we have $x_s \in B_2^n(x_0 - rN_{\partial K}(x_0), r)$. This follows provided that there is s_0 such that for all s with $0 < s \le s_0$

$$\|x_0 - x_s\| \le \tfrac{1}{2} r \cos^2\Theta. \quad (64)$$

See Figure 4.10.2.

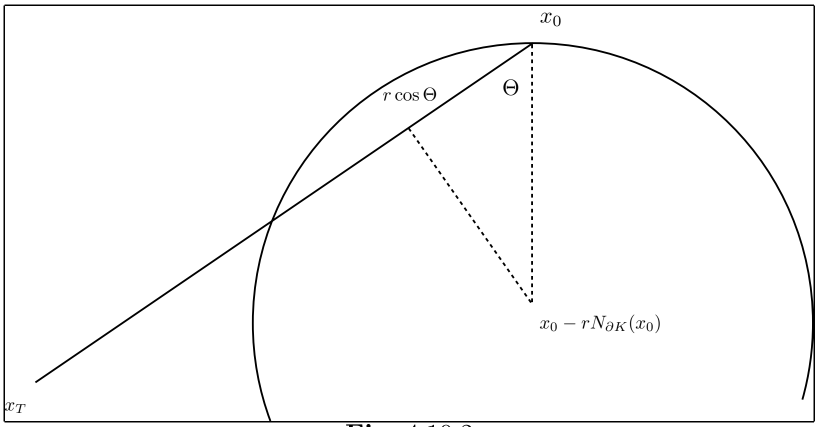

Fig. 4.10.2

We are going to verify this inequality. We consider the point $z \in [x_T, x_0]$ with $\|x_0 - z\| = \frac{1}{2} r \cos^2 \Theta$. Let H be any hyperplane with $z \in H$. Then

$$\mathbb{P}_f(\partial K \cap H^-) = \int_{\partial K \cap H^-} f(x) \mathrm{d}\mu_{\partial K}(x) \geq \left(\min_{x \in \partial K} f(x) \right) \mathrm{vol}_{n-1}(\partial K \cap H^-).$$

The set $K \cap H^-$ contains a cap of $B_2^n(x_0 - rN_{\partial K}(x_0), r)$ with height greater than $\frac{3}{8} r \cos^2 \Theta$. We verify this.

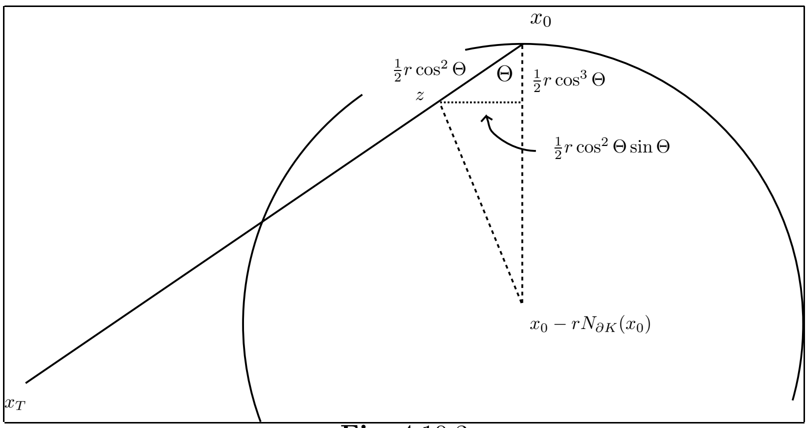

Fig. 4.10.3

By Figure 4.10.3 we have

$$
\begin{aligned}
\|z-(x_0-rN_{\partial K}(x_0)\| &= \sqrt{|r-\tfrac{1}{2}r\cos^3\Theta|^2+\tfrac{1}{4}r^2\cos^4\Theta\sin^2\Theta} \\
&= \sqrt{r^2-r^2\cos^3\Theta+\tfrac{1}{4}r^2\cos^6\Theta+\tfrac{1}{4}r^2\cos^4\Theta\sin^2\Theta} \\
&= \sqrt{r^2-r^2\cos^3\Theta+\tfrac{1}{4}r^2\cos^4\Theta} \\
&\le r\sqrt{1-\tfrac{3}{4}\cos^3\Theta}.
\end{aligned}
$$

Therefore the height of a cap is greater than

$$
r-\|z-(x_0-rN_{\partial K}(x_0)\| \ge r\left(1-\sqrt{1-\tfrac{3}{4}\cos^3\Theta}\right) \ge \tfrac{3}{8}r\cos^3\Theta.
$$

By Lemma 1.3 a cap of a Euclidean ball of radius r with height $h=\frac{3}{8}r\cos^3\Theta$ has surface area greater than

$$
\begin{aligned}
&\mathrm{vol}_{n-1}(B_2^{n-1})r^{\frac{n-1}{2}}\left(2h-\frac{h^2}{r}\right)^{\frac{n-1}{2}} \\
&\qquad= \mathrm{vol}_{n-1}(B_2^{n-1})r^{\frac{n-1}{2}}\left(\tfrac{3}{4}r\cos^3\Theta-\tfrac{9}{64}r\cos^6\Theta\right)^{\frac{n-1}{2}} \\
&\qquad\ge \mathrm{vol}_{n-1}(B_2^{n-1})r^{n-1}\left(\tfrac{1}{4}\cos^3\Theta\right)^{\frac{n-1}{2}}.
\end{aligned}
$$

By our choice of s_0 (63) we get

$$
\mathbb{P}_f(\partial K\cap H^-) \ge \left(\min_{x\in\partial K} f(x)\right)\mathrm{vol}_{n-1}(B_2^{n-1})r^{n-1}\left(\tfrac{1}{4}\cos^3\Theta\right)^{\frac{n-1}{2}} > s_0.
$$

Therefore we have for all s with $0<s<s_0$ that $z\in K_{s_0}$. By convexity we get

$$
\partial K_s\cap[z,x_0]\neq\emptyset.
$$

Thus (64) is shown.

Next we show that for all s with $0<s<s_0$ we have

$$
\sqrt{1-\frac{8R}{3r^3}\left(s\frac{\max_{x\in\partial K}f(x)}{\left(\min_{x\in\partial K}f(x)\right)^2\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}} \le \langle N_{\partial K}(x_0),N_{\partial K_s}(x_s)\rangle. \tag{65}
$$

By the same consideration for showing (64) we get for all s with $0<s<s_0$

$$
\Delta_r(s)\le\tfrac{3}{8}r\cos^3\Theta
$$

and by Lemma 1.3

$$
\begin{aligned}
s &= \mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))) \\
&\ge \left(\min_{x\in\partial K}f(x)\right)\mathrm{vol}_{n-1}(B_2^{n-1})r^{\frac{n-1}{2}}\left(2\Delta_r(s)-\frac{(\Delta_r(s))^2}{r}\right)^{\frac{n-1}{2}}.
\end{aligned}
$$

Since $\Delta_r(s) \le \frac{3}{8} r \cos^3 \Theta$

$$\begin{aligned} s &\ge \left(\min_{x\in\partial K} f(x)\right) \mathrm{vol}_{n-1}(B_2^{n-1}) r^{\frac{n-1}{2}} \left(2\Delta_r(s) - \Delta_r(s)\tfrac{3}{8}\cos^3\Theta\right)^{\frac{n-1}{2}} \\ &\ge \left(\min_{x\in\partial K} f(x)\right) \mathrm{vol}_{n-1}(B_2^{n-1}) (r\Delta_r(s))^{\frac{n-1}{2}}. \end{aligned}$$

Thus we have

$$s \ge \left(\min_{x\in\partial K} f(x)\right) \mathrm{vol}_{n-1}(B_2^{n-1}) (r\Delta_r(s))^{\frac{n-1}{2}}$$

or equivalently

$$\Delta_r(s) \le \frac{1}{r}\left(\frac{s}{\min_{x\in\partial K} f(x) \mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}. \tag{66}$$

Next we show

$$\tfrac{3}{4}\Delta(s) \le \Delta_r(s)$$

where $\Delta(s)$ is the distance of x_0 to the hyperplane $H(x_s, N_{\partial K}(x_0))$

$$\Delta(s) =< N_{\partial K}(x_0), x_0 - x_s > .$$

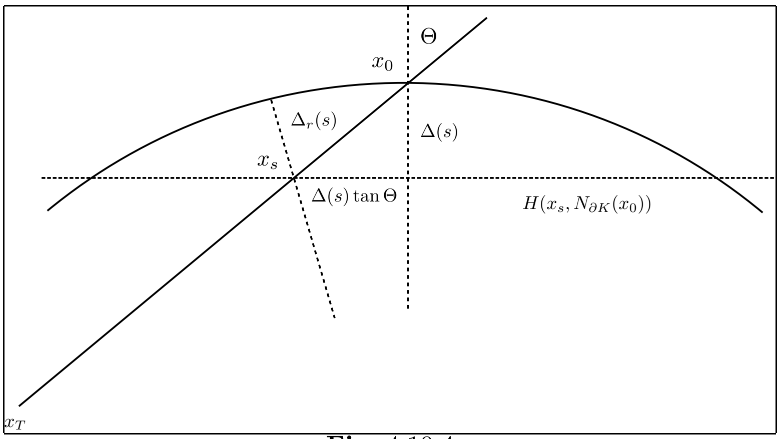

Fig. 4.10.4

By the Pythagorean Theorem, see Figure 4.10.4,

$$(r - \Delta_r(s))^2 = (r - \Delta(s))^2 + (\Delta(s)\tan\Theta)^2.$$

Thus

$$\begin{aligned}\Delta_r(s) &= r - \sqrt{(r-\Delta(s))^2 + (\Delta(s)\tan\Theta)^2} \\ &= r\left(1 - \sqrt{1 - \frac{1}{r^2}(2r\Delta(s) - \Delta^2(s) - (\Delta(s)\tan\Theta)^2)}\right).\end{aligned}$$

We use $\sqrt{1-t} \leq 1 - \frac{1}{2}t$

$$\begin{aligned}\Delta_r(s) &\geq \frac{1}{2r}\left(2r\Delta(s) - \Delta^2(s) - (\Delta(s)\tan\Theta)^2\right) \\ &= \Delta_r(s)\left[1 - \tfrac{1}{2}\frac{\Delta_r(s)}{r}(1+\tan^2\Theta)\right].\end{aligned}$$

By (64) we get $\Delta(s) = \|x_0 - x_s\|\cos\Theta \leq \frac{1}{2}r\cos^3\Theta$ and thus $\Delta(s) \leq \frac{1}{2}r\cos^3\Theta$. With this

$$\begin{aligned}\Delta_r(s) &= \Delta_r(s)\left[1 - \tfrac{1}{2}\frac{\Delta_r(s)}{r}(1+\tan^2\Theta)\right] \\ &= \Delta_r(s)\left[1 - \frac{1}{2r}(1+\tan^2\Theta)\tfrac{1}{2}r\cos^3\Theta\right] \\ &= \Delta_r(s)\left[1 - \tfrac{1}{4}\cos^4\Theta\right] \geq \tfrac{3}{4}\Delta_r(s).\end{aligned}$$

By formula (51) of the proof of Lemma 4.4 we have

$$\sqrt{1 - \frac{2R\Delta(s)}{r^2}\left(\frac{\max_{x\in\partial K} f(x)}{\min_{x\in\partial K} f(x)}\right)^{\frac{2}{n-1}}} \leq \langle N_{\partial K}(x_0), N_{\partial K_s}(x_s)\rangle.$$

By $\frac{3}{4}\Delta(s) \leq \Delta_r(s)$

$$\sqrt{1 - \frac{8R\Delta_r(s)}{3r^2}\left(\frac{\max_{x\in\partial K} f(x)}{\min_{x\in\partial K} f(x)}\right)^{\frac{2}{n-1}}} \leq \langle N_{\partial K}(x_0), N_{\partial K_s}(x_s)\rangle.$$

By (66) we get

$$\sqrt{1 - \frac{8R}{3r^3}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2 \operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}} \leq \langle N_{\partial K}(x_0), N_{\partial K_s}(x_s)\rangle.$$

Thus we have shown (65).

Next we show that for all $y \in \partial B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s))$

$$1 - \frac{\Delta_r(s)}{r} \leq \left\langle N_{\partial K_s}(x_s), \frac{y - (x_0 - RN_{\partial K}(x_0))}{\|y - (x_0 - RN_{\partial K}(x_0))\|}\right\rangle. \tag{67}$$

For this we show first that for all s with $0 < s < s_0$

$$\Delta_R(s) \leq \frac{R}{r}\Delta_r(s). \tag{68}$$

By our choice (63) of s_0 and by (65)

$$\langle N_{\partial K_s}(x_s), N_{\partial K}(x_0)\rangle \geq \sqrt{1-\tfrac{1}{12}\cos^3\Theta}$$

and by (64) we have $\|x_s - x_0\| < \frac{1}{2} r\cos^2\Theta$. Therefore we have for all s with $0 < s < s_0$ that the hyperplane $H(x_s, N_{\partial K_s}(x_s))$ intersects the line segment

$$[x_0, x_0 - rN_{\partial K}(x_0)].$$

Let r_1 be the distance of x_0 to the point defined by the intersection

$$[x_0, x_0 - rN_{\partial K}(x_0)] \cap H(x_s, N_{\partial K_s}(x_s)).$$

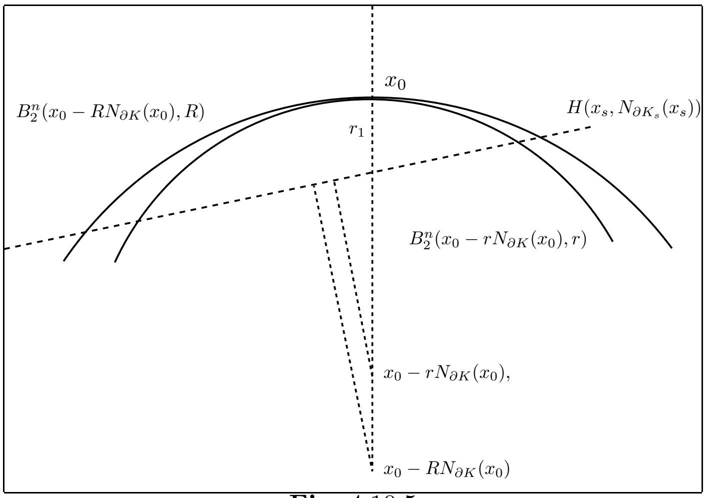

Fig. 4.10.5

We get by Figure 4.10.5

$$\frac{r - \Delta_r(s)}{R - \Delta_R(s)} = \frac{r - r_1}{R - r_1} \leq \frac{r}{R}.$$

The right hand side inequality follows from the monotonicity of the function $(r-t)/(R-t)$. Thus

$$r - \Delta_r(s) \leq \frac{r}{R}(R - \Delta_R(s)) = r - \frac{r}{R}\Delta_R(s)$$

and therefore

$$\frac{r}{R}\Delta_R(s) \leq \Delta_r(s).$$

For all $y \in \partial B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s))$ the cosine of the angle between $N_{\partial K_s}(x_s)$ and $y - (x_0 - RN_{\partial K}(x_0))$ is greater than $1 - \frac{\Delta_R(s)}{R}$.

This holds since y is an element of a cap of a Euclidean ball with radius R and with height $\Delta_R(s)$. Thus we have

$$1-\frac{\Delta_R(s)}{R}\le\left\langle N_{\partial K_s}(x_s),\frac{y-(x_0-RN_{\partial K}(x_0))}{\|y-(x_0-RN_{\partial K}(x_0))\|}\right\rangle .$$

By (68)

$$1-\frac{\Delta_r(s)}{r}\le\left\langle N_{\partial K_s}(x_s),\frac{y-(x_0-RN_{\partial K}(x_0))}{\|y-(x_0-RN_{\partial K}(x_0))\|}\right\rangle$$

and we have verified (67).

We show now that this inequality implies that for all s with $0<s<s_0$ and all $y\in\partial B_2^n(x_0-RN_{\partial K}(x_0),R)\cap H^-(x_s,N_{\partial K_s}(x_s))$

$$1-\Delta_r(s)\frac{R^2}{r^3}\le\left\langle N_{\partial K_s}(x_s),\frac{y-(x_0-rN_{\partial K}(x_0))}{\|y-(x_0-rN_{\partial K}(x_0))\|}\right\rangle . \tag{69}$$

Let α be the angle between $N_{\partial K_s}(x_s)$ and $y-(x_0-RN_{\partial K}(x_0))$ and let β be the angle between $N_{\partial K_s}(x_s)$ and $y-(x_0-rN_{\partial K}(x_0))$.

$$\begin{aligned}\cos\alpha &=\left\langle N_{\partial K_s}(x_s),\frac{y-(x_0-RN_{\partial K}(x_0))}{\|y-(x_0-RN_{\partial K}(x_0))\|}\right\rangle\\ \cos\beta &=\left\langle N_{\partial K_s}(x_s),\frac{y-(x_0-rN_{\partial K}(x_0))}{\|y-(x_0-rN_{\partial K}(x_0))\|}\right\rangle\end{aligned}$$

We put

$$a=\|y-(x_0-rN_{\partial K}(x_0))\| \qquad\qquad b=\|y-x_0\|.$$

See Figure 4.10.6.

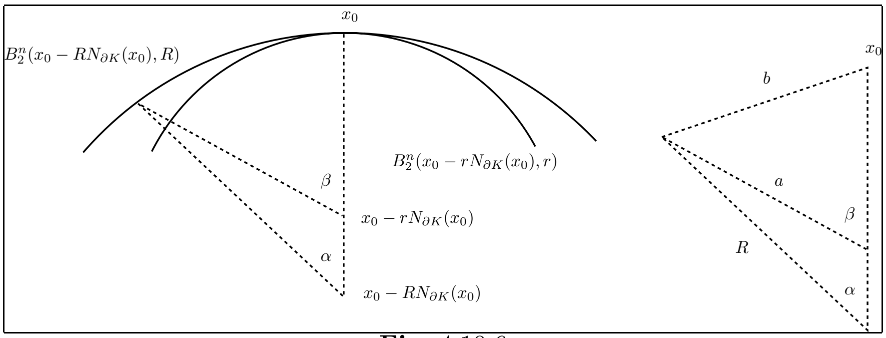

Fig. 4.10.6

By elementary trigonometric formulas we get

$$b^2=2R^2(1-\cos\alpha) \qquad\qquad b^2=a^2+r^2-2ar\cos\beta$$

and

$$a^2 = R^2 + (R-r)^2 - 2R(R-r)\cos\alpha = r^2 + 2R(R-r)(1-\cos\alpha).$$

From these equations we get

$$\begin{aligned}\cos\beta &= \frac{a^2+r^2-b^2}{2ar} = \frac{a^2+r^2-2R^2(1-\cos\alpha)}{2ar}\\ &= \frac{2r^2-2Rr(1-\cos\alpha)}{2r\sqrt{r^2+2R(R-r)(1-\cos\alpha)}} = \frac{r-R(1-\cos\alpha)}{\sqrt{r^2+2R(R-r)(1-\cos\alpha)}}.\end{aligned}$$

Thus

$$\cos\beta = \frac{1-\frac{R}{r}(1-\cos\alpha)}{\sqrt{1+2R(\frac{R}{r^2}-\frac{1}{r})(1-\cos\alpha)}}.$$

By (67) we have $1-\cos\alpha \le \frac{\Delta_r(s)}{r}$ and therefore

$$\begin{aligned}\cos\beta &\ge \frac{1-\frac{R\Delta_r(s)}{r^2}}{\sqrt{1+2R(\frac{R}{r^2}-\frac{1}{r})\frac{\Delta_r(s)}{r}}} \ge \frac{1-\frac{R\Delta_r(s)}{r^2}}{1+R(\frac{R}{r^2}-\frac{1}{r})\frac{\Delta_r(s)}{r}}\\ &= 1-\frac{\frac{R^2}{r^3}\Delta_r(s)}{1+R(\frac{R}{r^2}-\frac{1}{r})\frac{\Delta_r(s)}{r}} \ge 1-\frac{R^2}{r^3}\Delta_r(s).\end{aligned}$$

Thus we have proved (69). From (69) it follows now easily that for all s with $0<s<s_0$ and all $y \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s))$

$$1-\Delta_r(s)\frac{R^2}{r^3} \le \left\langle N_{\partial K_s}(x_s), \frac{y-(x_0-rN_{\partial K}(x_0))}{\|y-(x_0-rN_{\partial K}(x_0))\|}\right\rangle. \tag{70}$$

This follows because the cap $K \cap H^-(x_s, N_{\partial K_s}(x_s))$ is contained in the cap $B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s))$. Using now (66)

$$\begin{aligned}1-\frac{R^2}{r^4}\left(\frac{s}{\min_{x\in\partial K} f(x)\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}& \\ \le \left\langle N_{\partial K_s}(x_s), \frac{y-(x_0-rN_{\partial K}(x_0))}{\|y-(x_0-rN_{\partial K}(x_0))\|}\right\rangle&.\end{aligned} \tag{71}$$

For all s with $0<s<s_0$ and all $y \in \partial K \cap H^-(x_s, N_{\partial K_s}(x_s))$ the angle between $y-(x_0-rN_{\partial K}(x_0))$ and $N_{\partial K}(y)$ cannot be greater than the angle between $y-(x_0-rN_{\partial K}(x_0))$ and $N_{\partial K}(x_0)$. This follows from Figure 4.10.7.

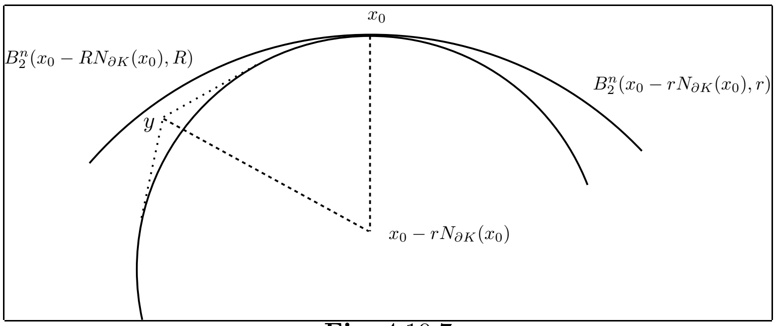

Fig. 4.10.7

A supporting hyperplane of K through y cannot intersect $B_2^n(x_0 - rN_{\partial K}(x_0), r)$. Therefore the angle between $y-(x_0 - rN_{\partial K}(x_0))$ and $N_{\partial K}(y)$ is smaller than the angle between $y-(x_0 - rN_{\partial K}(x_0))$ and the normal of a supporting hyperplane of $B_2^n(x_0 - rN_{\partial K}(x_0), r)$ that contains y.

Let α_1 denote the angle between $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$, α_2 the angle between $N_{\partial K_s}(x_s)$ and $y-(x_0-rN_{\partial K}(x_0))$, and α_3 the angle between $N_{\partial K}(x_0)$ and $y-(x_0-rN_{\partial K}(x_0))$. Then by (65) and (71) we have

$$\begin{aligned}\alpha_3 &\leq \alpha_1 + \alpha_2 \leq \tfrac{\pi}{2}\sin\alpha_1 + \tfrac{\pi}{2}\sin\alpha_2 \\ &\leq \tfrac{\pi}{2}\sqrt{\frac{8R}{3r^3}}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2 \operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \\ &\qquad\qquad + \tfrac{\pi}{\sqrt{2}}\frac{R}{r^2}\left(\frac{s}{\min_{x\in\partial K} f(x)\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}} \\ &\leq 10\frac{R}{r^2}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2 \operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.\end{aligned}$$

Let α_4 be the angle between $N_{\partial K}(y)$ and $y-(x_0-rN_{\partial K}(x_0))$. By the above consideration $\alpha_4 \leq \alpha_3$. Thus

$$\alpha_4 \leq 10\frac{R}{r^2}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2 \operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.$$

Let α_5 be the angle between $N_{\partial K_s}(x_s)$ and $N_{\partial K}(y)$. Then

$$\begin{aligned}
\sin\alpha_5 &\le \alpha_5 \le \alpha_2+\alpha_4\\
&\le 10\frac{R}{r^2}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}\\
&\qquad +\tfrac{\pi}{\sqrt{2}}\frac{R}{r^2}\left(\frac{s}{\min_{x\in\partial K} f(x)\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}\\
&\le 30\frac{R}{r^2}\left(s\frac{\max_{x\in\partial K} f(x)}{(\min_{x\in\partial K} f(x))^2\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{1}{n-1}}.
\end{aligned}$$

□

Lemma 4.11. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f\mathrm{d}\mu=1$. Assume that for all t with $0<t\le T$ we have $K_t\subseteq\overset{\circ}{K}$. Let $x_s\in\partial K_s$ be given by the equation $\{x_s\}=[x_0,x_T]\cap\partial K_s$. Suppose that there are r,R with $0<r,R<\infty$ and*

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R).$$

Let the normals $N_{\partial K_s}(x_s)$ be such that

$$s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))).$$

Let s_0 be as in Lemma 4.10. Then we have for all s with $0<s<s_0$

$$\begin{aligned}
&\int_{\partial K\cap H_s}\frac{1}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)\\
&\qquad\ge c^n\frac{r^n\left(\min_{x\in\partial K} f(x)\right)^{\frac{2}{n-1}}\left(\operatorname{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}}{R^{n-1}\max_{x\in\partial K} f(x)}\ s^{\frac{n-3}{n-1}}
\end{aligned}$$

where c is an absolute constant and $H_s=H(x_s,N_{\partial K_s}(x_s))$.

Proof. By Lemma 4.10 we have

$$\begin{aligned}
&\int_{\partial K\cap H_s}\frac{1}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)\\
&\ge\frac{r^2}{30R}\left(\frac{(\min_{x\in\partial K} f(x))^2\operatorname{vol}_{n-1}(B_2^{n-1})}{s\ \max_{x\in\partial K} f(x)}\right)^{\frac{1}{n-1}}\operatorname{vol}_{n-2}(\partial K\cap H_s)\\
&\ge\frac{r^2}{30R}\left(\frac{(\min_{x\in\partial K} f(x))^2\operatorname{vol}_{n-1}(B_2^{n-1})}{s\ \max_{x\in\partial K} f(x)}\right)^{\frac{1}{n-1}}\\
&\qquad\times\operatorname{vol}_{n-2}(\partial B_2^n(x_0-rN_{\partial K}(x_0),r)\cap H_s). \qquad (72)
\end{aligned}$$

Now we estimate the radius of the $n-1$-dimensional Euclidean ball $B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H_s$ from below. As in Lemma 4.10 $\Delta_r(s)$ is the height of the cap

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_s, N_{\partial K_s}(x_s))$$

and $\Delta_R(s)$ the one of

$$B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s)).$$

By (68) we have $\Delta_R(s) \le \frac{R}{r}\Delta_r(s)$. Moreover,

$$\begin{aligned} s &= \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))) \\ &= \int_{\partial K \cap H_s^-} f(x) \mathrm{d}\mu_{\partial K}(x) \le \max_{x \in \partial K} f(x) \mathrm{vol}_{n-1}(\partial K \cap H_s^-). \end{aligned} \tag{73}$$

Since $K \cap H_s^- \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H_s^-$ we have

$$\begin{aligned} \mathrm{vol}_{n-1}(\partial K \cap H_s^-) &\le \mathrm{vol}_{n-1}(\partial(K \cap H_s^-)) \\ &\le \mathrm{vol}_{n-1}(\partial(B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H_s^-)) \\ &\le 2\mathrm{vol}_{n-1}(\partial B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H_s^-). \end{aligned}$$

By Lemma 1.3 we get

$$\mathrm{vol}_{n-1}(\partial K \cap H_s^-) \le 2\sqrt{1 + \frac{2\Delta_R(s)R}{(R - \Delta_R(s))^2}} \mathrm{vol}_{n-1}(B_2^{n-1})(2R\Delta_R(s))^{\frac{n-1}{2}}.$$

As we have seen in the proof of Lemma 4.10 we have $\Delta_r(s) \le \frac{1}{2}r$. Together with $\Delta_R(s) \le \frac{R}{r}\Delta_r(s)$ we get $\Delta_R(s) \le \frac{1}{2}R$. This gives us

$$\mathrm{vol}_{n-1}(\partial K \cap H_s^-) \le 2\sqrt{5}\mathrm{vol}_{n-1}(B_2^{n-1})(2R\Delta_R(s))^{\frac{n-1}{2}}$$

and

$$\begin{aligned} \frac{R}{r}\Delta_r(s) &\ge \Delta_R(s) \ge \frac{1}{2R}\left(\frac{\mathrm{vol}_{n-1}(\partial K \cap H_s^-)}{2\sqrt{5}\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}} \\ &\ge \frac{1}{2R}\left(\frac{s}{2\sqrt{5}\mathrm{vol}_{n-1}(B_2^{n-1})\max_{x \in \partial K} f(x)}\right)^{\frac{2}{n-1}}. \end{aligned}$$

By this and by $\Delta_r(s) \le \frac{1}{2}r$ the radius of $B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H_s$ is greater than

$$\begin{aligned} \sqrt{2r\Delta_r(s) - \Delta_r(s)^2} &\ge \sqrt{r\Delta_r(s)} \\ &\ge \frac{r}{\sqrt{2}R}\left(\frac{s}{2\sqrt{5}\mathrm{vol}_{n-1}(B_2^{n-1})\max_{x \in \partial K} f(x)}\right)^{\frac{1}{n-1}}. \end{aligned}$$

Therefore, by (72)

$$\int_{\partial K\cap H_s}\frac{1}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)$$

$$\geq\frac{r^2}{30R}\left(\frac{(\min_{x\in\partial K}f(x))^2\,\mathrm{vol}_{n-1}(B_2^{n-1})}{s\ \max_{x\in\partial K}f(x)}\right)^{\frac{1}{n-1}}\mathrm{vol}_{n-2}(\partial B_2^{n-1})$$

$$\left(\frac{r}{\sqrt{2}R}\right)^{n-2}\left(\frac{s}{2\sqrt{5}\mathrm{vol}_{n-1}(B_2^{n-1})\max_{x\in\partial K}f(x)}\right)^{\frac{n-2}{n-1}}.$$

By (73) the latter expression is greater than or equal to

$$c^n\frac{r^n\,(\min_{x\in\partial K}f(x))^{\frac{2}{n-1}}\,(\mathrm{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}{R^{n-1}\max_{x\in\partial K}f(x)}\ s^{\frac{n-3}{n-1}}$$

where c is an absolute constant. □

Lemma 4.12. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K}f\mathrm{d}\mu=1$. Assume that for all t with $0<t\leq T$ we have $K_t\subseteq\overset{\circ}{K}$. Let $x_s\in\partial K_s$ be given by the equation $\{x_s\}=[x_0,x_T]\cap\partial K_s$. Suppose that there are r,R with $0<r,R<\infty$ and*

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R).$$

Let the normals $N_{\partial K_s}(x_s)$ be such that

$$s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))).$$

Let s_0 be as in Lemma 4.10. Let β be such that $B_2^n(x_T,\beta)\subseteq K_{s_0}\subseteq K\subseteq B_2^n(x_T,\frac{1}{\beta})$ and let $H_s=H(x_s,N_{\partial K_s}(x_s))$. Then there are constants a and b with $0\leq a,b<1$ that depend only on r, R, and f such that we have for all N

$$N^{\frac{2}{n-1}}\int_0^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)}\frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}$$

$$\times\left(\frac{\|x_s-x_T\|}{\|x_0-x_T\|}\right)^n\frac{<x_0-x_T,N_{\partial K}(x_0)>}{<x_s-x_T,N_{\partial K_s}(x_s)>}\mathrm{d}s$$

$$\leq c_n\frac{R^{n-1}\max_{x\in\partial K}f(x)\left[(1-a)^{-\frac{2}{n-1}}+(1-b)^{-\frac{2}{n-1}}\right]}{\beta^2r^n\,(\min_{x\in\partial K}f(x))^{\frac{n+1}{n-1}}}$$

where c_n is a constant that depends only on the dimension n. The constants a and b are the same as in Lemma 4.8. They depend only on n, r, R and f.

Lemma 4.12 provides an uniform estimate. The constants do not depend on the boundary point x_0.

Proof. As in Lemma 4.10 Θ denotes the angle between the vectors $N_{\partial K}(x_0)$ and $x_0 - x_T$. Θ_s is the angle between the vectors $N_{\partial K_s}(x_s)$ and $x_s - x_T$ which is the same as the angle between $N_{\partial K_s}(x_s)$ and $x_0 - x_T$. Thus $< \frac{x_0 - x_T}{\|x_0 - x_T\|}, N_{\partial K}(x_0) >= \cos\Theta$ and $< \frac{x_s - x_T}{\|x_s - x_T\|}, N_{\partial K_s}(x_s) >= \cos\Theta_s$. By Lemma 2.3.(ii) K_s has volume strictly greater than 0 if we choose s small enough. Since $K_t \subseteq \overset{\circ}{K}$ the point x_T is an interior point of K. For small enough s_0 the set K_{s_0} has nonempty interior and therefore there is a $\beta > 0$ such that

$$B_2^n(x_T, \beta) \subseteq K_{s_0} \subseteq K \subseteq B_2^n(x_T, \tfrac{1}{\beta}).$$

Then for all s with $0 < s \leq s_0$

$$\beta^2 \leq \left\langle \frac{x_0 - x_T}{\|x_0 - x_T\|}, N_{\partial K}(x_0) \right\rangle \leq 1 \quad \text{and} \quad \beta^2 \leq \left\langle \frac{x_s - x_T}{\|x_s - x_T\|}, N_{\partial K_s}(x_s) \right\rangle \leq 1.$$

Thus

$$\frac{\|x_s - x_T\| < x_0 - x_T, N_{\partial K}(x_0) >}{\|x_0 - x_T\| < x_s - x_T, N_{\partial K_s}(x_s) >} \leq \frac{1}{\beta^2}.$$

As $\frac{\|x_s - x_T\|}{\|x_0 - x_T\|} \leq 1$,

$$\begin{aligned} N^{\frac{2}{n-1}} \int_0^{s_0} & \frac{\mathbb{P}_f^N \{(x_1, \dots, x_N) |\ x_s \notin [x_1, \dots, x_N]\}}{\int_{\partial(K \cap H_s)} \frac{f(y) \mathrm{d}\mu_{\partial(K \cap H_s)}(y)}{(1 - <N_{\partial K_s}(x_s), N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \\ & \times \left(\frac{\|x_s - x_T\|}{\|x_0 - x_T\|} \right)^n \frac{< x_0 - x_T, N_{\partial K}(x_0) >}{< x_s - x_T, N_{\partial K_s}(x_s) >} \mathrm{d}s \\ \leq N^{\frac{2}{n-1}} \frac{1}{\beta^2} & \int_0^{s_0} \frac{\mathbb{P}_f^N \{(x_1, \dots, x_N) |\ x_s \notin [x_1, \dots, x_N]\}}{\int_{\partial(K \cap H_s)} \frac{f(y) \mathrm{d}\mu_{\partial(K \cap H_s)}(y)}{(1 - <N_{\partial K_s}(x_s), N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}s. \end{aligned}$$

By Lemma 4.8 and Lemma 4.11 the last expression is less than

$$\begin{aligned} N^{\frac{2}{n-1}} & \frac{R^{n-1} \max_{x \in \partial K} f(x)}{\beta^2 c^n r^n \left(\min_{x \in \partial K} f(x)\right)^{\frac{2}{n-1}} \left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}} \\ & \times \int_0^{s_0} \left[2^n \left(a - as + s\right)^N + 2^n (1 - s + bs)^N \right] \ s^{-\frac{n-3}{n-1}} \mathrm{d}s. \end{aligned} \tag{74}$$

We estimate now the integral

$$\begin{aligned} \int_0^{s_0} & \left[2^n \left(a - as + s\right)^N + 2^n (1 - s + bs)^N \right] \ s^{-\frac{n-3}{n-1}} \mathrm{d}s \\ & = 2^n \int_0^{s_0} [1 - (1-a)(1-s)]^N s^{-\frac{n-3}{n-1}} + [1 - (1-b)s]^N s^{-\frac{n-3}{n-1}} \mathrm{d}s. \end{aligned}$$

For $s_0 \le \frac{1}{2}$ (we may assume this) we have $1-(1-a)(1-s) \le 1-(1-a)s$. Therefore the above expression is smaller than

$$\begin{aligned} 2^n \int_0^{s_0} & [1-(1-a)s]^N s^{-\frac{n-3}{n-1}} + [1-(1-b)s]^N s^{-\frac{n-3}{n-1}} \mathrm{d}s \\ &= 2^n (1-a)^{-\frac{2}{n-1}} \int_0^{(1-a)s_0} [1-s]^N s^{-\frac{n-3}{n-1}} \mathrm{d}s \\ &\quad + 2^n (1-b)^{-\frac{2}{n-1}} \int_0^{(1-b)s_0} [1-s]^N s^{-\frac{n-3}{n-1}} \mathrm{d}s. \end{aligned}$$

Since $s_0 \le \frac{1}{2}$ and $0 < a, b < 1$ the last expression is smaller than

$$2^n \left[(1-a)^{-\frac{2}{n-1}} + (1-b)^{-\frac{2}{n-1}}\right] B\left(N+1, \tfrac{2}{n-1}\right)$$

where B denotes the Beta function. We have

$$\lim_{x \to \infty} \frac{\Gamma(x+\alpha)}{\Gamma(x)} x^{-\alpha} = 1.$$

Thus

$$\begin{aligned} \lim_{N\to\infty} & B(N+1, \tfrac{2}{n-1})(N+1)^{\frac{2}{n-1}} \\ &= \lim_{N\to\infty} \frac{\Gamma(N+1)\Gamma(\frac{2}{n-1})}{\Gamma(N+1+\frac{2}{n-1})} (N+1)^{\frac{2}{n-1}} = \Gamma(\tfrac{2}{n-1}) \end{aligned}$$

and

$$B(N+1, \tfrac{2}{n-1}) \le 2^{2+\frac{2}{n-1}} \frac{\Gamma(\frac{2}{n-1})}{N^{\frac{2}{n-1}}}.$$

We get

$$\begin{aligned} \int_0^{s_0} & \left[2^n (a - as + s)^N + 2^n (1-s+bs)^N\right] \ s^{-\frac{n-3}{n-1}} ds \\ &\le 2^n \left[(1-a)^{-\frac{2}{n-1}} + (1-b)^{-\frac{2}{n-1}}\right] 2^{2+\frac{2}{n-1}} \frac{\Gamma(\frac{2}{n-1})}{N^{\frac{2}{n-1}}}. \end{aligned}$$

Therefore, by (74)

$$\begin{aligned} N^{\frac{2}{n-1}} \int_0^{s_0} & \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N) |\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)} \frac{f(y) \mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-(<N(x_s),N(y)>)^2)^{\frac{1}{2}}}} \\ &\times \left(\frac{\|x_s - x_T\|}{\|x_0 - x_T\|}\right)^n \frac{< x_0 - x_T, N_{\partial K}(x_0) >}{< x_s - x_T, N_{\partial K_s}(x_s) >} \mathrm{d}s \\ \le N^{\frac{2}{n-1}} & \frac{R^{n-1} \max_{x\in\partial K} f(x)}{\beta^2 c^n r^n \left(\min_{x\in\partial K} f(x)\right)^{\frac{2}{n-1}} \left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}} \\ & \qquad 2^n \left[(1-a)^{-\frac{2}{n-1}} + (1-b)^{-\frac{2}{n-1}}\right] 2^{2+\frac{2}{n-1}} \frac{\Gamma(\frac{2}{n-1})}{N^{\frac{2}{n-1}}}. \end{aligned}$$

With a new constant c_n that depends only on the dimension n the last expression is less than

$$c_n \frac{R^{n-1}\max_{x\in\partial K} f(x)\left[(1-a)^{-\frac{2}{n-1}}+(1-b)^{-\frac{2}{n-1}}\right]}{\beta^2 r^n\left(\min_{x\in\partial K} f(x)\right)^{\frac{2}{n-1}}}.$$

□

Lemma 4.13. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $f:\partial K\to\mathbb{R}$ be a strictly positive, continuous function with $\int_{\partial K} f \mathrm{d}\mu = 1$. Assume that for all t with $0<t\le T$ we have $K_t \subseteq \overset{\circ}{K}$. Let $x_s\in\partial K_s$ be given by the equation $\{x_s\}=[x_0,x_T]\cap\partial K_s$. Suppose that there are r,R with $0<r,R<\infty$ and*

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq K\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R).$$

Let the normals $N_{\partial K_s}(x_s)$ be such that

$$s=\mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))).$$

Let s_0 be as in Lemma 4.10. Then there are $c_1,c_2,c_3>0$, N_0, and u_0 such that we have for all $u>u_0$ and $N>N_0$

$$N^{\frac{2}{n-1}}\int_{\frac{u}{N}}^{T}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\mathrm{d}s\le c_1e^{-u}+c_2e^{-c_3N}$$

where $H_s=H(x_s,N_{\partial K_s}(x_s))$. The constants u_0, N_0, c_1, c_2 and c_3 depend only on n, r, R and f.

Proof. First we estimate the integral from s_0 to $\frac{u}{N}$. As in the proof of Lemma 4.12 we show

$$\begin{aligned}&N^{\frac{2}{n-1}}\int_{\frac{u}{N}}^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\mathrm{d}s\\ &\le N^{\frac{2}{n-1}}\frac{R^{n-1}\max_{x\in\partial K} f(x)}{\beta^2c^nr^n\left(\min_{x\in\partial K} f(x)\right)^{\frac{2}{n-1}}\left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}}\\ &\qquad 2^n\left[(1-a)^{-\frac{2}{n-1}}+(1-b)^{-\frac{2}{n-1}}\right]\int_{\frac{u}{N}}^{s_0}[1-s]^Ns^{-\frac{n-3}{n-1}}\mathrm{d}s.\end{aligned}$$

We estimate the integral

$$\int_{\frac{u}{N}}^{s_0}[1-s]^Ns^{-\frac{n-3}{n-1}}\mathrm{d}s\le\int_{\frac{u}{N}}^{s_0}e^{-sN}s^{-\frac{n-3}{n-1}}\mathrm{d}s=N^{-\frac{2}{n-1}}\int_{u}^{s_0N}e^{-s}s^{-\frac{n-3}{n-1}}\mathrm{d}s.$$

If we require that $u_0 \geq 1$ then the last expression is not greater than

$$N^{-\frac{2}{n-1}} \int_u^{s_0 N} \mathrm{e}^{-s} \mathrm{d}s \leq N^{-\frac{2}{n-1}} \int_u^{\infty} \mathrm{e}^{-s} \mathrm{d}s = N^{-\frac{2}{n-1}} \mathrm{e}^{-u}.$$

Thus

$$\begin{aligned} N^{\frac{2}{n-1}} \int_{\frac{u}{N}}^{s_0} & \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N) \mid x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K \cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K \cap H_s}(y)} \mathrm{d}s \\ \leq & \frac{R^{n-1} \max_{x\in\partial K} f(x)}{\beta^2 c^n r^n \left(\min_{x\in\partial K} f(x)\right)^{\frac{2}{n-1}} \left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{\frac{2}{n-1}}} \\ & \qquad \times 2^n \left[(1-a)^{-\frac{2}{n-1}} + (1-b)^{-\frac{2}{n-1}}\right] \mathrm{e}^{-u}. \end{aligned}$$

Now we estimate the integral from s_0 to T

$$N^{\frac{2}{n-1}} \int_{s_0}^{T} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N) \mid x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K \cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}} \mathrm{d}\mu_{\partial K \cap H_s}(y)} \mathrm{d}s.$$

The same arguments that we have used in the proof of Lemma 4.9 in order to show formula (62) give that the latter expression is less than

$$\frac{2^n N^{\frac{2}{n-1}} \mathrm{vol}_{n-1}(\partial K)(T-s_0)}{c_2 \min_{x\in\partial K} f(x)} \left(1 - \min_{x\in\partial K} f(x) c_1^{n-1} 2^{-n} \mathrm{vol}_{n-1}(\partial B_2^n)\right)^N$$

where c_1 is the distance between ∂K and ∂K_{s_0}. Choosing now new constants c_1 and c_2 finishes the proof. □

Lemma 4.14. *Let H be a hyperplane in $\mathbb{R}^n$ that contains 0. Then in both halfspaces there is a 2^n-tant i.e. there is a sequence of signs θ such that*

$$\{x | \forall i, 1 \leq i \leq n : sgn(x_i) = \theta_i\}.$$

Moreover, if H^+ is the halfspace that contains the above set then

$$H^+ \subset \bigcup_{i=1}^{n} \{x | sgn(x_i) = \theta_i\}.$$

The following lemma is an extension of a localization principle introduced by Bárány [Ba1] for random polytopes whose vertices are chosen from the inside of the convex body. The measure in that case is the normalized Lebesgue measure on the convex body.

For large numbers N of chosen points the probability that a point is an element of a random polytope is almost 1 provided that this point is not too close to the boundary. So it leaves us to compute the probability for those points that are in the vicinity of the boundary. The localization principle now says that in order to compute the probability that a point close to the boundary is contained in a random polytope it is enough to consider only those points that are in a small neighborhood of the point under consideration. As a neighborhood we choose a cap of the convex body.

The arguments are similar to the ones used in [Sch1].

Lemma 4.15. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder with a base that is an ellipsoid). Let $f : \partial K \to \mathbb{R}$ be a continuous, strictly positive function with $\int_{\partial K} f \mathrm{d}\mu_{\partial K} = 1$. Assume that for all t with $0 < t \leq T$ we have $K_t \subseteq \overset{\circ}{K}$. We define the point x_s by $\{x_s\} = [x_T, x_0] \cap \partial K_s$ and*

$$\Delta(s) =< N_{\partial K}(x_0), x_0 - x_s >$$

is the distance between the planes $H(x_0, N_{\partial K}(x_0))$ and $H(x_s, N_{\partial K}(x_0))$. Suppose that there are r, R with $0 < r, R < \infty$ and

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \subseteq K \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R).$$

Then, there is c_0 such that for all c with $c \geq c_0$ and b with $b > 2$ there is $s_{c,b} > 0$ such that we have for all s with $0 < s \leq s_{c,b}$ and for all $N \in \mathbb{N}$ with

$$N \geq \tfrac{1}{bs} \mathrm{vol}_{n-1}(\partial K)$$

that

$$\begin{aligned} \big|\mathbb{P}_f^N\{(x_1, \ldots, x_N)| \quad & x_s \notin [x_1, \ldots, x_N]\} - \\ & \mathbb{P}_f^N\{(x_1, \ldots, x_N)| \ x_s \notin [\{x_1, \ldots, x_N\} \cap H^-]\}\big| \\ & \quad \leq 2^{n-1} \exp(-\tfrac{c_1}{b}\sqrt{c}) \end{aligned}$$

where $H = H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ and $c_1 = c_1(n)$ is a constant that only depends on the dimension n.

In particular, for all $\epsilon > 0$ and all $k \in \mathbb{N}$ there is $N_0 \in \mathbb{N}$ such that we have for all $N \geq N_0$ and all $x_s \in [x_0, x_T]$

$$\begin{aligned} \big|\mathbb{P}_f^N\{(x_1, \ldots, x_N)| \quad & x_s \notin [x_1, \ldots, x_N]\} - \\ & \mathbb{P}_f^{N+k}\{(x_1, \ldots, x_{N+k})| \ x_s \notin [x_1, \ldots, x_{N+k}]\}\big| \leq \epsilon. \end{aligned}$$

The numbers $s_{c,b}$ may depend on the boundary points x_0.

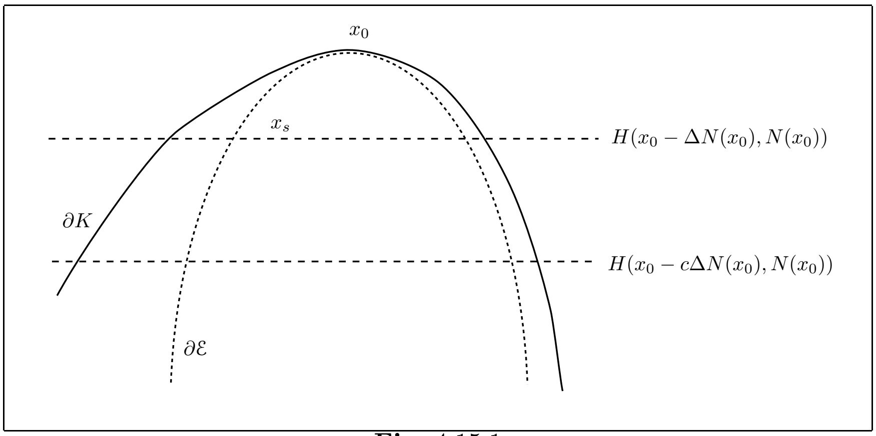

Fig. 4.15.1

Subsequently we apply Lemma 4.15 to a situation where b is already given and we choose c sufficiently big so that

$$2^{n-1} \exp(-\tfrac{c_1}{b}\sqrt{c})$$

is as small as we desire.

Proof. Let c and b be given. Since f is continuous for any given $\epsilon > 0$ we can choose $s_{c,b}$ so small that we have for all s with $0 < s \leq s_{c,b}$ and all $x \in \partial K \cap H^-(x_0 - c\Delta(s) N_{\partial K}(x_0), N_{\partial K}(x_0))$

$$|f(x) - f(x_0)| < \epsilon.$$

We may assume that $x_0 = 0$, $N_{\partial K}(x_0) = -e_n$. Let

$$\mathcal{E} = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^{n-1} \left| \frac{x_i}{a_i} \right|^2 + \left| \frac{x_n}{a_n} - 1 \right|^2 \leq 1 \right. \right\}$$

be the standard approximating ellipsoid at x_0 (see Lemma 1.2). Thus the principal axes are multiples of e_i, $i = 1, \dots, n$.

We define the operator $T_\eta : \mathbb{R}^n \to \mathbb{R}^n$

$$T_\eta(x_1, \dots, x_n) = (\eta x_1, \dots, \eta x_{n-1}, x_n).$$

By Lemma 1.2 for any $\epsilon > 0$ we may choose $s_{c,b}$ so small that we have

$$\begin{aligned} T_{1-\epsilon}(\mathcal{E} \cap H^-(x_0 - c\Delta(s_{c,b}) N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ \subseteq K \cap H^-(x_0 - c\Delta(s_{c,b}) N_{\partial K}(x_0), N_{\partial K}(x_0)) \qquad (75) \\ \subseteq T_{1+\epsilon}(\mathcal{E} \cap H^-(x_0 - c\Delta(s_{c,b}) N_{\partial K}(x_0), N_{\partial K}(x_0))). \end{aligned}$$

For s with $0 < s \leq s_{c,b}$ we denote the lengths of the principal axes of the $n-1$-dimensional ellipsoid

$$T_{1+\epsilon}(\mathcal{E}) \cap H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)))$$

by λ_i, $i = 1, \ldots, n-1$, so that the principal axes are $\lambda_i e_i$, $i = 1, \ldots, n-1$. We may assume (for technical reasons) that for all s with $0 < s \leq s_{c,b}$

$$x_0 - c\Delta(s)N_{\partial K}(x_0) \pm \lambda_i e_i \notin K \qquad i = 1, \ldots, n-1. \tag{76}$$

This is done by choosing (if necessary) a slightly bigger ϵ.

For any sequence $\Theta = (\Theta_i)_{i=1}^n$ of signs $\Theta_i = \pm 1$ we put

$$\begin{aligned} \mathrm{corn}_K(\Theta) = \partial K \cap H^+(x_s, N_{\partial K}(x_0)) \\ \cap \left\{ \bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \right\}. \end{aligned} \tag{77}$$

We have

$$\mathrm{corn}_K(\Theta) \subseteq H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)). \tag{78}$$

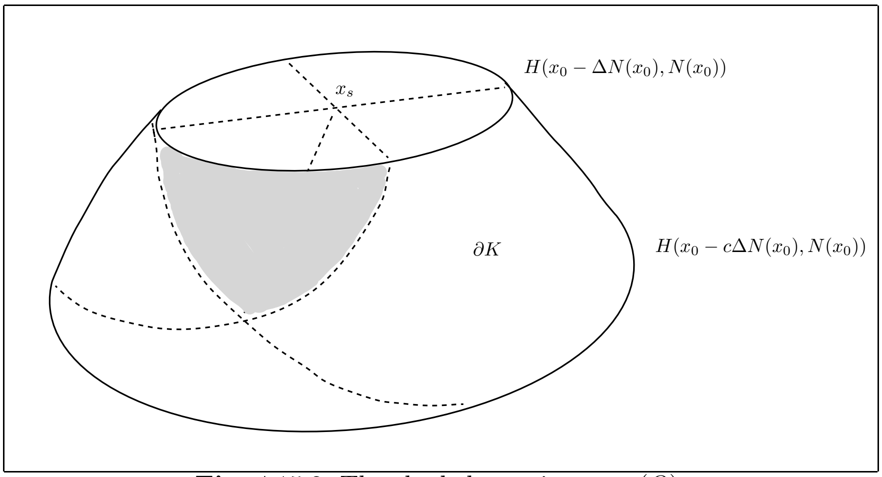

Fig. 4.15.2: The shaded area is $\mathrm{corn}_K(\Theta)$.

We refer to these sets as corner sets (see Figure 4.15.2). The hyperplanes

$$H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i)$$

$\Theta_i = \pm 1$ and $i = 1, \ldots, n-1$ are chosen in such a way that x_s and

$$x_0 + \Theta_i \lambda_i e_i + c\Delta(s)e_n = \Theta_i \lambda_i e_i + c\Delta(s)e_n$$

($x_0 = 0$) are elements of the hyperplanes. We check this. By definition x_s is an element of this hyperplane. We have

$$< x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i >$$
$$= (\Theta_i < x_s, e_i > -\lambda_i) < x_s, e_n > +\Theta_i(c-1)\Delta(s) < x_s, e_i > .$$

Since $N_{\partial K}(x_0) = -e_n$ we have $\Delta(s) =< x_s, e_n >$ and

$$\begin{aligned} &< x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i > \\ &\quad = (\Theta_i < x_s, e_i > -\lambda_i)\Delta(s) + \Theta_i(c-1)\Delta(s) < x_s, e_i > \\ &\quad = \Delta(s)\{(\Theta_i < x_s, e_i > -\lambda_i) + \Theta_i(c-1) < x_s, e_i >\} \\ &\quad = \Delta(s)\{-\lambda_i + \Theta_i c < x_s, e_i >\} \end{aligned}$$

and

$$\begin{aligned} &< \Theta_i\lambda_i e_i + c\Delta(s)e_n, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i > \\ &\quad = \lambda_i(c-1)\Delta(s) + c\Delta(s)(\Theta_i < x_s, e_i > -\lambda_i) \\ &\quad = -\lambda_i\Delta(s) + \Theta_i c\Delta(s) < x_s, e_i > . \end{aligned}$$

These two equalities show that for all i with $i = 1, \ldots, n-1$

$$\Theta_i\lambda_i e_i + c\Delta(s)e_n \in H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i).$$

We conclude that for all i with $i = 1, \ldots, n-1$ and all s, $0 < s \le s_{c,b}$,

$$\begin{aligned} &K \cap H^+(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ &\quad \cap H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) = \emptyset. \end{aligned} \tag{79}$$

We verify this. Since

$$x_0 + \Theta_i\lambda_i e_i + c\Delta(s)e_n \in H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i)$$

we have

$$\begin{aligned} &H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ &\quad \cap H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \\ &\qquad = \left\{ x_0 + \Theta_i\lambda_i e_i + c\Delta(s)e_n + \sum_{j \ne i,n} a_j e_j \,\middle|\, a_j \in \mathbb{R} \right\}. \end{aligned}$$

On the other hand, by (75)

$$\begin{aligned} &K \cap H^-(x_0 - c\Delta(s_{c,b})N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ &\quad \subseteq T_{1+\epsilon}(\mathcal{E} \cap H^-(x_0 - c\Delta(s_{c,b})N_{\partial K}(x_0), N_{\partial K}(x_0))) \end{aligned}$$

and by (76)

$$x_0 - c\Delta(s)N_{\partial K}(x_0) + \lambda_i e_i \notin K \qquad i = 1, \ldots, n-1.$$

From this we conclude that

$$H(x_0 - c\Delta(s)N_{\partial K}(x_0)) \\ \cap H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \cap K = \emptyset.$$

Using this fact and the convexity of K we deduce (78).

We want to show now that we have for all s with $0 < s \leq s_{c,b}$ and $H = H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$

$$\begin{aligned} &\{(x_1,\dots,x_N)|\ x_s \notin [\{x_1,\dots,x_N\} \cap H^-]\} \\ &\qquad \setminus\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\} \\ &= \{(x_1,\dots,x_N)|\ x_s \notin [\{x_1,\dots,x_N\} \cap H^-] \text{ and } x_s \in [x_1,\dots,x_N]\} \\ &\subseteq \bigcup_{\Theta} \{(x_1,\dots,x_N)|\ x_1,\dots,x_N \in \partial K \setminus \mathrm{corn}_K(\Theta)\}. \end{aligned} \tag{80}$$

In order to do this we show first that for $H = H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ we have

$$\begin{aligned} &\{(x_1,\dots,x_N)|\ x_s \notin [\{x_1,\dots,x_N\} \cap H^-] \text{ and } x_s \in [x_1,\dots,x_N]\} \\ &\subseteq \{(x_1,\dots,x_N)|\exists H_{x_s}, \text{hyperplane} : x_s \in H_{x_s}, H_{x_s}^- \cap K \cap H^+ \neq \emptyset \\ &\qquad \text{and } \{x_1,\dots,x_N\} \cap H^- \subseteq \overset{\circ}{H^+}_{x_s}\}. \end{aligned} \tag{81}$$

We show this now. We have $x_s \notin [\{x_1,\dots,x_N\} \cap H^-]$ and $x_s \in [x_1,\dots,x_N]$. We observe that there is $z \in K \cap \overset{\circ}{H^+} (x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ such that

$$[z, x_s] \cap [\{x_1,\dots,x_N\} \cap H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))] = \emptyset. \tag{82}$$

We verify this. Assume that $x_1 \dots, x_k \in H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ and $x_{k+1} \dots, x_N \in \overset{\circ}{H^+} (x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$. Since $x_s \in [x_1,\dots,x_N]$ there are nonnegative numbers a_i, $i = 1,\dots,N$, with $\sum_{i=1}^N a_i = 1$ and

$$x_s = \sum_{i=1}^N a_i x_i.$$

Since $x_s \notin [\{x_1,\dots,x_N\} \cap H^-]$ we have $\sum_{i=k+1}^N a_i > 0$ and since $x_s \in H^-(x_0 - \Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ we have $\sum_{i=1}^k a_i > 0$. Now we choose

$$y = \frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i} \qquad \text{and} \qquad z = \frac{\sum_{i=k+1}^N a_i x_i}{\sum_{i=k+1}^N a_i}.$$

Thus we have $y \in [x_1,\dots,x_k]$, $z \in [x_{k+1},\dots,x_N]$, and

$$x_s = \alpha y + (1-\alpha) z$$

where $\alpha = \sum_{i=1}^{k} a_i$.

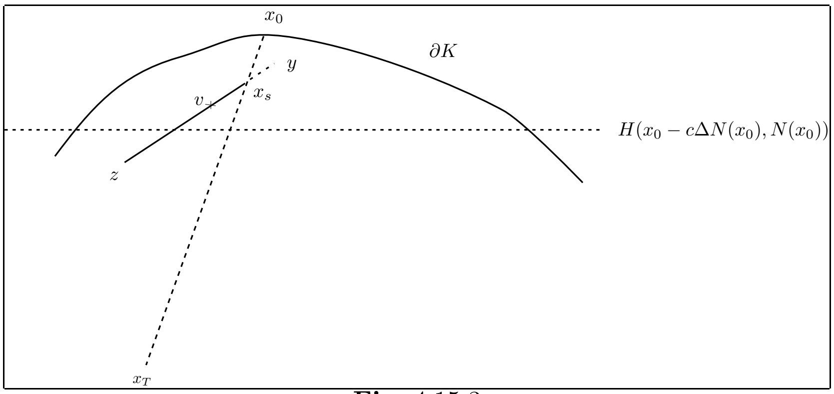

Fig. 4.15.3

We claim that $[z, x_s] \cap [x_1, \dots, x_k] = \emptyset$. Suppose this is not the case. Then there is $v \in [z, x_s]$ with $v \in [x_1, \dots, x_k]$. We have $v \neq z$ and $v \neq x_s$. Thus there is β with $0 < \beta < 1$ and $v = \beta z + (1-\beta) x_s$. Therefore we get

$$v = \beta z + (1-\beta)x_s = \beta(\tfrac{1}{1-\alpha} x_s - \tfrac{\alpha}{1-\alpha} y) + (1-\beta) x_s = \tfrac{1-\alpha+\alpha\beta}{1-\alpha} x_s - \tfrac{\alpha\beta}{1-\alpha} y$$

and thus

$$x_s = \tfrac{1-\alpha}{1-\alpha+\alpha\beta} v + \tfrac{\alpha\beta}{1-\alpha+\alpha\beta} y.$$

Thus x_s is a convex combination of y and v. Since $v \in [x_1, \dots, x_k]$ and $y \in [x_1, \dots, x_k]$ we conclude that $x_s \in [x_1, \dots, x_k]$ which is not true. Therefore we have reached a contradiction and

$$[z, x_s] \cap [x_1, \dots, x_k] = \emptyset.$$

We have verified (82).

Now we conclude that

$$\{x_s + t(z - x_s) |\ t \geq 0\} \cap [x_1, \dots, x_k] = \emptyset.$$

We have

$$\{x_s + t(z - x_s) |\ t \geq 0\} = [z, x_s] \cup \{x_s + t(z - x_s) |\ t > 1\}.$$

We know already that $[z, x_s]$ and $[x_1, \dots, x_k]$ are disjoint. On the other hand we have

$$\{x_s + t(z - x_s) |\ t > 1\} \subseteq \overset{\circ}{H^+} (x_0 - c\Delta(s) N_{\partial K}(x_0), N_{\partial K}(x_0)).$$

This is true since $x_s \in \overset{\circ}{H}^{-}(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ and

$$z \in \overset{\circ}{H}^{+}(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)). \tag{83}$$

Since $\{x_1, \ldots, x_k\} \subseteq H^{-}(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$ we conclude that the sets

$$\{x_s + t(z - x_s)|\ t > 1\} \qquad \text{and} \qquad [x_1, \ldots, x_k]$$

are disjoint. Now we apply the theorem of Hahn-Banach to the convex, closed set $\{x_s + t(z - x_s)|\ t \geq 0\}$ and the compact, convex set $[x_1, \ldots, x_k]$. There is a hyperplane H_{x_s} that separates these two sets strictly. We pass to a parallel hyperplane that separates these two sets and is a support hyperplane of $\{x_s + t(z - x_s)|\ t \geq 0\}$. Let us call this new hyperplane now H_{x_s}. We conclude that $x_s \in H_{x_s}$. We claim that H_{x_s} satisfies (81).

We denote the halfspace that contains z by $H^{-}_{x_s}$. Then

$$[x_1, \ldots, x_k] \subseteq \overset{\circ}{H}{}^{+}_{x_s} .$$

Thus we have $x_s \in H_{x_s}$, $H^{-}_{x_s} \cap K \cap H^{+}(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)) \supset \{z\} \neq \emptyset$, and

$$[x_1, \ldots, x_k] \subseteq \overset{\circ}{H}{}^{+}_{x_s} .$$

Therefore we have shown (81)

$$\begin{aligned} \{(x_1, \ldots, x_N)|\ x_s \notin [\{x_1, \ldots, x_N\} \cap H^{-}] \text{ and } x_s \in [x_1, \ldots, x_N]\} \\ \subseteq \{(x_1, \ldots, x_N)| \exists H_{x_s} : x_s \in H_{x_s}, H^{-}_{x_s} \cap K \cap H^{+} \neq \emptyset \\ \text{and } \{x_1, \ldots, x_N\} \cap H^{-} \subseteq \overset{\circ}{H}{}^{+}_{x_s}\} \end{aligned}$$

where $H = H(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))$. Now we show that

$$\begin{aligned} \{(x_1, \ldots, x_N)| \exists H_{x_s} : x_s \in H_{x_s}, H^{-}_{x_s} \cap K \cap H^{+} \neq \emptyset \\ \text{and } \{x_1, \ldots, x_N\} \cap H^{-} \subseteq \overset{\circ}{H}{}^{+}_{x_s}\} \\ \subseteq \bigcup_{\Theta} \{(x_1, \ldots, x_N)| x_1, \ldots, x_N \in \partial K \setminus \mathrm{corn}_K(\Theta)\} \end{aligned} \tag{84}$$

which together with (81) gives us (80).

We show that for every H_{x_s} with $x_s \in H_{x_s}$ and $H^{-}_{x_s} \cap K \cap H^{+} \neq \emptyset$ there is a sequence of signs Θ so that we have

$$\mathrm{corn}_K(\Theta) \subseteq H^{-}_{x_s} \qquad \text{and} \qquad \mathrm{corn}_K(-\Theta) \subseteq H^{+}_{x_s}. \tag{85}$$

This implies that for all sequences $(x_1, \ldots, x_N)$ that are elements of the left hand side set of (4.15.5) there is a Θ such that for all $k = 1, \ldots, N$

$$x_k \notin \mathrm{corn}_K(\Theta).$$

Indeed,

$$\begin{aligned} \{x_1, \dots, x_N\} \cap H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)) &\subseteq \overset{\circ}{H}{}^+_{x_s} \\ \mathrm{corn}_K(\Theta) \cap H^+(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)) &= \emptyset. \end{aligned}$$

This proves (84). We choose Θ so that (85) is fulfilled. We have for all $i = 1, \dots, n-1$

$$\begin{aligned} &H(x_s, N_{\partial K}(x_0)) \cap H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \\ &\quad = \{x \in \mathbb{R}^n | < x, e_n >=< x_s, e_n > \text{ and } < x - x_s, \Theta_i e_i > \geq 0\}. \end{aligned}$$

Indeed, $N_{\partial K}(x_0) = -e_n$ and

$$H(x_s, N_{\partial K}(x_0)) = \{x \in \mathbb{R}^n | < x, e_n >=< x_s, e_n >\}$$

and

$$\begin{aligned} &H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \\ &\quad = \{x \in \mathbb{R}^n | < x - x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i > \geq 0\}. \end{aligned}$$

On the intersection of the two sets we have $< x - x_s, e_n >= 0$ and thus

$$\begin{aligned} 0 &\leq < x - x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i > \\ &=< x - x_s, \Theta_i(c-1)\Delta(s)e_i > . \end{aligned}$$

Since $c-1$ and $\Delta(s)$ are positive we can divide and get

$$0 \leq < x - x_s, \Theta_i e_i > .$$

Therefore, the hyperplanes

$$H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \qquad i = 1, \dots, n-1$$

divide the hyperplane $H(x_s, N_{\partial K}(x_0))$ into 2^{n-1}-tants, i.e. 2^{n-1} sets of equal signs. x_s is considered as the origin in the hyperplane $H(x_s, N_{\partial K}(x_0))$. By Lemma 4.14 there is Θ such that

$$\begin{aligned} &H(x_s, N_{\partial K}(x_0)) \cap H^+_{x_s} \\ &\quad \supseteq H(x_s, N_{\partial K}(x_0)) \\ &\qquad \cap \left\{ \bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \right\} \end{aligned}$$

and

$$\begin{aligned}&H(x_s,N(x_0))\cap H^-_{x_s}\\&\qquad\supseteq H(x_s,N(x_0))\\&\qquad\qquad\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(-\Theta_i<x_s,e_i>-\lambda_i)e_n-\Theta_i(c-1)\Delta(s)e_i)\right\}.\end{aligned}$$

For a given H_{x_s} we choose this Θ and claim that

$$\mathrm{corn}_K(\Theta)\subseteq H^-_{x_s}. \tag{86}$$

Suppose this is not the case. We consider the hyperplane $\tilde{H}_{x_s}$ with

$$H_{x_s}\cap H(x_s,N_{\partial K}(x_0))=\tilde{H}_{x_s}\cap H(x_s,N_{\partial K}(x_0))$$

and

$$\bigcap_{i=1}^{n-1}H(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\subseteq\tilde{H}_{x_s}.$$

The set on the left hand side is a 1-dimensional affine space. We obtain $\tilde{H}_{x_s}$ from H_{x_s} by rotating H_{x_s} around the "axis" $H_{x_s}\cap H(x_s,N_{\partial K}(x_0))$. Then we have

$$H^+(x_s,N_{\partial K}(x_0))\cap H^-_{x_s}\subseteq H^+(x_s,N_{\partial K}(x_0))\cap\tilde{H}^-_{x_s}.$$

Indeed, from the procedure by which we obtain $\tilde{H}_{x_s}$ from H_{x_s} it follows that one set has to contain the other. Moreover, since $\mathrm{corn}_K(\Theta)\subseteq\tilde{H}^-_{x_s}$, but $\mathrm{corn}_K(\Theta)\not\subseteq H^-_{x_s}$ we verify the above inclusion. On the other hand, by our choice of Θ and by Lemma 4.14

$$\tilde{H}^-_{x_s}\subseteq\bigcup_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i).$$

By (76) none of the halfspaces

$$H^+(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\qquad i=1,\dots,n-1$$

contains an element of

$$K\cap H^+(x_0-c\Delta(s)N_{\partial K}(x_0),N_{\partial K}(x_0))$$

and therefore $H^-_{x_s}$ also does not contain such an element. But we know that H_{x_s} contains such an element by (83) giving a contradiction. Altogether we have shown (80) with $H=H(x_0-c\Delta(s)N_{\partial K}(x_0),N_{\partial K}(x_0))$

$$\begin{aligned}&\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-]\text{ and }x_s\in[x_1,\dots,x_N]\}\\&\qquad\subseteq\bigcup_{\Theta}\{(x_1,\dots,x_N)|\ x_1,\dots,x_N\in\partial K\setminus\mathrm{corn}_K(\Theta)\}.\end{aligned}$$

This gives us

$$
\begin{aligned}
&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-] \text{ and } x_s\in[x_1,\dots,x_N]\}\\
&\quad\le\sum_{\Theta}\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_1,\dots,x_N\in\partial K\setminus \mathrm{corn}_K(\Theta)\}\\
&\quad=\sum_{\Theta}\left(1-\int_{\mathrm{corn}_K(\Theta)}f(x)d\mu(x)\right)^N\\
&\quad\le\sum_{\Theta}\left(1-(f(x_0)-\epsilon)\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))\right)^N. \qquad (87)
\end{aligned}
$$

Now we establish an estimate for $\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))$. Let p be the orthogonal projection onto the hyperplane $H(x_0,N_{\partial K}(x_0))=H(0,-e_n)$. By the definition (77) of the set $\mathrm{corn}_K(\Theta)$

$$
\begin{aligned}
&p\Bigg(K\cap H^+(x_s,N_{\partial K}(x_0)))\\
&\qquad\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\right\}\Bigg)\\
&=p\Bigg(\partial\Bigg(K\cap H^+(x_s,N_{\partial K}(x_0)))\\
&\qquad\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\right\}\Bigg)\Bigg)\\
&\subseteq p(\mathrm{corn}_K(\Theta))\cup p(K\cap H(x_s,N_{\partial K}(x_0))). \qquad (88)
\end{aligned}
$$

This holds since $u\in H(x_0,N_{\partial K}(x_0))$ can only be the image of a point

$$
\begin{aligned}
&w\in\partial\Bigg(K\cap H^+(x_s,N_{\partial K}(x_0)))\\
&\qquad\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\right\}\Bigg)
\end{aligned}
$$

if $<N(w),N_{\partial K}(x_0)>=<N(w),-e_n>\ \ge 0$. This holds only for $w\in\mathrm{corn}_K(\Theta)$ or $w\in H(x_s,N_{\partial K}(x_0))\cap K$. Indeed, the other normals are

$$
-(\Theta_i<x_s,e_i>-\lambda_i)e_n-\Theta_i(c-1)\Delta(s)e_i \qquad i=1,\dots,n-1
$$

and for $i=1,\dots,n-1$

$$
<-(\Theta_i<x_s,e_i>-\lambda_i)e_n-\Theta_i(c-1)\Delta(s)e_i,-e_n>=\Theta_i<x_s,e_i>-\lambda_i.
$$

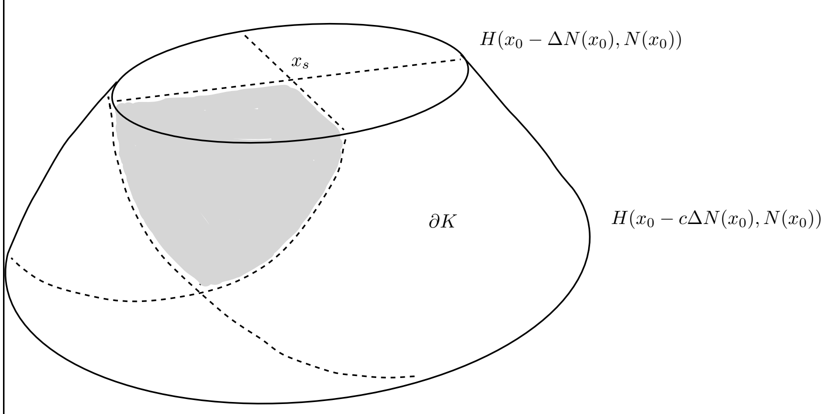

Fig. 4.15.4

By (76) we have for all $i = 1, \ldots, n-1$ that $| < x_s, e_i > | < \lambda_i$. This implies that $\Theta_i < x_s, e_i > -\lambda_i < 0$.

Since

$$\mathrm{vol}_{n-1}(p(\mathrm{corn}_K(\Theta))) \leq \mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))$$

and

$$\mathrm{vol}_{n-1}(p(K \cap H(x_s, N_{\partial K}(x_0)))) = \mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K}(x_0)))$$

we get from (88)

$$\begin{aligned} &\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))) \qquad (89) \\ &\quad \geq \mathrm{vol}_{n-1}\Bigg(p\Bigg(K \cap H^+(x_s, N_{\partial K}(x_0))) \\ &\qquad\qquad \cap \left\{ \bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \right\}\Bigg)\Bigg) \\ &\quad -\mathrm{vol}_{n-1}(K \cap H(x_s, N_{\partial K}(x_0))). \end{aligned}$$

Now we use that the indicatrix of Dupin at x_0 exists. Let $\mathcal{E}$ be the standard approximating ellipsoid (Lemma 1.2) whose principal axes have lengths a_i, $i = 1, \ldots, n$. By Lemma 1.2 and Lemma 1.3 for all $\epsilon > 0$ there is s_0 such that for all s with $0 < s \leq s_0$ the set

$$K \cap H(x_s, N_{\partial K}(x_0))$$

is contained in an $n-1$-dimensional ellipsoid whose principal axes have lengths less than

$$(1+\epsilon)a_i\sqrt{\frac{2\Delta(s)}{a_n}} \qquad\qquad i=1,\dots,n-1.$$

We choose $s_{c,b}$ to be smaller than this s_0. Therefore for all s with $0<s\le s_{c,b}$

$$\begin{aligned}&\mathrm{vol}_{n-1}(K\cap H(x_s,N_{\partial K}(x_0)))\\ &\qquad\qquad\le(1+\epsilon)^{n-1}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-1}{2}}\left(\prod_{i=1}^{n-1}a_i\right)\mathrm{vol}_{n-1}(B_2^{n-1}).\end{aligned}$$

Thus we deduce from (89)

$$\begin{aligned}&\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))) \\ &\ge\mathrm{vol}_{n-1}\Bigg(p\Bigg(K\cap H^+(x_s,N_{\partial K}(x_0))) \\ &\qquad\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\right\}\Bigg)\Bigg) \\ &\quad-(1+\epsilon)^{n-1}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-1}{2}}\left(\prod_{i=1}^{n-1}a_i\right)\mathrm{vol}_{n-1}(B_2^{n-1}).\end{aligned} \tag{90}$$

Now we get an estimate for the first summand of the right hand side. Since $\mathcal{E}$ is an approximating ellipsoid we have by Lemma 1.2 that for all $\epsilon>0$ there is s_0 such that we have for all s with $0<s\le s_0$

$$x_0-\Delta(s)N_{\partial K}(x_0)+(1-\epsilon)\Theta_i a_i\sqrt{\frac{2\Delta(s)}{a_n}}e_i\in K \qquad\qquad i=1,\dots,n-1.$$

Again, we choose $s_{c,b}$ to be smaller than this s_0.

Let θ be the angle between $N_{\partial K}(x_0)=-e_n$ and $x_0-x_T=-x_T$. Then

$$\|x_s\|=\Delta(s)(\cos\theta)^{-1}. \tag{91}$$

Consequently,

$$\|(x_0-\Delta(s)N_{\partial K}(x_0))-x_s\|=\Delta(s)\tan\theta.$$

Therefore, for all $\epsilon>0$ there is s_0 such that we have for all s with $0<s\le s_0$

$$x_s+(1-\epsilon)\Theta_i a_i\sqrt{\tfrac{2\Delta(s)}{a_n}}e_i\in K \qquad\qquad i=1,\dots,n-1.$$

Moreover, for $i=1,\dots,n-1$

$$x_s + (1-\epsilon)\Theta_i a_i \sqrt{\tfrac{2\Delta(s)}{a_n}} e_i \in K \cap H^+(x_s, N_{\partial K}(x_0)) \qquad (92)$$
$$\cap \left\{ \bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i) \right\}.$$

Indeed, by the above these points are elements of K. Since $N_{\partial K}(x_0) = -e_n$

$$x_s + (1-\epsilon)\Theta_i a_i \sqrt{\tfrac{2\Delta(s)}{a_n}} e_i \in K \cap H(x_s, N_{\partial K}(x_0)).$$

For $i \neq j$

$$\left\langle x_s + (1-\epsilon)\Theta_j a_j \sqrt{\tfrac{2\Delta(s)}{a_n}} e_j, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i \right\rangle$$
$$= \langle x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i \rangle$$

and for $i = j$

$$\left\langle x_s + (1-\epsilon)\Theta_i a_i \sqrt{\tfrac{2\Delta(s)}{a_n}} e_i, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i \right\rangle$$
$$= \langle x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i \rangle$$
$$+ (1-\epsilon)(c-1) a_i \sqrt{\tfrac{2\Delta(s)}{a_n}} \Delta(s).$$

Since the second summand is nonnegative we get for all j with $j = 1, \ldots, n-1$

$$x_s + (1-\epsilon)\Theta_j a_j \sqrt{\frac{2\Delta(s)}{a_n}} e_j \in$$
$$\bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i).$$

There is a unique point z in $H^+(x_s, N_{\partial K}(x_0))$ with

$$\{z\} = \partial K \cap \left\{ \bigcap_{i=1}^{n-1} H(x_s, (\Theta_i < x_s, e_i > -\lambda_i)e_n + \Theta_i (c-1)\Delta(s) e_i) \right\}. \qquad (93)$$

This holds since the intersection of the hyperplanes is 1-dimensional. We have that

$$\mathrm{vol}_{n-1}\left(\left[p(z), p(x_s) + \left((1-\epsilon)\Theta_1 a_1 \sqrt{\tfrac{2\Delta(s)}{a_n}} e_1 \right), \ldots, \right.\right.$$
$$\left.\left. p(x_s) + \left((1-\epsilon)\Theta_{n-1} a_{n-1} \sqrt{\tfrac{2\Delta(s)}{a_n}} e_{n-1} \right) \right] \right)$$

$$= \mathrm{vol}_{n-1}\left(p\left[z, x_s + \left((1-\epsilon)\Theta_1 a_1 \sqrt{\tfrac{2\Delta(s)}{a_n}} e_1\right), \dots, x_s + \left((1-\epsilon)\Theta_{n-1} a_{n-1} \sqrt{\tfrac{2\Delta(s)}{a_n}} e_{n-1}\right)\right]\right)$$

$$\leq \mathrm{vol}_{n-1}\Bigg(p\Bigg(K \cap H^+(x_s, N_{\partial K}(x_0)) \cap \Bigg\{\bigcap_{i=1}^{n-1} H^-(x_s, (\Theta_i < x_s, e_i > -\lambda_i) e_n + \Theta_i (c-1)\Delta(s) e_i)\Bigg\}\Bigg)\Bigg). \tag{94}$$

The $(n-1)$-dimensional volume of the simplex

$$\left[p(z), p(x_s) + \left((1-\epsilon)\Theta_1 a_1 \sqrt{\tfrac{2\Delta(s)}{a_n}} e_1\right), \dots, p(x_s) + \left((1-\epsilon)\Theta_{n-1} a_{n-1} \sqrt{\tfrac{2\Delta(s)}{a_n}} e_{n-1}\right)\right]$$

equals

$$\frac{d}{n-1} \mathrm{vol}_{n-2}\left(\left[(1-\epsilon) a_1 \sqrt{\tfrac{2\Delta(s)}{a_n}} e_1, \dots, (1-\epsilon) a_{n-1} \sqrt{\tfrac{2\Delta(s)}{a_n}} e_{n-1}\right]\right)$$

where d is the distance of $p(z)$ from the plane spanned by

$$p(x_s) + (1-\epsilon)\Theta_i a_i \sqrt{\frac{2\Delta(s)}{a_n}} e_i \qquad\qquad i = 1, \dots, n-1$$

in the space $\mathbb{R}^{n-1}$. We have

$$\begin{aligned} &\mathrm{vol}_{n-2}\left(\left[(1-\epsilon) a_1 \sqrt{\tfrac{2\Delta(s)}{a_n}} e_1, \dots, (1-\epsilon) a_{n-1} \sqrt{\tfrac{2\Delta(s)}{a_n}} e_{n-1}\right]\right) \\ &\quad = (1-\epsilon)^{n-2} \left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-2}{2}} \mathrm{vol}_{n-2}\left([a_1 e_1, \dots, a_{n-1} e_{n-1}]\right) \\ &\quad = \frac{1}{(n-2)!} (1-\epsilon)^{n-2} \left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-2}{2}} \prod_{i=1}^{n-1} a_i \left(\sum_{i=1}^{n-1} |a_i|^{-2}\right)^{\frac{1}{2}}. \end{aligned}$$

From this and (94)

$$\frac{d}{(n-1)!}(1-\epsilon)^{n-2}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-2}{2}}\prod_{i=1}^{n-1}a_i\left(\sum_{i=1}^{n-1}|a_i|^{-2}\right)^{\frac{1}{2}}$$
$$\le \mathrm{vol}_{n-1}\Bigg(p\Bigg(K\cap H^+(x_s,N_{\partial K}(x_0))$$
$$\cap\left\{\bigcap_{i=1}^{n-1}H^-(x_s,(\Theta_i<x_s,e_i>-\lambda_i)e_n+\Theta_i(c-1)\Delta(s)e_i)\right\}\Bigg)\Bigg).$$

From this inequality and (90)

$$\begin{aligned}&\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta)) \qquad (95)\\ &\ge \frac{d}{(n-1)!}(1-\epsilon)^{n-2}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-2}{2}}\prod_{i=1}^{n-1}a_i\left(\sum_{i=1}^{n-1}|a_i|^{-2}\right)^{\frac{1}{2}}\\ &\qquad -(1+\epsilon)^{n-1}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-1}{2}}\prod_{i=1}^{n-1}a_i\ \mathrm{vol}_{n-1}(B_2^{n-1}).\end{aligned}$$

We claim that there is a constant c_2 that depends only on K (and not on s and c) such that we have for all c and s with $0<s\le s_{c,b}$

$$d\ge c_2\sqrt{c\Delta(s)}. \qquad (96)$$

d equals the distance of $p(z)$ from the hyperplane that passes through 0 and that is parallel to the one spanned by

$$p(x_s)+(1-\epsilon)\Theta_i a_i\sqrt{\tfrac{2\Delta(s)}{a_n}}e_i \qquad i=1,\dots,n-1$$

in $\mathbb{R}^{n-1}$ minus the distance of 0 to the hyperplane spanned by

$$p(x_s)+(1-\epsilon)\Theta_i a_i\sqrt{\tfrac{2\Delta(s)}{a_n}}e_i \qquad i=1,\dots,n-1.$$

Clearly, the last quantity is smaller than

$$\|p(x_s)\|+\sqrt{\tfrac{2\Delta(s)}{a_n}}\max_{1\le i\le n-1}a_i$$

which can be estimated by (91)

$$\begin{aligned}\|p(x_s)\|+\sqrt{\tfrac{2\Delta(s)}{a_n}}\max_{1\le i\le n-1}a_i &\le \|x_s\|+\sqrt{\tfrac{2\Delta(s)}{a_n}}\max_{1\le i\le n-1}a_i\\ &=\Delta(s)(\cos\theta)^{-1}+\sqrt{\tfrac{2\Delta(s)}{a_n}}\max_{1\le i\le n-1}a_i.\end{aligned}$$

It is left to show that the distance of $p(z)$ to the hyperplane that passes through 0 and that is parallel to the one spanned by

$$p(x_s) + (1-\epsilon)\Theta_i a_i \sqrt{\tfrac{2\Delta(s)}{a_n}} e_i \qquad\qquad i = 1, \dots, n-1$$

is greater than a constant times $\sqrt{c\Delta(s)}$. Indeed, there is c_0 such that for all c with $c > c_0$ the distance d is of the order $\sqrt{c\Delta(s)}$.

Since z is an element of all hyperplanes

$$H(x_s, (\Theta_i x_s(i) - \lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \qquad\qquad i = 1, \dots, n-1$$

we have for all $i = 1, \dots, n-1$

$$< z - x_s, (\Theta_i x_s(i) - \lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i >= 0$$

which implies that we have for all $i = 1, \dots, n-1$

$$z(i) - x_s(i) = (z(n) - x_s(n)) \frac{\lambda_i - \Theta_i x_s(i)}{\Theta_i(c-1)\Delta(s)}. \tag{97}$$

Instead of z we consider $\tilde{z}$ given by

$$\{\tilde{z}\} = \partial T_{1-\epsilon}(\mathcal{E}) \cap \left\{ \bigcap_{i=1}^{n-1} H(x_s, (\Theta_i x_s(i) - \lambda_i)e_n + \Theta_i(c-1)\Delta(s)e_i) \right\}. \tag{98}$$

We also have

$$\tilde{z}(i) - x_s(i) = (\tilde{z}(n) - x_s(n)) \frac{\lambda_i - \Theta_i x_s(i)}{\Theta_i(c-1)\Delta(s)}. \tag{99}$$

By (75)

$$\begin{aligned} &T_{1-\epsilon}(\mathcal{E} \cap H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0))) \\ &\qquad\qquad \subseteq K \cap H^-(x_0 - c\Delta(s)N_{\partial K}(x_0), N_{\partial K}(x_0)). \end{aligned}$$

Therefore we have for all $i = 1, \dots, n$ that $|\tilde{z}(i)| \leq |z(i)|$. We will show that we have for all $i = 1, \dots, n-1$ that $c_3\sqrt{c\Delta(s)} \leq |\tilde{z}(i)|$. (We need this estimate for one coordinate only, but get it for all $i = 1, \dots, n-1$. $\tilde{z}(n)$ is of the order $\Delta(s)$.)

We have

$$1 = \sum_{i=1}^{n-1} \left| \frac{\tilde{z}(i)}{a_i(1-\epsilon)} \right|^2 + \left| \frac{\tilde{z}(n)}{a_n} - 1 \right|^2$$

and equivalently

$$\begin{aligned} 2\frac{\tilde{z}(n)}{a_n} &= \sum_{i=1}^{n-1} \left| \frac{\tilde{z}(i)}{a_i(1-\epsilon)} \right|^2 + \left| \frac{\tilde{z}(n)}{a_n} \right|^2 \\ &= \sum_{i=1}^{n-1} \left| \frac{\tilde{z}(i) - x_s(i) + x_s(i)}{a_i(1-\epsilon)} \right|^2 + \left| \frac{\tilde{z}(n) - x_s(n) + x_s(n)}{a_n} \right|^2. \end{aligned}$$

By triangle-inequality

$$\sqrt{2\frac{\tilde{z}(n)}{a_n}} - \sqrt{\sum_{i=1}^{n-1}\left|\frac{x_s(i)}{a_i(1-\epsilon)}\right|^2 + \left|\frac{x_s(n)}{a_n}\right|^2} \leq \sqrt{\sum_{i=1}^{n-1}\left|\frac{\tilde{z}(i)-x_s(i)}{a_i(1-\epsilon)}\right|^2 + \left|\frac{\tilde{z}(n)-x_s(n)}{a_n}\right|^2}.$$

By (99)

$$\sqrt{2\frac{\tilde{z}(n)}{a_n}} - \sqrt{\sum_{i=1}^{n-1}\left|\frac{x_s(i)}{a_i(1-\epsilon)}\right|^2 + \left|\frac{x_s(n)}{a_n}\right|^2} \leq |\tilde{z}(n)-x_s(n)|\sqrt{\sum_{i=1}^{n-1}\left|\frac{\lambda_i - \Theta_i x_s(i)}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2 + \left|\frac{1}{a_n}\right|^2}.$$

Since $\tilde{z} \in H^+(x_s, N_{\partial K}(x_0))$ we have $\tilde{z}(n) \geq \Delta(s)$. By (91) we have for all $i = 1, \dots, n$ that $|x_s(i)| \leq \|x_s\| \leq \Delta(s)(\cos\theta)^{-1}$. Therefore, for small enough s

$$\sqrt{\frac{\tilde{z}(n)}{a_n}} \leq |\tilde{z}(n)-x_s(n)|\sqrt{\sum_{i=1}^{n-1}\left|\frac{\lambda_i - \Theta_i x_s(i)}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2 + \left|\frac{1}{a_n}\right|^2}.$$

Since $\tilde{z}(n) \geq x_s(n) \geq 0$

$$\frac{1}{a_n} \leq |\tilde{z}(n)-x_s(n)|\left(\sum_{i=1}^{n-1}\left|\frac{\Theta_i < x_s, e_i > -\lambda_i}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2 + \frac{1}{a_n}\right).$$

For sufficiently small s we have $|\tilde{z}(n)-x_s(n)| \leq \frac{1}{2}$ and therefore

$$\frac{1}{2a_n} \leq |\tilde{z}(n)-x_s(n)|\sum_{i=1}^{n-1}\left|\frac{\Theta_i < x_s, e_i > -\lambda_i}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2$$

and

$$\begin{aligned}\sqrt{\frac{1}{2a_n}} &\leq \sqrt{\tilde{z}(n)-x_s(n)}\left(\sum_{i=1}^{n-1}\left|\frac{\Theta_i < x_s, e_i > -\lambda_i}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{\tilde{z}(n)-x_s(n)}\left\{\left(\sum_{i=1}^{n-1}\left|\frac{\Theta_i < x_s, e_i >}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2\right)^{\frac{1}{2}} \right. \\ &\qquad \left. + \left(\sum_{i=1}^{n-1}\left|\frac{\lambda_i}{(c-1)\Delta(s)a_i(1-\epsilon)}\right|^2\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

Therefore

$$\sqrt{\frac{1}{2a_n}} \le \frac{\sqrt{|\tilde{z}(n)-x_s(n)|}}{(c-1)(1-\epsilon)\Delta(s)}\left\{\left(\sum_{i=1}^{n-1}\left|\frac{x_s(i)}{a_i}\right|^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{n-1}\left|\frac{\lambda_i}{a_i}\right|^2\right)^{\frac{1}{2}}\right\}$$

$$\le \frac{\sqrt{|\tilde{z}(n)-x_s(n)|}}{(c-1)(1-\epsilon)\Delta(s)}\left\{\frac{\left(\sum_{i=1}^{n-1}|x_s(i)|^2\right)^{\frac{1}{2}}}{\min_{1\le i\le n-1}a_i}+\left(\sum_{i=1}^{n-1}\left|\frac{\lambda_i}{a_i}\right|^2\right)^{\frac{1}{2}}\right\}.$$

By (91) we have $\|x_s\| = \Delta(s)(\cos\theta)^{-1}$. From the definition of λ_i, $i = 1,\dots,n-1$, (following formula (75)) and Lemma 1.3 we get $\lambda_i \le (1+\epsilon)a_i\sqrt{\frac{c\Delta(s)}{a_n}}$. Therefore we get

$$\sqrt{\frac{1}{2a_n}} \le \frac{\sqrt{|\tilde{z}(n)-x_s(n)|}}{(c-1)(1-\epsilon)\Delta(s)}\left\{\frac{\Delta(s)(\cos\theta)^{-1}}{\min_{1\le i\le n-1}a_i}+(1+\epsilon)\sqrt{(n-1)\frac{c\Delta(s)}{a_n}}\right\}.$$

Thus there is a constant c_3 such that for all c with $c \ge 2$ and s with $0 < s \le s_{c,b}$

$$\frac{1}{a_n} \le \frac{c_3}{c\Delta(s)}|\tilde{z}(n)-x_s(n)|.$$

By this inequality and (99)

$$|\tilde{z}(i)-x_s(i)| = |\tilde{z}(n)-x_s(n)|\frac{|\Theta_i < x_s,e_i > -\lambda_i|}{(c-1)\Delta(s)} \ge c_4|\Theta_i < x_s,e_i > -\lambda_i|.$$

By (91) we have $\|x_s\| = \Delta(s)(\cos\theta)^{-1}$ and from the definition of λ_i, $i = 1,\dots,n-1$, we get $\lambda_i \ge (1-\epsilon)a_i\sqrt{\frac{c\Delta(s)}{a_n}}$. Therefore $\tilde{z}(i)$ is of the order of λ_i which is in turn of the order of $\sqrt{c\Delta(s)}$.

The orthogonal projection p maps $(z_1,\dots,z_n)$ onto $(z_1,\dots,z_{n-1},0)$. The distance d of $p(z)$ to the $n-2$-dimensional hyperplane that passes through 0 and that is parallel to the one spanned by

$$p(x_s) + (1-\epsilon)a_i\sqrt{\frac{2\Delta(s)}{a_n}}e_i \qquad\qquad i = 1,\dots,n-1$$

equals $| < p(z),\xi > |$ where ξ is the normal to this plane. We have

$$\xi = \left(\frac{\frac{1}{a_i}}{\left(\sum_{i=1}^{n-1}a_i^{-2}\right)^{\frac{1}{2}}}\right)_{i=1}^{n-1}$$

and get $| < p(z),\xi > | \ge c_4\sqrt{c\Delta(s)}$. Thus we have proved (96). By (95) and (96) there is a constant c_0 such that for all c with $c \ge c_0$

$$
\begin{aligned}
&\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta)) \\
&\quad\geq \frac{c_4\sqrt{c\Delta(s)}}{(n-1)!}(1-\epsilon)^{n-2}\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-2}{2}}\prod_{i=1}^{n-1}a_i\left(\sum_{i=1}^{n-1}|a_i|^{-2}\right)^{\frac{1}{2}} \\
&\qquad\qquad -(1+\epsilon)^{n-1}\mathrm{vol}_{n-1}(B_2^{n-1})\left(\frac{2\Delta(s)}{a_n}\right)^{\frac{n-1}{2}}\prod_{i=1}^{n-1}a_i \\
&\quad\geq c_5\sqrt{c}\Delta(s)^{\frac{n-1}{2}}
\end{aligned}
$$

where c_5 depends only on K. Finally, by the latter inequality and by (87)

$$
\begin{aligned}
&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-] \text{ and } x_s\in[x_1,\dots,x_N]\} \\
&\qquad\leq \sum_{\Theta}\Big(1-(f(x_0)-\epsilon)\mathrm{vol}_{n-1}(\mathrm{corn}_K(\Theta))\Big)^N \\
&\qquad\leq 2^{n-1}\Big(1-(f(x_0)-\epsilon)c_5\sqrt{c}\Delta(s)^{\frac{n-1}{2}}\Big)^N \\
&\qquad\leq 2^{n-1}\exp\left(-N(f(x_0)-\epsilon)c_5\sqrt{c}\Delta(s)^{\frac{n-1}{2}}\right).
\end{aligned}
$$

By hypothesis we have $\frac{1}{bN}\mathrm{vol}_{n-1}(\partial K)\leq s$. We have

$$
\begin{aligned}
s&\leq \mathbb{P}_f(\partial K\cap H^-(x_s,N_{\partial K}(x_0))) \\
&\leq (f(x_0)+\epsilon)\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K}(x_0))).
\end{aligned}
$$

By Lemma 1.3 we get

$$s\leq c_6 f(x_0)\Delta(s)^{\frac{n-1}{2}}$$

and therefore

$$\frac{N}{\mathrm{vol}_{n-1}(\partial K)}\geq\frac{1}{bs}\geq\frac{1}{c_6 b f(x_0)\Delta(s)^{\frac{n-1}{2}}}.$$

Therefore

$$
\begin{aligned}
&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-] \text{ and } x_s\in[x_1,\dots,x_N]\} \\
&\qquad\qquad\leq 2^{n-1}\exp\left(-c_7\frac{\sqrt{c}}{b}\right).
\end{aligned}
$$

Now we derive

$$
\begin{aligned}
\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ &\ x_s\notin[x_1,\dots,x_N]\}- \\
&\mathbb{P}_f^{N+k}\{(x_1,\dots,x_{N+k})|\ x_s\notin[x_1,\dots,x_{N+k}]\}\big|\leq\epsilon.
\end{aligned}
$$

It is enough to show

$$
\begin{aligned}
&\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\in[\{x_1,\dots,x_N\}\cap H^-]\}- \\
&\qquad\mathbb{P}_f^{N+k}\{(x_1,\dots,x_{N+k})|\ x_s\in[\{x_1,\dots,x_{N+k}\}\cap H^-]\}\big|\leq\epsilon.
\end{aligned}
$$

We have

$$\begin{aligned}\{(x_1,\dots,x_{N+k})|\ x_s\in[\{x_1,\dots,x_{N+k}\}\cap H^-]\}\\ =\{(x_1,\dots,x_{N+k})|\ x_s\in[\{x_1,\dots,x_N\}\cap H^-]\}\\ \cup\{(x_1,\dots,x_{N+k})|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-]\text{ and}\\ x_s\in[\{x_1,\dots,x_{N+k}\}\cap H^-]\}.\end{aligned}$$

Clearly, the above set is contained in

$$\begin{aligned}\{(x_1,\dots,x_{N+k})|\ x_s\in[\{x_1,\dots,x_N\}\cap H^-]\}\\ \cup\{(x_1,\dots,x_{N+k})|\exists i,1\le i\le k: x_{N+i}\in H^-\cap\partial K\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\mathbb{P}_f^{N+k}\{(x_1,\dots,x_{N+k})|\ x_s\in[\{x_1,\dots,x_{N+k}\}\cap H^-]\}\\ &\le\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\in[\{x_1,\dots,x_N\}\cap H^-]\}\\ &\qquad+\mathbb{P}_f^k\{(x_{N+1},\dots,x_{N+k})|\exists i,1\le i\le k: x_{N+i}\in H^-\cap\partial K\}\\ &=\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\in[\{x_1,\dots,x_N\}\cap H^-]\}\\ &\qquad+k\int_{\partial K\cap H^-}f(x)\mathrm{d}\mu.\end{aligned}$$

We choose H so that $k\int_{\partial K\cap H^-}f(x)\mathrm{d}\mu$ is sufficiently small. □

Lemma 4.16. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Let $\mathcal{E}$ be the standard approximating ellipsoid at x_0. Let $f:\partial K\to\mathbb{R}$ be a continuous, strictly positive function with $\int_{\partial K}f\mathrm{d}\mu=1$ and K_s be the surface body with respect to the measure $f\mathrm{d}\mu_{\partial K}$ and $\mathcal{E}_s$ the surface body with respect to the measure with the constant density $(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$ on $\partial\mathcal{E}$. Suppose that the indicatrix of Dupin at x_0 exists and is an ellipsoid (and not a cylinder with an ellipsoid as base). We define x_s, y_s and z_s by*

$$\{x_s\}=[x_0,x_T]\cap\partial K_s\qquad\{z_s\}=[x_0,z_T]\cap\partial\mathcal{E}_s$$

$$\{y_s\}=[x_0,x_T]\cap H(z_s,N_{\partial K}(x_0)).$$

(i) For every $\epsilon>0$ and all $\ell\in\mathbb{N}$ there are $c_0>1$ and $s_0>0$ so that we have for all $k\in\mathbb{N}$ with $1\le k\le\ell$, all s and all c with $0<cs<s_0$ and $c_0\le c$, and all hyperplanes H that are orthogonal to $N_{\partial K}(x_0)$ and that satisfy $\mathrm{vol}_{n-1}(\partial K\cap H^-)=cs$

$$\Big|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}-\mathbb{P}^k_{\partial K\cap H^-}\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]\}\Big|<\epsilon$$

where $\mathbb{P}_{f,\partial K\cap H^-}$ *is the normalized restriction of the measure* $\mathbb{P}_f$ *to the set* $\partial K\cap H^-$.

(ii) For every $\epsilon>0$ *and all* $\ell\in\mathbb{N}$ *there are* $c_0>1$ *and* $s_0>0$ *so that we have for all* $k\in\mathbb{N}$ *with* $1\le k\le\ell$, *all* s *and all* c *with* $0<cs<s_0$ *and* $c_0\le c$, *and all hyperplanes* H *that are orthogonal to* $N_{\partial K}(x_0)$ *and that satisfy* $\mathrm{vol}_{n-1}(\partial K\cap H^-)=cs$

$$\left|\mathbb{P}^k_{\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}-\right.$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ x_s\in[z_1,\dots,z_k]\}\right|<\epsilon.$$

(iii) For every $\epsilon>0$ *and all* $\ell\in\mathbb{N}$ *there are* $c_0>1$ *and* $s_0>0$ *so that we have for all* $k\in\mathbb{N}$ *with* $1\le k\le\ell$, *all* s *and all* c *with* $0<cs<s_0$ *and* $c_0\le c$, *and all hyperplanes* H *that are orthogonal to* $N_{\partial K}(x_0)$ *and that satisfy* $\mathrm{vol}_{n-1}(\partial K\cap H^-)=cs$

$$\left|\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\quad z_s\in[z_1,\dots,z_k]\}-\right.$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ y_s\in[z_1,\dots,z_k]\}\right|<\epsilon.$$

(iv) For every $\epsilon>0$ *and all* $\ell\in\mathbb{N}$ *there are* $c_0>1$, $s_0>0$, *and* $\delta>0$ *so that we have for all* $k\in\mathbb{N}$ *with* $1\le k\le\ell$, *all* s,s' *and all* c *with* $0<cs,cs'<s_0$, $(1-\delta)s\le s'\le(1+\delta)s$, *and* $c_0\le c$, *and all hyperplanes* H_s *that are orthogonal to* $N_{\partial\mathcal{E}}(x_0)$ *and that satisfy* $\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H_s^-)=cs$

$$\left|\mathbb{P}^k_{\partial\mathcal{E}\cap H_s^-}\{(z_1,\dots,z_k)|\quad z_s\in[z_1,\dots,z_k]\}-\right.$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap H_{s'}^-}\{(z_1,\dots,z_k)|\ z_{s'}\in[z_1,\dots,z_k]\}\right|<\epsilon.$$

(v) For every $\epsilon>0$ *and all* $\ell\in\mathbb{N}$ *there are* $c_0>1$ *and* $\Delta_0>0$ *so that we have for all* $k\in\mathbb{N}$ *with* $1\le k\le\ell$, *all* Δ, *all* $\gamma\ge1$ *and all* c *with* $0<c\gamma\Delta<\Delta_0$ *and* $c_0\le c$, *and*

$$\left|\mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ x_0-\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\}-\right.$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\gamma\Delta}}\{(x_1,\dots,x_k)|\ x_0-\gamma\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\}\right|<\epsilon$$

where $H_{c\Delta}=H_{c\Delta}(x_0-c\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))$.

(vi) For every $\epsilon>0$ *and all* $\ell\in\mathbb{N}$ *there are* $c_0>1$ *and* $s_0>0$ *so that we have for all* $k\in\mathbb{N}$ *with* $1\le k\le\ell$, *all* s *with* $0<cs<s_0$, *all* c *with* $c_0\le c$, *and all hyperplanes* H *and* $\tilde H$ *that are orthogonal to* $N_{\partial K}(x_0)$ *and that satisfy*

$$\mathbb{P}_f(\partial K\cap H^-)=cs\qquad\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap\tilde H^-)}{\mathrm{vol}_{n-1}(\partial\mathcal{E})}=cs$$

that

$$\left|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}-\right.$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde H^-}\{(z_1,\dots,z_k)|\ z_s\in[z_1,\dots,z_k]\}\right|<\epsilon.$$

(The hyperplanes H and $\tilde{H}$ may not be very close, depending on the value $f(x_0)$.)

Proof. (i) This is much simpler than the other cases. We define $\Phi_{x_s}:\partial K\times\cdots\times\partial K\to\mathbb{R}$ by

$$\Phi_{x_s}(x_1,\dots,x_k)=\begin{cases}0 & x_s\notin[x_1,\dots,x_k]\\ 1 & x_s\in[x_1,\dots,x_k].\end{cases}$$

Then we have

$$\begin{aligned}&\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]\}\\ &=(\mathbb{P}_f(\partial K\cap H^-))^{-k}\times\\ &\quad\int_{\partial K\cap H^-}\cdots\int_{\partial K\cap H^-}\Phi_{x_s}(x_1,\dots,x_k)\prod_{i=1}^k f(x_i)\mathrm{d}\mu_{\partial K}(x_1)\cdots\mathrm{d}\mu_{\partial K}(x_k).\end{aligned}$$

By continuity of f for every $\delta>0$ we find s_0 so small that we have for all s with $0<s\le s_0$ and all $x\in\partial K\cap H^-(x_s,N_{\partial K}(x_0))$

$$|f(x_0)-f(x)|<\delta.$$

(ii) We may suppose that $x_0=0$ and that e_n is orthogonal to K at x_0. Let $T_s:\mathbb{R}^n\to\mathbb{R}^n$ be given by

$$T_s(x(1),\dots,x(n))=(sx(1),\dots,sx(n-1),x(n)). \tag{100}$$

Then, by Lemma 1.2, for every $\delta>0$ there is a hyperplane H orthogonal to e_n such that for

$$\mathcal{E}_1=T_{\frac{1}{1+\delta}}(\mathcal{E})\qquad \mathcal{E}_2=T_{1+\delta}(\mathcal{E})$$

we have

$$\mathcal{E}_1\cap H^-=T_{\frac{1}{1+\delta}}(\mathcal{E})\cap H^-\subseteq K\cap H^-\subseteq T_{1+\delta}(\mathcal{E})\cap H^-=\mathcal{E}_2\cap H^-.$$

Since the indicatrix of Dupin at x_0 is an ellipsoid and not a cylinder and since f is continuous with $f(x_0)>0$ we conclude that there is s_0 such that

$$T_{\frac{1}{1+\delta}}(\mathcal{E})\cap H^-(x_{s_0},N_{\partial K}(x_0))\subseteq K\cap H^-\subseteq T_{1+\delta}(\mathcal{E})\cap H^-(x_{s_0},N_{\partial K}(x_0)). \tag{101}$$

We have that

$$\begin{aligned}\mathbb{P}^k_{\partial K\cap H^-}\{(x_1,\dots,x_k)|\ &x_s\in[x_1,\dots,x_k]\}\\ &=\mathbb{P}^k_{\partial K\cap H^-}\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]^\circ\}.\end{aligned}$$

This follows from Lemma 4.2. Therefore it is enough to verify the claim for this set. The set

$$\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]^\circ, x_1,\dots,x_k\in\partial K\cap H^-\}$$

is an open subset of the k-fold product $(\partial K\cap H^-)\times\cdots\times(\partial K\cap H^-)$. Indeed, since x_s is in the interior of the polytope $[x_1,\dots,x_k]$ we may move the vertices slightly and x_s is still in the interior of the polytope.

Therefore this set is an intersection of $(\partial K\cap H^-)\times\cdots\times(\partial K\cap H^-)$ with an open subset $\mathcal{O}$ of $\mathbb{R}^{kn}$. Such a set $\mathcal{O}$ can be written as the countable union of cubes whose pairwise intersections have measure 0. Cubes are sets $B^n_\infty(x_0,r)=\{x|\max_i|x(i)-x_0(i)|\le r\}$. Thus there are cubes $B^n_\infty(x^j_i,r^j_i)$, $1\le i\le k$, $j\in\mathbb{N}$, in $\mathbb{R}^n$ such that

$$\mathcal{O}=\bigcup_{j=1}^{\infty}\prod_{i=1}^{k}B^n_\infty(x^j_i,r^j_i) \tag{102}$$

and for $j\neq m$

$$\begin{aligned}\mathrm{vol}_{kn}&\left(\prod_{i=1}^{k}B^n_\infty(x^j_i,r^j_i)\cap\prod_{i=1}^{k}B^n_\infty(x^m_i,r^m_i)\right)\\&=\prod_{i=1}^{k}\mathrm{vol}_n(B^n_\infty(x^j_i,r^j_i)\cap B^n_\infty(x^m_i,r^m_i))=0.\end{aligned}$$

Therefore, for every pair j,m with $j\neq m$ there is i, $1\le i\le k$, such that

$$B^n_\infty(x^j_i,r^j_i)\cap B^n_\infty(x^m_i,r^m_i) \tag{103}$$

is contained in a hyperplane that is orthogonal to one of the vectors $e_1,\dots,e_n$. We put

$$W_j=\prod_{i=1}^{k}\left(B^n_\infty(x^j_i,r^j_i)\cap\partial K\cap H^-\right)\qquad j\in\mathbb{N} \tag{104}$$

and get

$$\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]^\circ, x_1,\dots,x_k\in\partial K\cap H^-\}=\bigcup_{j=1}^{\infty}W_j. \tag{105}$$

Then we have for $j\neq m$ that

$$\mathrm{vol}_{k(n-1)}(W_j\cap W_m)=0.$$

Indeed,

$$W_j \cap W_m = \left\{ (x_1, \ldots, x_k) | \forall i : x_i \in \partial K \cap B_\infty^n(x_i^j, r_i^j) \cap B_\infty^n(x_i^m, r_i^m) \cap H^- \right\}.$$

There is at least one i_0 such that

$$B_\infty^n(x_{i_0}^j, r_{i_0}^j) \cap B_\infty^n(x_{i_0}^m, r_{i_0}^m)$$

is contained in a hyperplane L that is orthogonal to one of the vectors $e_1, \ldots, e_n$. Therefore

$$\mathrm{vol}_{n-1}(\partial K \cap B_\infty^n(x_{i_0}^j, r_{i_0}^j) \cap B_\infty^n(x_{i_0}^m, r_{i_0}^m)) \leq \mathrm{vol}_{n-1}(\partial K \cap L).$$

The last expression is 0 if the hyperplane is chosen sufficiently close to x_0. Indeed, $\partial K \cap L$ is either a face of K or $\partial K \cap L = \partial(K \cap L)$. In the latter case $\mathrm{vol}_{n-1}(\partial K \cap L) = \mathrm{vol}_{n-1}(\partial(K \cap H)) = 0$. If H is sufficiently close to x_0, then L does not contain a $n-1$-dimensional face of K. This follows from the fact that the indicatrix exists and is an ellipsoid and consequently all normals are close to $N_{\partial K}(x_0) = e_n$ but not equal.

Let $rp : \partial K \to \partial \mathcal{E}$ where $rp(x)$ is the unique point with

$$\{rp(x)\} = \{x_s + t(x - x_s) | t \geq 0\} \cap \partial \mathcal{E}. \tag{106}$$

For s_0 small enough we have for all s with $0 < s \leq s_0$ that $x_s \in \mathcal{E}$. In this case rp is well defined. $Rp : \partial K \times \cdots \times \partial K \to \partial \mathcal{E} \times \cdots \times \partial \mathcal{E}$ is defined by

$$Rp(x_1, \ldots, x_k) = (rp(x_1), \ldots, rp(x_k)). \tag{107}$$

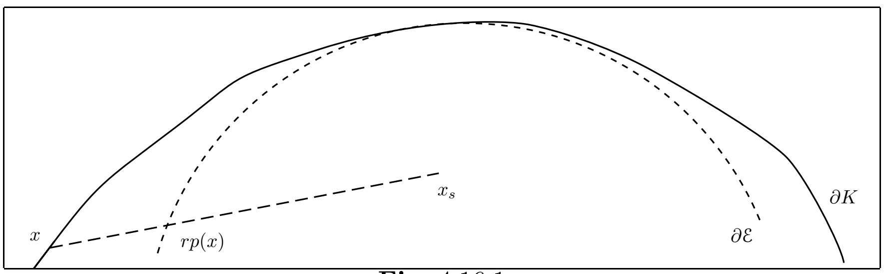

Fig. 4.16.1

There is a map $\alpha : \partial K \to (-\infty, 1)$ such that

$$rp(x) = x - \alpha(x)(x - x_s). \tag{108}$$

Since x_s is an interior point of K the map α does not attain the value 1. For every $\epsilon > 0$ there is s_0 such that we have for all s and c with $0 < cs \leq s_0$ and $c \geq c_0$ and for all hyperplanes H that are orthogonal to $N_{\partial K}(x_0) = e_n$ and that satisfy $\mathrm{vol}_{n-1}(\partial K \cap H^-) = cs$ and all cubes $B_\infty^n(x_i^j, r_i^j)$ that satisfy (104) and (105)

$$\mathrm{vol}_{n-1}(\partial K \cap B_\infty^n(x_i^j, r_i^j)) \le (1+\epsilon)\mathrm{vol}_{n-1}(rp(\partial K \cap B_\infty^n(x_i^j, r_i^j))). \quad (109)$$

To show this we have to establish that there is s_0 such that for all $x \in \partial K \cap H^-(x_{s_0}, N_{\partial K}(x_0))$ and all s with $0 < s \le s_0$

$$\|x - rp(x)\| \le \epsilon \|x_s - rp(x)\| \quad (110)$$

$$(1-\epsilon) \le \frac{\left\langle N_{\partial K}(x), \frac{x-x_s}{\|x-x_s\|}\right\rangle}{\left\langle N_{\partial \mathcal{E}}(rp(x)), \frac{x-x_s}{\|x-x_s\|}\right\rangle} \le (1+\epsilon). \quad (111)$$

Indeed, the volume of a surface element changes under the map rp by the factor

$$\left(\frac{\|rp(x)-x_s\|}{\|x-x_s\|}\right)^{n-1} \frac{\left\langle N_{\partial K}(x), \frac{x-x_s}{\|x-x_s\|}\right\rangle}{\left\langle N_{\partial \mathcal{E}}(rp(x)), \frac{x-x_s}{\|x-x_s\|}\right\rangle}.$$

We establish (111). We have

$$\begin{aligned}\frac{\langle N_{\partial K}(x), x-x_s\rangle}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle} &= 1 + \frac{\langle N_{\partial K}(x) - N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle} \\ &\le 1 + \frac{\|N_{\partial K}(x) - N_{\partial \mathcal{E}}(rp(x))\| \ \|x-x_s\|}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle}.\end{aligned}$$

We have

$$\|N_{\partial K}(x) - N_{\partial \mathcal{E}}(rp(x))\| \le \epsilon \|x - x_0\|.$$

This can be shown in the same way as (33) (Consider the plane $H(x, N_{\partial K}(x_0))$. The distance of this plane to x_0 is of the order $\|x-x_0\|^2$.) Thus we have

$$\frac{\langle N_{\partial K}(x), x-x_s\rangle}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle} \le 1 + \frac{\epsilon\|x-x_0\|\|x-x_s\|}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle} \le 1 + \frac{\epsilon c_0 \|x-x_s\|^2}{\langle N_{\partial \mathcal{E}}(rp(x)), x-x_s\rangle}.$$

It is left to show

$$| < N_{\partial \mathcal{E}}(rp(x)), x-x_s > | \ge c_0 \|x-x_s\|^2.$$

If x is close to x_0 then this estimate reduces to $\|x-x_s\| \ge \|x-x_s\|^2$ which is obvious. If x is not close to x_0 then $\|x-x_s\|^2$ is of the order of the height of the cap $\partial \mathcal{E} \cap H^-(rp(x), N_{\partial K}(x_0))$. Therefore, it is enough to show

$$| < N_{\partial \mathcal{E}}(rp(x)), x-x_s > | \ge c_0 | < N_{\partial K}(x_0), rp(x) - x_0 > |.$$

We consider the map $T: \mathbb{R}^n \to \mathbb{R}^n$ that transforms the standard approximating ellipsoid into a Euclidean ball (5)

$$T(x) = \left(\frac{x_1}{a_1}\left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}, \dots, \frac{x_{n-1}}{a_{n-1}}\left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}, x_n\right).$$

Thus it is enough to show

$$|< T^{-1t} N_{\partial \mathcal{E}}(rp(x)), Tx - Tx_s > | \geq c_0 | < N_{\partial K}(x_0), rp(x) - x_0 > |.$$

Since $Tx_0 = x_0 = 0$ and $T^{-1t}(N_{\partial K}(x_0)) = N_{\partial K}(x_0) = e_n$ the above inequality is equivalent to

$$|< T^{-1t} N_{\partial \mathcal{E}}(rp(x)), Tx - Tx_s > | \geq c_0 | < N_{\partial K}(x_0), T(rp(x)) - x_0 > |.$$

Allowing another constant c_0, the following is equivalent to the above

$$\left| \left\langle \frac{T^{-1t} N_{\partial \mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial \mathcal{E}}(rp(x))\|}, Tx - Tx_s \right\rangle \right| \geq c_0 | < N_{\partial K}(x_0), T(rp(x)) - x_0 > |.$$

Thus we have reduced the estimate to the case of a Euclidean ball.

The hyperplane $H(T(rp(x)), N_{\partial K}(x_0))$ intersects the line

$$\{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\}$$

at the point z with $\|x_0 - z\| = | < N_{\partial K}(x_0), T(rp(x)) - x_0 > |$. Let the radius of $T(\mathcal{E})$ be r. See Figure 4.16.2.

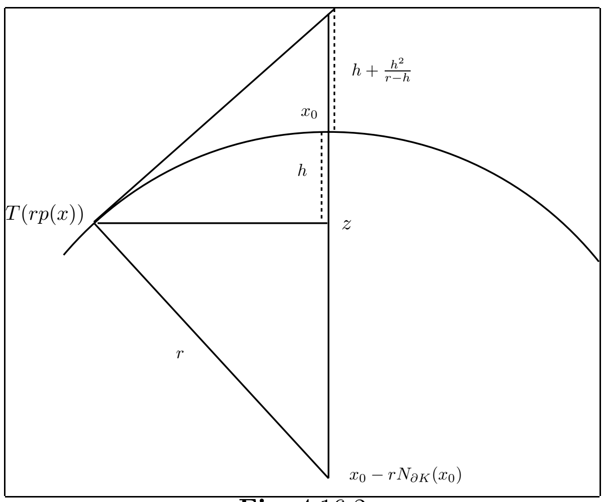

Fig. 4.16.2

We may assume that $< T^{-1t} N_{\partial \mathcal{E}}(rp(x)), N_{\partial K}(x_0) > \geq \frac{1}{2}$. Therefore we have by Figure 4.16.2 ($h = \|x_0 - z\|$)

$$\begin{aligned}
&\left| \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, T(rp(x)) - x_0 \right\rangle \right| \\
&\qquad = \left(\|x_0 - z\| + \frac{\|x_0 - z\|^2}{r - \|x_0 - z\|} \right) \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, N_{\partial K}(x_0) \right\rangle \\
&\qquad \geq \|x_0 - z\| \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, N_{\partial K}(x_0) \right\rangle \\
&\qquad \geq \tfrac{1}{2} | < N_{\partial K}(x_0), T(rp(x)) - x_0 > |
\end{aligned}$$

where r is the radius of $T(\mathcal{E})$. Since there is a constant c_0 such that

$$\begin{aligned}
&\left| \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, T(x) - T(x_s) \right\rangle \right| \\
&\qquad\qquad \geq c_0 \left| \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, T(rp(x)) - x_0 \right\rangle \right|
\end{aligned}$$

we get

$$\left| \left\langle \frac{T^{-1t} N_{\partial\mathcal{E}}(rp(x))}{\|T^{-1t} N_{\partial\mathcal{E}}(rp(x))\|}, T(x) - T(x_s) \right\rangle \right| \geq \tfrac{1}{2} c_0 | < N_{\partial K}(x_0), T(rp(x)) - x_0 > |.$$

The left hand inequality of (111) is shown in the same way.

Now we verify (110).

Again we apply the affine transform T to K that transforms the indicatrix of Dupin at x_0 into a Euclidean sphere (5). T leaves x_0 and $N_{\partial K}(x_0)$ invariant.

An affine transform maps a line onto a line and the factor by which a segment of a line is stretched is constant. We have

$$\frac{\|x - rp(x)\|}{\|x_s - rp(x)\|} = \frac{\|T(x) - T(rp(x))\|}{\|T(x_s) - T(rp(x))\|}.$$

Thus we have

$$\begin{aligned}
&B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(T(x_{s_0}), N_{\partial K}(x_0)) \\
&\qquad \subseteq T(K) \cap H^-(T(x_{s_0}), N_{\partial K}(x_0)) \\
&\qquad\qquad \subseteq B_2^n(x_0 - (1+\epsilon) r N_{\partial K}(x_0), (1+\epsilon) r) \cap H^-(T(x_{s_0}), N_{\partial K}(x_0)).
\end{aligned}$$

The center of the $n-1$-dimensional sphere

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H(T(rp(x)), N_{\partial K}(x_0))$$

is

$$x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0)$$

and the height of the cap

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(T(rp(x)), N_{\partial K}(x_0))$$

is

$$|< x_0 - T(rp(x)), N_{\partial K}(x_0) >|.$$

Therefore, for sufficiently small s_0 and all s with $0 < s \leq s_0$ we get that the radius of the cap $\|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\|$ satisfies

$$\begin{aligned}&\sqrt{r|< x_0 - T(rp(x)), N_{\partial K}(x_0) >|} \\ &\qquad \leq \|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\|.\end{aligned} \tag{112}$$

We show that there is a constant $c_0 > 0$ so that we have for all s with $0 < s \leq s_0$ and all $x \in \partial K \cap H^-(x_{s_0}, N_{\partial K}(x_0))$

$$\|T(rp(x)) - T(x_s)\| \geq c_0 \sqrt{r|< x_0 - T(rp(x)), N_{\partial K}(x_0) >|}. \tag{113}$$

Let α be the angle between $N_{\partial K}(x_0)$ and $x_0 - T(x_T)$. We first consider the case

$$\begin{aligned}&\|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ &\qquad \geq 2(1 + (\cos\alpha)^{-1})|< x_0 - T(x_s), N_{\partial K}(x_0) >|.\end{aligned} \tag{114}$$

(This case means: x_0 is not too close to $T(rp(x))$.) Then we have

$$\begin{aligned}&\|T(rp(x)) - T(x_s)\| \\ &\quad \geq \|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ &\qquad - \|x_0 - T(x_s)\| - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >| \\ &\quad = \|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ &\qquad - (\cos\alpha)^{-1}|< x_0 - T(x_s), N_{\partial K}(x_0) >| \\ &\qquad - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >|.\end{aligned}$$

By the assumption (114)

$$\begin{aligned}&\|T(rp(x)) - T(x_s)\| \\ &\quad \geq \tfrac{1}{2}\|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ &\qquad + (1 + (\cos\alpha)^{-1})|< x_0 - T(x_s), N_{\partial K}(x_0) >| \\ &\qquad - (\cos\alpha)^{-1}|< x_0 - T(x_s), N_{\partial K}(x_0) >| \\ &\qquad - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >| \\ &\quad = \tfrac{1}{2}\|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ &\qquad + |< x_0 - T(x_s), N_{\partial K}(x_0) >| - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >|.\end{aligned}$$

By (112)

$$\begin{aligned}&\|T(rp(x)) - T(x_s)\| \\ &\quad \geq \tfrac{1}{2}\sqrt{r|< x_0 - T(rp(x)), N_{\partial K}(x_0) >|} \\ &\qquad + |< x_0 - T(x_s), N_{\partial K}(x_0) >| - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >| \\ &\quad \geq \tfrac{1}{2}\sqrt{r|< x_0 - T(rp(x)), N_{\partial K}(x_0) >|} \\ &\qquad - |< x_0 - T(rp(x)), N_{\partial K}(x_0) >|.\end{aligned}$$

We get for sufficiently small s_0 that for all s with $0 < s \le s_0$

$$\|T(rp(x)) - T(x_s)\| \ge \tfrac{1}{4}\sqrt{r| < x_0 - T(rp(x)), N_{\partial K}(x_0) > |}.$$

The second case is

$$\begin{aligned} \|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\| \\ < 2(1 + (\cos\alpha)^{-1})| < x_0 - T(x_s), N_{\partial K}(x_0) > |. \end{aligned} \tag{115}$$

(In this case, x_0 is close to $T(rp(x))$.) $\|T(rp(x)) - T(x_s)\|$ can be estimated from below by the least distance of $T(x_s)$ to the boundary of $B_2^n(x_0 - rN_{\partial K}(x_0), r)$. This, in turn, can be estimated from below by

$$c'| < x_0 - T(x_s), N_{\partial K}(x_0) > |.$$

Thus we have

$$\|T(rp(x)) - T(x_s)\| \ge c'| < x_0 - T(x_s), N_{\partial K}(x_0) > |.$$

On the other hand, by our assumption (115)

$$\begin{aligned} &\|T(rp(x)) - T(x_s)\| \\ &\ge \frac{c'}{2(1 + (\cos\alpha)^{-1})} \times \\ &\qquad \|T(rp(x)) - (x_0 - < x_0 - T(rp(x)), N_{\partial K}(x_0) > N_{\partial K}(x_0))\|. \end{aligned}$$

By (112)

$$\|T(rp(x)) - T(x_s)\| \ge \frac{c'}{2(1 + (\cos\alpha)^{-1})}\sqrt{r| < x_0 - T(rp(x)), N_{\partial K}(x_0) > |}.$$

This establishes (113).

Now we show that for s_0 sufficiently small we have for all s with $0 < s \le s_0$ and all x

$$\|T(x) - T(rp(x))\| \le 2\sqrt{2\epsilon(1+\epsilon)r| < x_0 - T(rp(x)), N_{\partial K}(x_0) > |}. \tag{116}$$

Instead of $T(x)$ we consider the points z and z' with

$$\{z\} = B_2^n(x_0 - (1+\epsilon)rN_{\partial K}(x_0), (1+\epsilon)r) \cap \{T(x_s) + t(T(x) - T(x_s)) | t \ge 0\}$$

$$\{z'\} = B_2^n(x_0 - (1-\epsilon)rN_{\partial K}(x_0), (1-\epsilon)r) \cap \{T(x_s) + t(T(x) - T(x_s)) | t \ge 0\}.$$

We have

$$\|T(x) - T(rp(x))\| \le \max\{\|z - T(rp(x))\|, \|z' - T(rp(x))\|\}.$$

We may assume that $\|x - x_s\| \ge \|rp(x) - x_s\|$. This implies $\|T(x) - T(rp(x))\| \le \|z - T(rp(x))\|$. $\|z - T(rp(x))\|$ is smaller than the diameter of the cap

$$B_2^n(x_0-(1+\epsilon)rN_{\partial K}(x_0),(1+\epsilon)r)$$
$$\cap H^-(T(rp(x)),N_{\partial B_2^n(x_0-rN_{\partial K}(x_0),r)}(T(rp(x))))$$

because z and $T(rp(x))$ are elements of this cap. See Figure 4.16.3.

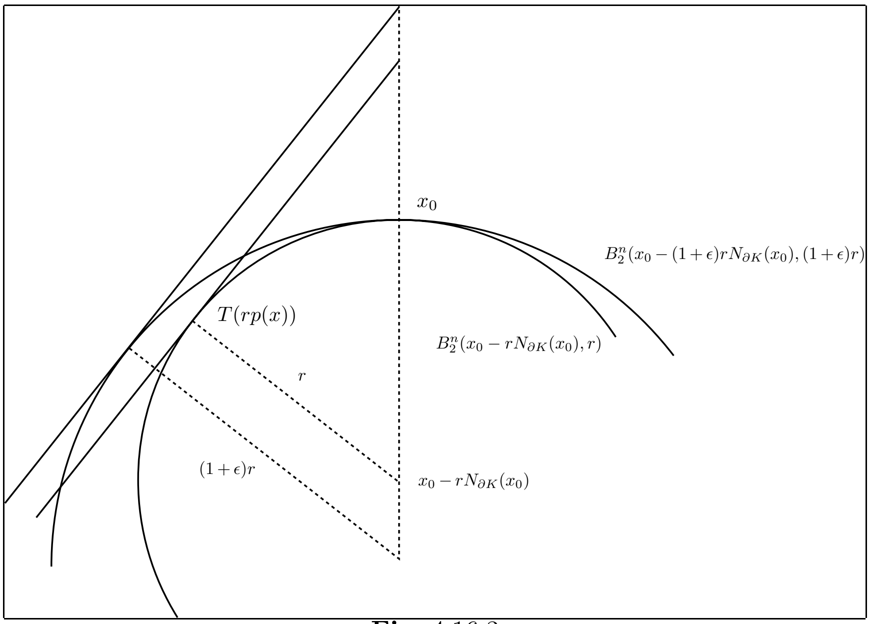

Fig. 4.16.3

We compute the radius of this cap. The two triangles in Figure 4.16.3 are homothetic with respect to the point x_0. The factor of homothety is $1+\epsilon$. The distance between the two tangents to $B_2^n(x_0-(1+\epsilon)rN_{\partial K}(x_0),(1+\epsilon)r)$ and $B_2^n(x_0-rN_{\partial K}(x_0),r)$ is

$$\epsilon|<x_0-T(rp(x)),N_{\partial K}(x_0)>|.$$

Consequently the radius is less than

$$\sqrt{2\epsilon(1+\epsilon)r|<x_0-T(rp(x)),N_{\partial K}(x_0)>|}.$$

Thus we have established (116). The inequalities (113) and (116) give (110).

From the inequalities (110) and (111) we get for $x\in\partial K\cap B_\infty^n(x_i^j,r_i^j)$ and r_i^j sufficiently small

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial K\cap\ & B_\infty^n(x_i^j,r_i^j))\\ &\le\frac{\left|\frac{\|x_s-x\|}{\|x_s-rp(x)\|}\right|^{n-1}}{\langle N_{\partial K}(x),N_{\partial\mathcal{E}}(rp(x))\rangle}\mathrm{vol}_{n-1}(rp(\partial K\cap B_\infty^n(x_i^j,r_i^j)))\\ &\le(1+\epsilon)\frac{(1+\epsilon)^{n-1}}{1-\epsilon}\mathrm{vol}_{n-1}(rp(\partial K\cap B_\infty^n(x_i^j,r_i^j))).\end{aligned}$$

It follows that for a new s_0

$$\begin{aligned}\mathrm{vol}_{k(n-1)}(W_j) &= \prod_{i=1}^{k} \mathrm{vol}_{n-1}(\partial K \cap B_\infty^n(x_i^j, r_i^j)) \\ &\le (1+\epsilon)^k \prod_{i=1}^{k} \mathrm{vol}_{n-1}(rp(\partial K \cap B_\infty^n(x_i^j, r_i^j))) \\ &= (1+\epsilon)^k \mathrm{vol}_{k(n-1)}(Rp(W_j)).\end{aligned}$$

And again with a new s_0

$$\mathrm{vol}_{k(n-1)}(W_j) \le (1+\epsilon)\mathrm{vol}_{k(n-1)}(Rp(W_j)). \tag{117}$$

We also have for all $x_i \in \partial K$, $i = 1, \dots, k$

$$\begin{aligned}Rp(\{(x_1,\dots,x_k)| \quad & x_s \in [x_1,\dots,x_k]^\circ \text{ and } x_i \in \partial K\}) \\ & \subseteq \{(z_1,\dots,z_k)|\ x_s \in [z_1,\dots,z_k] \text{ and } z_i \in \partial \mathcal{E}\}.\end{aligned} \tag{118}$$

We verify this. Let a_i, $i = 1, \dots, k$, be nonnegative numbers with $\sum_{i=1}^k a_i = 1$ and

$$x_s = \sum_{i=1}^{k} a_i x_i.$$

We choose

$$b_i = \frac{a_i}{(1-\alpha(x_i))(1+\sum_{j=1}^k \frac{\alpha(x_j)a_j}{1-\alpha(x_j)})}$$

where $\alpha(x_i)$, $i = 1, \dots, k$, are defined by (108). We claim that $\sum_{i=1}^k b_i = 1$ and

$$x_s = \sum_{i=1}^{k} b_i rp(x_i).$$

We have

$$\begin{aligned}\sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} \frac{a_i}{(1-\alpha(x_i))(1+\sum_{j=1}^k \frac{\alpha(x_j)a_j}{1-\alpha(x_j)})} \\ &= \sum_{i=1}^{k} \frac{a_i(1+\frac{\alpha(x_i)}{1-\alpha(x_i)})}{1+\sum_{j=1}^k \frac{\alpha(x_j)a_j}{1-\alpha(x_j)}} = 1.\end{aligned}$$

Moreover, by (108) we have $rp(x_i) = x_i - \alpha(x_i)(x_i - x_s)$

$$\begin{aligned}
\sum_{i=1}^{k} b_i rp(x_i) &= \sum_{i=1}^{k} b_i (x_i - \alpha(x_i)(x_i - x_s)) \\
&= \sum_{i=1}^{k} \frac{a_i (x_i - \alpha(x_i)(x_i - x_s))}{(1-\alpha(x_i))(1+\sum_{j=1}^{k} \frac{\alpha(x_j)a_j}{1-\alpha(x_j)})} \\
&= \sum_{i=1}^{k} \frac{a_i x_i}{1+\sum_{j=1}^{k} \frac{\alpha(x_j)a_j}{1-\alpha(x_j)}} + \sum_{i=1}^{k} \frac{a_i \alpha(x_i) x_s}{(1-\alpha(x_i))(1+\sum_{j=1}^{k} \frac{\alpha(x_j)a_j}{1-\alpha(x_j)})} \\
&= \frac{x_s}{1+\sum_{j=1}^{k} \frac{\alpha(x_j)a_j}{1-\alpha(x_j)}} + \frac{\sum_{i=1}^{k} \frac{a_i\alpha(x_i)}{1-\alpha(x_i)} x_s}{1+\sum_{j=1}^{k} \frac{\alpha(x_j)a_j}{1-\alpha(x_j)}} = x_s.
\end{aligned}$$

Thus we have established (118)

$$\begin{aligned}
Rp(\{(x_1,\dots,x_k)| \quad & x_s \in [x_1,\dots,x_k]^\circ \text{ and } x_i \in \partial K\}) \\
& \subseteq \{(z_1,\dots,z_k)|\ x_s \in [z_1,\dots,z_k] \text{ and } z_i \in \partial \mathcal{E}\}.
\end{aligned}$$

Next we verify that there is a hyperplane $\tilde{H}$ that is parallel to H and such that

$$\mathrm{vol}_{n-1}(\partial K \cap \tilde{H}^-) \le (1+\epsilon)\mathrm{vol}_{n-1}(\partial K \cap H^-) \tag{119}$$

and

$$\begin{aligned}
Rp(\{(x_1,\dots,x_k)|\ x_s \in [x_1,\dots,x_k]^\circ, x_i \in \partial K \cap H^-\}) \\
\subseteq \{(z_1,\dots,z_k)|\ x_s \in [z_1,\dots,z_k] \text{ and } z_i \in \partial\mathcal{E} \cap \tilde{H}^-\}.
\end{aligned} \tag{120}$$

This is done by arguments similar to the ones above. Thus we get with a new s_0

$$\begin{aligned}
&\mathbb{P}^k_{\partial K \cap H^-}\{(x_1,\dots,x_k)|\ x_s \in [x_1,\dots,x_k]\} = \frac{\mathrm{vol}_{k(n-1)}\left(\bigcup_{j=1}^{\infty} W_j\right)}{(\mathrm{vol}_{n-1}(\partial K \cap H^-))^k} \\
&\le (1+\epsilon)\frac{\mathrm{vol}_{k(n-1)}\left(\bigcup_{j=1}^{\infty} Rp(W_j)\right)}{(\mathrm{vol}_{n-1}(\partial K \cap H^-))^k} \\
&\le (1+\epsilon)\frac{\mathrm{vol}_{k(n-1)}\{(z_1,\dots,z_k)|\ x_s \in [z_1,\dots,z_k] \text{ and } z_i \in \partial\mathcal{E} \cap \tilde{H}^-\}}{(\mathrm{vol}_{n-1}(\partial K \cap H^-))^k} \\
&\le (1+\epsilon)\frac{\mathrm{vol}_{k(n-1)}\{(z_1,\dots,z_k)|\ x_s \in [z_1,\dots,z_k] \text{ and } z_i \in \partial\mathcal{E} \cap H^-\}}{(\mathrm{vol}_{n-1}(\partial K \cap H^-))^k} + k\epsilon.
\end{aligned}$$

$\mathrm{vol}_{n-1}(\partial K \cap H^-)$ and $\mathrm{vol}_{n-1}(\partial \mathcal{E} \cap H^-)$ differ only by a factor between $1-\epsilon$ and $1+\epsilon$ if we choose s_0 small enough. Therefore, for sufficiently small s_0 we have for all s with $0 < s \le s_0$

$$\mathbb{P}^k_{\partial K\cap H^-}\{(x_1,\dots,x_k)|\ x_s\in[x_1,\dots,x_k]\}$$
$$\le(1+\epsilon)\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ x_s\in[z_1,\dots,z_k]\}+\epsilon.$$

(iii) We show now that for sufficiently small s_0 we have

$$|\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\quad y_s\in[z_1,\dots,z_k]\}$$
$$-\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ z_s\in[z_1,\dots,z_k]\}|<\epsilon.$$

The arguments are very similar to those for the first inequality. We consider the standard approximating ellipsoid $\mathcal{E}$ and the map $tp:\partial\mathcal{E}\to\partial\mathcal{E}$ mapping $x\in\partial\mathcal{E}$ onto the unique point $tp(x)$ with

$$\{tp(x)\}=\partial\mathcal{E}\cap\{y_s+t(x-z_s)|\ t\ge0\}.$$

See Figure 4.16.4.

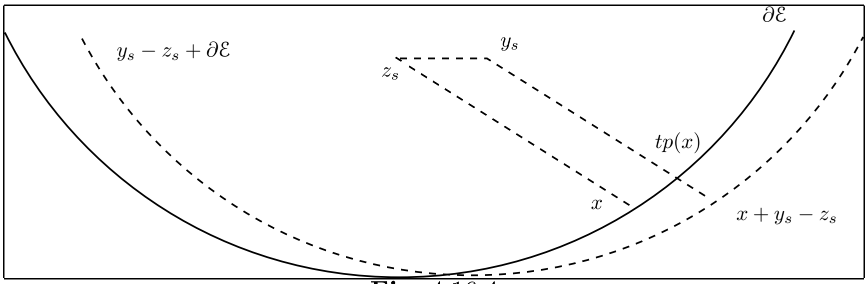

Fig. 4.16.4

We define $Tp:\partial\mathcal{E}\times\cdots\partial\mathcal{E}\to\partial\mathcal{E}\times\cdots\times\partial\mathcal{E}$ by $Tp(z_1,\dots,z_k)=(tp(z_1),\dots,tp(z_k))$. Then we have

$$Tp(\{(z_1,\dots,z_k)|\quad z_s\in[z_1,\dots,z_k]\text{ and }z_i\in\partial\mathcal{E}\})$$
$$\subseteq\{(y_1,\dots,y_k)|\ y_s\in[y_1,\dots,y_k]\text{ and }y_i\in\partial\mathcal{E}\}.$$

The calculation is the same as for the inequality (ii). The map tp changes the volume of a surface-element at the point x by the factor

$$\left(\frac{\|y_s-tp(x)\|}{\|y_s-(x+y_s-z_s)\|}\right)^{n-1}\frac{\left\langle\frac{y_s-tp(x)}{\|y_s-tp(x)\|},N_{\partial\mathcal{E}}(x)\right\rangle}{\left\langle\frac{y_s-tp(x)}{\|y_s-tp(x)\|},N_{\partial\mathcal{E}}(tp(x))\right\rangle}\tag{121}$$
$$=\left(\frac{\|y_s-tp(x)\|}{\|x-z_s\|}\right)^{n-1}\frac{\left\langle\frac{y_s-tp(x)}{\|y_s-tp(x)\|},N_{\partial\mathcal{E}}(x)\right\rangle}{\left\langle\frac{y_s-tp(x)}{\|y_s-tp(x)\|},N_{\partial\mathcal{E}}(tp(x))\right\rangle}.$$

We have to show that this expression is arbitrarily close to 1 provided that s is sufficiently small. Since we consider an ellipsoid

$$\frac{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(x) \right\rangle}{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(tp(x)) \right\rangle} \tag{122}$$

is sufficiently close to 1 provided that s is sufficiently small. We check this. We have

$$\frac{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(x) \right\rangle}{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(tp(x)) \right\rangle} = 1 + \frac{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(tp(x)) - N_{\partial\mathcal{E}}(x) \right\rangle}{\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(tp(x)) \right\rangle}.$$

We show that (122) is close to 1 first for the case that $\mathcal{E}$ is a Euclidean ball. We have $\|N_{\partial\mathcal{E}}(x) - N_{\partial\mathcal{E}}(tp(x))\| \leq c_0\|x_0 - z_s\|$ for some constant c_0 because $\|N_{\partial\mathcal{E}}(x) - N_{\partial\mathcal{E}}(tp(x))\| \leq \|y_s - z_s\|$ and $\|y_s - z_s\| \leq c_0\|x_0 - z_s\|$. The inequality $\|y_s - z_s\| \leq c_0\|x_0 - z_s\|$ holds because $\{z_s\} = [x_0, z_T] \cap \partial\mathcal{E}_s$ and $\{y_s\} = [x_0, x_T] \cap H(z_s, N_{\partial K}(x_0))$.

On the other hand, there is a constant c_0 such that for all s

$$\left\langle \frac{y_s - tp(x)}{\|y_s - tp(x)\|}, N_{\partial\mathcal{E}}(tp(x)) \right\rangle \geq c_0\sqrt{\|x_0 - z_s\|}.$$

These two inequalities give that (122) is close to 1 in the case that $\mathcal{E}$ is a Euclidean ball. In order to obtain these inequalities for the case of an ellipsoid we apply the diagonal map A that transforms the Euclidean ball into the ellipsoid. A leaves e_n invariant. Lemma 2.6 gives the first inequality and the second inequality gives

$$\left\langle A\left(\frac{y_s - tp(x)}{\|y_s - tp(x)\|}\right), A^{-1t}(N_{\partial\mathcal{E}}(tp(x))) \right\rangle \geq c_0\sqrt{\|x_0 - z_s\|}.$$

This gives that (122) is close to 1 for ellipsoids. Therefore, in order to show that the expression (121) converges to 1 for s to 0 it is enough to show that for all x

$$\left(\frac{\|y_s - tp(x)\|}{\|x - z_s\|}\right)^{n-1} \tag{123}$$

is arbitrarily close to 1 provided that s is small. In order to prove this we show for all x

$$1 - c_1\|z_s - x_0\|^{\frac{1}{6}} \leq \frac{\|y_s - tp(x)\|}{\|y_s - (x + y_s - z_s)\|} \leq 1 + c_2\|z_s - x_0\|^{\frac{1}{6}} \tag{124}$$

or, equivalently, that there is a constant c_3 such that

$$\frac{\|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \leq c_3\|z_s - x_0\|^{\frac{1}{6}}. \tag{125}$$

We verify the equivalence. By triangle inequality

$$1 + c_3\|z_s - x_0\|^{\frac{1}{6}} \geq \frac{\|y_s - tp(x)\| + \|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \geq \frac{\|y_s - (x + y_s - z_s)\|}{\|y_s - tp(x)\|}$$

which gives the left hand inequality of (124). Again, by triangle inequality

$$1 - c_3\|z_s - x_0\|^{\frac{1}{6}} \leq \frac{\|y_s - tp(x)\| - \|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \leq \frac{\|y_s - (x + y_s - z_s)\|}{\|y_s - tp(x)\|}$$

which gives the right hand inequality of (124).

We show (125). We begin by showing that

$$\frac{\|tp(x_0) - (x_0 + y_s - z_s)\|}{\|y_s - tp(x_0)\|} \leq c_3\|z_s - x_0\|^{\frac{1}{2}}. \tag{126}$$

See Figure 4.16.5.

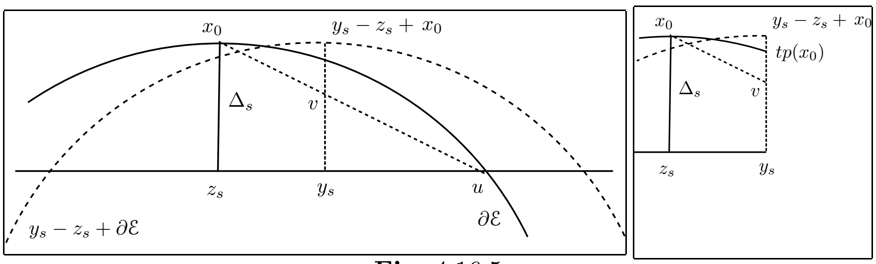

Fig. 4.16.5

Clearly, by Figure 4.16.5

$$\|tp(x_0) - (x_0 + y_s - z_s)\| \leq \|v - (x_0 + y_s - z_s)\|.$$

There is ρ such that for all s with $0 < s \leq s_0$ we have $\|z_s - u\| \geq \rho\sqrt{\|x_0 - z_s\|}$. Let θ be the angle between $x_0 - x_T$ and $N_{\partial K}(x_0)$. By this and $\|z_s - y_s\| = (\tan\theta)\|x_0 - z_s\|$

$$\begin{aligned}\|tp(x_0) - (x_0 + y_s - z_s)\| &\leq \|v - (x_0 + y_s - z_s)\| \\ &= \|z_s - y_s\|\frac{\|x_0 - z_s\|}{\|z_s - u\|} \leq \frac{\tan\theta}{\rho}\|x_0 - z_s\|^{\frac{3}{2}}.\end{aligned}$$

It follows

$$\|y_s - tp(x_0)\| = \|z_s - x_0\| - \|tp(x_0) - (x_0 + y_s - z_s)\| \geq \|z_s - x_0\| - \frac{\tan\theta}{\rho}\|x_0 - z_s\|^{\frac{3}{2}}.$$

This proves (126) which is the special case $x = x_0$ for (125).

Now we treat the general case of (125). We consider three cases: One case being $x \in H^-(z_s, N_{\partial K}(x_0))$ and $\|y_s - w_1\| \le \|x_0 - z_s\|^{\frac{2}{3}}$, another $x \in H^-(z_s, N_{\partial K}(x_0))$ and $\|y_s - w_1\| \ge \|x_0 - z_s\|^{\frac{2}{3}}$ and the last $x \in H^+(z_s, N_{\partial K}(x_0))$.

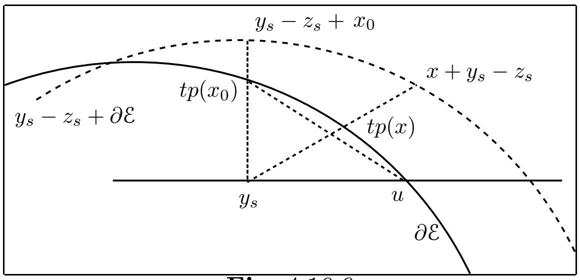

Fig. 4.16.6

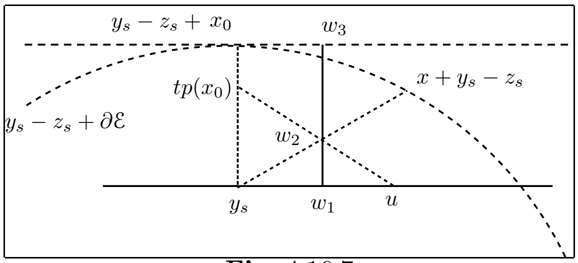

Fig. 4.16.7

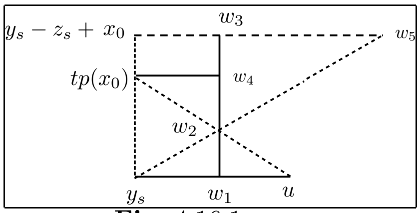

Fig. 4.16.1

First we consider the case that $x \in H^-(z_s, N_{\partial K}(x_0))$ and

$$\|y_s - w_1\| \le \|x_0 - z_s\|^{\frac{2}{3}}.$$

We observe that (see Figure 4.16.5 and 4.16.7)

$$\begin{aligned} \|y_s - tp(x)\| &\ge \|y_s - w_2\| \\ \|tp(x) - (x + y_s - z_s)\| &\le \|w_2 - (x + y_s - z_s)\|\|w_2 - w_5\|. \end{aligned}$$

Thus we get

$$\frac{\|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \le \frac{\|w_2 - w_5\|}{\|y_s - w_2\|} = \frac{\|w_3 - w_2\|}{\|w_1 - w_2\|}. \tag{127}$$

Comparing the triangles $(tp(x_0), w_4, w_2)$ and $(tp(x_0), u, y_s)$ we get

$$\frac{\|w_2 - w_4\|}{\|tp(x_0) - w_4\|} = \frac{\|tp(x_0) - y_s\|}{\|y_s - u\|}.$$

Since $\|tp(x_0) - w_4\| = \|y_s - w_1\|$

$$\|w_2 - w_4\| = \|y_s - w_1\| \frac{\|tp(x_0) - y_s\|}{\|y_s - u\|}.$$

By the assumption $\|y_s - w_1\| \leq \|x_0 - z_s\|^{\frac{2}{3}}$, by $\|tp(x_0) - y_s\| \leq \|x_0 - z_s\|$ and by $\|y_s - u\| \geq c_0\sqrt{\|x_0 - z_s\|}$ we get with a new constant c_0

$$\|w_2 - w_4\| \leq c_0 \|x_0 - z_s\|^{\frac{7}{6}}$$

and with a new c_0

$$\begin{aligned} \|w_2 - w_3\| &= \|w_2 - w_4\| + \|w_3 - w_4\| \\ &= \|w_2 - w_4\| + \|tp(x_0) - (y_s - z_s + x_0)\| \\ &\leq c_0(\|x_0 - z_s\|^{\frac{7}{6}} + \|z_s - x_0\|^{\frac{3}{2}}). \end{aligned}$$

From this and $\|w_1 - w_3\| = \|z_s - x_0\|$ we conclude

$$\|w_1 - w_2\| \geq \|z_s - x_0\| - c_0\|x_0 - z_s\|^{\frac{7}{6}}.$$

The inequality (127) gives now

$$\frac{\|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \leq \frac{\|w_3 - w_2\|}{\|w_1 - w_2\|} \leq \frac{c\|x_0 - z_s\|^{\frac{7}{6}}}{\|z_s - x_0\| - c\|x_0 - z_s\|^{\frac{7}{6}}}.$$

The second case is that $tp(x) \in H^-(z_s, N_{\partial K}(x_0))$ and

$$\|y_s - w_1\| \geq \|x_0 - z_s\|^{\frac{2}{3}}.$$

Compare Figure 4.16.9.

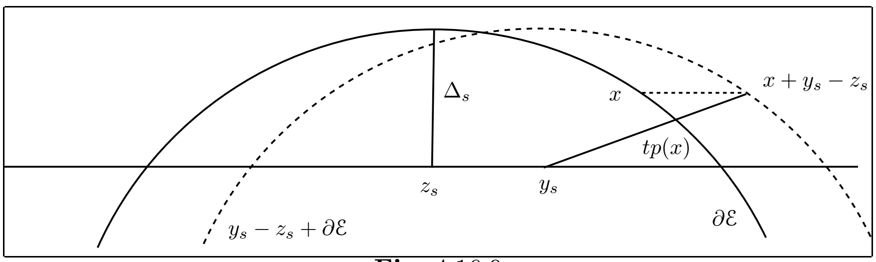

Fig. 4.16.9

Since $\|y_s - w_1\| \geq \|x_0 - z_s\|^{\frac{2}{3}}$ we get

$$\|y_s - tp(x)\| \geq \|x_0 - z_s\|^{\frac{2}{3}}.$$

We have $\|tp(x) - (x + y_s - z_s)\| \leq \|y_s - z_s\|$ because $x \in H^-(z_s, N_{\partial K}(x_0))$ (see Figure 4.16.9). Since $\|z_s - y_s\| \leq c_0\|x_0 - z_s\|$ we deduce $\|tp(x) - (x + y_s - z_s)\| \leq c_0\|x_0 - z_s\|$. Thus we get

$$\frac{\|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \leq \frac{c_0\|x_0 - z_s\|}{\|x_0 - z_s\|^{\frac{2}{3}}} = c_0\|x_0 - z_s\|^{\frac{1}{3}}.$$

The last case is $tp(x) \in H^+(z_s, N_{\partial K}(x_0))$ (See Figure 4.16.10). We have

$$\|y_s - tp(x)\| \geq \|y_s - u\| \geq \|z_s - u\| - \|y_s - z_s\|.$$

There are constants c_0 and ρ such that

$$\begin{aligned} \|y_s - tp(x)\| &\geq \rho\sqrt{\|x_0 - z_s\|} - c_0\|x_0 - z_s\| \\ \|tp(x) - (x + y_s - z_s)\| &\leq c_0\|x_0 - z_s\|. \end{aligned} \tag{128}$$

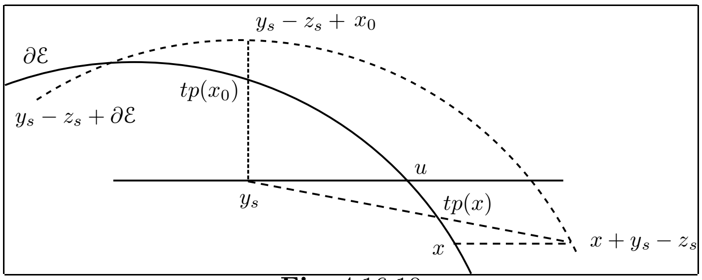

Fig. 4.16.10

The first inequality is apparent, the second is not. We show the second inequality.

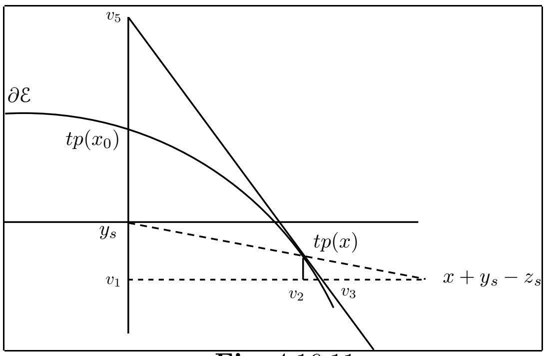

Fig. 4.16.11

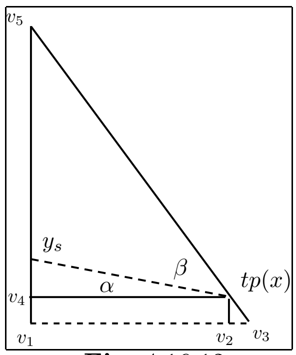

Fig. 4.16.12

We know that the distance between v_3 and $x + y_s - z_s$ is less than $\|y_s - z_s\|$ which is less than $c_0\|x_0 - z_s\|$ (See Figure 4.16.11). The angles α and β are given in Figure 4.16.12. We show that there is a constant c_0 such that $\beta \geq c_0\alpha$. We have

$$\tan\alpha = \frac{\|y_s - v_4\|}{\|v_1 - v_2\|} \qquad\qquad \tan(\alpha+\beta) = \frac{\|v_4 - v_5\|}{\|v_1 - v_2\|}.$$

We have

$$\frac{1}{1-\tan\alpha\tan\beta} + \frac{\frac{\tan\beta}{\tan\alpha}}{1-\tan\alpha\tan\beta} = \frac{\tan(\alpha+\beta)}{\tan\alpha} = \frac{\|v_4 - v_5\|}{\|y_s - v_4\|} = 1 + \frac{\|y_s - v_5\|}{\|y_s - v_4\|}$$

which gives

$$\frac{\tan\beta}{\tan\alpha} = -\tan\alpha\tan\beta + (1-\tan\alpha\tan\beta)\frac{\|y_s - v_5\|}{\|y_s - v_4\|}.$$

It is not difficult to show that there is a constant c such that for all s with $0 < s \leq s_0$

$$\|y_s - v_5\| \geq c\|y_s - v_4\|.$$

This gives

$$\frac{\tan\beta}{\tan\alpha} \geq -\tan\alpha\tan\beta + c(1-\tan\alpha\tan\beta).$$

For s_0 sufficiently small α and β will be as small as we require. Therefore, the right hand side is positive. Since the angles are small we have $\tan\alpha \sim \alpha$ and $\tan\beta \sim \beta$. From $\beta \geq c_0\alpha$ we deduce now that

$$\|tp(x) - (x + y_s - z_s)\| \leq c_0\|v_3 - (x + y_s - z_s)\| \leq c\|y_s - z_s\|.$$

We obtain by (128)

$$\frac{\|tp(x) - (x + y_s - z_s)\|}{\|y_s - tp(x)\|} \leq \frac{c\|y_s - z_s\|}{\rho\sqrt{\|x_0 - z_s\|} - c_0\|x_0 - z_s\|}.$$

There is a constant c such that $\|y_s - z_s\| \leq c_0\|x_0 - z_s\|$.

(iv) First we show

$$\Big|\mathbb{P}^k_{\partial\mathcal{E}\cap H_s^-}\{(z_1,\dots,z_k)| \quad z_s \in [z_1,\dots,z_k]\} - \mathbb{P}^k_{\partial\mathcal{E}\cap H_s^-}\{(z_1,\dots,z_k)| \; z_{s'} \in [z_1,\dots,z_k]\}\Big| < \epsilon.$$

Here the role of the maps rp and tp used in (ii) and (iii) is played by the map that maps $x \in \partial\mathcal{E}$ onto the element $[z_s, x + z_s - z_{s'}] \cap \partial\mathcal{E}$. See Figure 4.16.13.

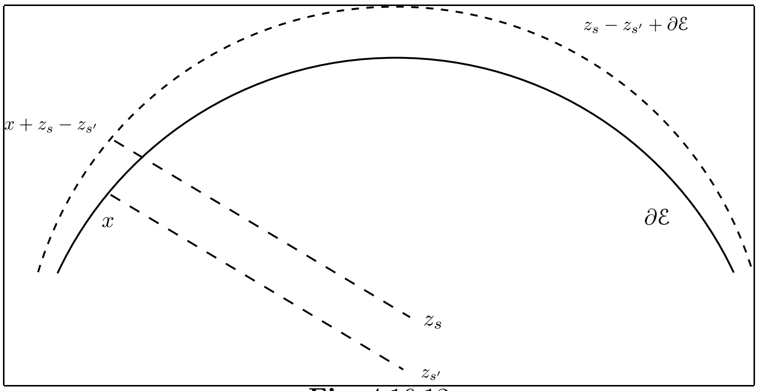

Fig. 4.16.13

Then we show

$$\left|\mathbb{P}^k_{\partial\mathcal{E}\cap H_s^-}\{(z_1,\dots,z_k)|\quad z_{s'}\in[z_1,\dots,z_k]\} - \mathbb{P}^k_{\partial\mathcal{E}\cap H_{s'}^-}\{(z_1,\dots,z_k)|\ z_{s'}\in[z_1,\dots,z_k]\}\right| < \epsilon.$$

This is easy to do. It is enough to choose δ small enough so that the probability that a random point z_i is chosen from $\partial\mathcal{E}\cap H_s^-\cap H_{s'}^+$ is very small, e.g. $\delta=\ell^{-2}$ suffices.

(v) We assume that $x_0=0$, $N_{\partial K}(x_0)=e_n$, and $\gamma\geq 1$. We consider the transform dil : $\partial\mathcal{E}\to\partial(\frac{1}{\gamma}\mathcal{E})$ defined by $dil(x)=\frac{1}{\gamma}x$. Then

$$\text{dil}(\partial\mathcal{E}\cap H^-_{c\gamma\Delta})=\partial(\tfrac{1}{\gamma}\mathcal{E})\cap H^-_{c\Delta}\qquad \text{dil}(x_0-\gamma\Delta N_{\partial K}(x_0))=x_0-\Delta N_{\partial K}(x_0)$$

where $H_\Delta = H(x_0-\Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$. A surface element on $\partial\mathcal{E}$ is mapped onto one of $\partial(\frac{1}{\gamma}\mathcal{E})$ whose volume is smaller by the factor γ^{-n+1}. Therefore we get

$$\left|\mathbb{P}^k_{\partial(\frac{1}{\gamma}\mathcal{E})\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ \ x_0-\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\} - \mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\gamma\Delta}}\{(x_1,\dots,x_k)|\ x_0-\gamma\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\}\right| < \epsilon. \tag{129}$$

Now we apply the map $pd:\mathbb{R}^n\to\mathbb{R}^n$ with

$$pd(x)=(tx(1),\dots,tx(n-1),x(n)).$$

We choose t such that the lengths of the principal radii of curvature of $pd(\partial(\frac{1}{\gamma}\mathcal{E}))$ at x_0 coincide with those of $\partial\mathcal{E}$ at x_0. Thus $pd(\partial(\frac{1}{\gamma}\mathcal{E}))$ approximates $\partial\mathcal{E}$ well at x_0 and we can apply Lemma 1.2. See Figure 4.16.14.

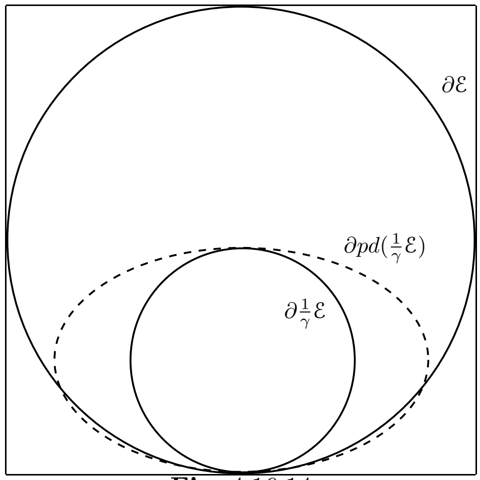

Fig. 4.16.14

The relation

$$x_0 - \Delta N_{\partial K}(x_0) \in [x_1, \dots, x_k]$$

holds if and only if

$$x_0 - \Delta N_{\partial K}(x_0) \in [pd(x_1), \dots, pd(x_k)].$$

Indeed, this follows from

$$x_0 - \Delta N_{\partial K}(x_0) = pd(x_0 - \Delta N_{\partial K}(x_0))$$

and

$$pd([x_1, \dots, x_k]) = [pd(x_1), \dots, pd(x_k)].$$

Let $x \in \partial(\frac{1}{\gamma}\mathcal{E})$ and let $N_{\partial(\frac{1}{\gamma}\mathcal{E})\cap H}(x)$ with $H = H(x, N_{\partial K}(x_0)) = H(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$ be the normal in H to $\partial(\frac{1}{\gamma}\mathcal{E}) \cap H$. Let α be the angle between $N_{\partial(\frac{1}{\gamma}\mathcal{E})}(x)$ and $N_{\partial(\frac{1}{\gamma}\mathcal{E})\cap H}(x)$.

Then a $n-2$-dimensional surface element in $\partial(\frac{1}{\gamma}\mathcal{E}) \cap H$ at x is mapped onto one in $\partial pd(\frac{1}{\gamma}\mathcal{E}) \cap H$ and the volume changes by a factor t^{n-2}. A $n-1$-dimensional surface element of $\partial(\frac{1}{\gamma}\mathcal{E})$ at x has the volume of a surface element of $\partial(\frac{1}{\gamma}\mathcal{E}) \cap H$ times $(\cos\alpha)^{-1} d\Delta$. When applying the map pd the tangent $\tan\alpha$ changes by the factor t (see Figure 4.16.15). Thus a $n-1$-dimensional surface element of $\partial(\frac{1}{\gamma}\mathcal{E})$ at x is mapped by pd onto one in $\partial pd(\frac{1}{\gamma}\mathcal{E})$ and its $n-1$-dimensional volume changes by the factor

$$t^{n-2} \cos\alpha \sqrt{1 + t^2 \tan^2\alpha} = t^{n-2} \sqrt{\cos^2\alpha + t^2 \sin^2\alpha}.$$

See Figure 4.16.15.

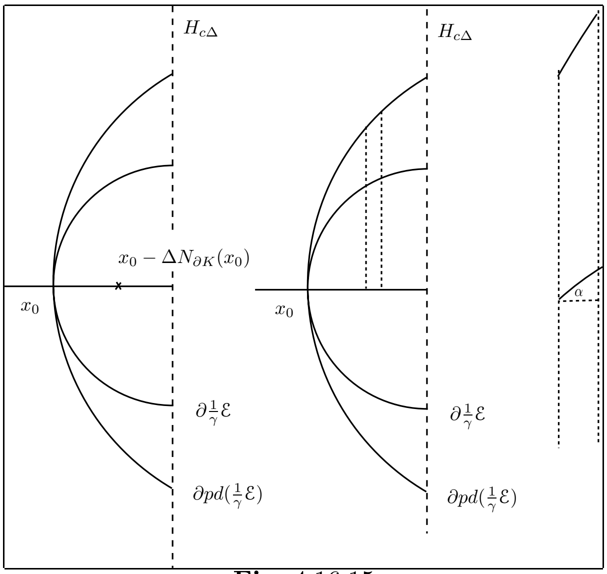

Fig. 4.16.15

If we choose Δ_0 sufficiently small then for all Δ with $0 < \Delta \leq \Delta_0$ the angle α will be very close to $\frac{\pi}{2}$. Thus, for every δ there is Δ_0 such that for all $x \in \partial(\frac{1}{\gamma}\mathcal{E}) \cap H^-(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0))$

$$(1-\delta)t^{n-1} \leq t^{n-2}\sqrt{\cos^2\alpha + t^2\sin^2\alpha} \leq (1+\delta)t^{n-1}.$$

Therefore, the image measure of the surface measure on $\partial(\frac{1}{\gamma}\mathcal{E})$ under the map *pd* has a density that deviates only by a small number from a constant function. More precisely, for every δ there is Δ_0 so that the density function differs only by δ from a constant function. By (i) of this lemma

$$\Big| \mathbb{P}^k_{\partial(\frac{1}{\gamma}\mathcal{E})\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ x_0 - \Delta N_{\partial K}(x_0) \in [x_1,\dots,x_k]\} - \mathbb{P}^k_{\partial pd(\frac{1}{\gamma}\mathcal{E})\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ x_0 - \Delta N_{\partial K}(x_0) \in [x_1,\dots,x_k]\}\Big| < \epsilon.$$

(In fact, we need only the continuity of this density function at x_0.) $\partial pd(\frac{1}{\gamma}\mathcal{E})$ and $\partial\mathcal{E}$ have the same principal curvature radii at x_0. Therefore, we can apply (ii) of this lemma and get

$$\Big| \mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ x_0 - \Delta N_{\partial K}(x_0) \in [x_1,\dots,x_k]\} - \mathbb{P}^k_{\partial(\frac{1}{\gamma}\mathcal{E})\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ x_0 - \Delta N_{\partial K}(x_0) \in [x_1,\dots,x_k]\}\Big| < \epsilon.$$

By (129)

$$\Big|\mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\Delta}}\{(x_1,\dots,x_k)|\ \ x_0-\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\}- \\ \mathbb{P}^k_{\partial\mathcal{E}\cap H^-_{c\gamma\Delta}}\{(x_1,\dots,x_k)|\ x_0-\gamma\Delta N_{\partial K}(x_0)\in[x_1,\dots,x_k]\}\Big|<\epsilon.$$

(vi) By (i) and (ii) of this lemma

$$\Big|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}- \\ \mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ x_s\in[z_1,\dots,z_k]\}\Big|<\epsilon$$

where H satisfies $\mathrm{vol}_{n-1}(\partial K\cap H^-)=cs$ and H is orthogonal to $N_{\partial K}(x_0)$. We choose $\tilde{s}$ so that

$$\{z_{\tilde{s}}\}=[x_0,z_T]\cap H(x_s,N_{\partial K}(x_0))\qquad\{z_{\tilde{s}}\}=[x_0,z_T]\cap\partial\mathcal{E}_{\tilde{s}}.$$

We have $(1-\epsilon)\tilde{s}\le\frac{s}{f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})}\le(1+\epsilon)\tilde{s}$. We verify this. For sufficiently small s_0 we have for all s with $0<s\le s_0$ and $H_s=H(x_s,N_{\partial K}(x_0))$

$$(1-\epsilon)s\le\int_{\partial K\cap H_s}f(x)\mathrm{d}\mu_{\partial K}\le(1+\epsilon)s.$$

(H and H_s are generally different.) By the continuity of f at x_0 we get for a new s_0 and all s with $0<s\le s_0$

$$(1-\epsilon)s\le f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H_s^-)\le(1+\epsilon)s.$$

Since

$$\tilde{s}=\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H_s^-)}{\mathrm{vol}_{n-1}(\partial\mathcal{E})}$$

we get the estimates on $\tilde{s}$.

By (iii) of this lemma

$$\Big|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}- \\ \mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ z_{\tilde{s}}\in[z_1,\dots,z_k]\}\Big|<\epsilon.$$

A perturbation argument allows us to assume that $\tilde{s}=\frac{s}{f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})}$. By (iv) we get for H with $\mathrm{vol}_{n-1}(\partial K\cap H^-)=cs$

$$\Big|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}- \\ \mathbb{P}^k_{\partial\mathcal{E}\cap H^-}\{(z_1,\dots,z_k)|\ z_{\tilde{s}}\in[z_1,\dots,z_k]\}\Big|<\epsilon.$$

Let L and $\tilde{L}$ be hyperplanes orthogonal to $N_{\partial K}(x_0)$ with $\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap L^-)=cs$ and $\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap\tilde{L}^-)=csf(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})$. By (v) of this lemma

$$\Big|\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{L}^-}\{(z_1,\dots,z_k)|\ z_s\in[z_1,\dots,z_k]\} \\ -\mathbb{P}^k_{\partial\mathcal{E}\cap L^-}\{(z_1,\dots,z_k)|\ z_{\tilde{s}}\in[z_1,\dots,z_k]\}\Big|<\epsilon.$$

In order to verify this it is enough to check that the quotient of the height of the cap $\partial\mathcal{E}\cap L^-$ and the distance of $z_{\tilde{s}}$ to x_0 equals up to a small error $(cf(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{2}{n-1}}$. Indeed, by Lemma 1.3 the height of the cap $\partial\mathcal{E}\cap\tilde{L}^-$ resp. the distance of z_s to x_0 equal up to a small error

$$\frac{1}{2}\left(\frac{csf(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})\sqrt{\kappa}}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}} \qquad \text{resp.} \qquad \frac{1}{2}\left(\frac{s\sqrt{\kappa}}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}.$$

For the height of the cap $\partial\mathcal{E}\cap L^-$ and the distance of $z_{\tilde{s}}$ to x_0

$$\frac{1}{2}\left(\frac{cs\sqrt{\kappa}}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}} \quad \text{resp.} \quad \frac{1}{2}\left(\frac{s\sqrt{\kappa}}{f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n-1}}.$$

Therefore the quotients are the same.

Since $\mathrm{vol}_{n-1}(\partial K\cap H^-) = cs$ and $\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap L^-) = cs$ and $\mathcal{E}$ is the standard approximating ellipsoid of K at x_0 we have

$$(1-\epsilon)cs \le \mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H^-) \le (1+\epsilon)cs$$

and

$$\begin{aligned}\Big|\mathbb{P}^k_{\partial\mathcal{E}\cap H^-}&\{(z_1,\dots,z_k)|\ z_{\tilde{s}}\in[z_1,\dots,z_k]\}\\ &-\mathbb{P}^k_{\partial\mathcal{E}\cap L^-}\{(z_1,\dots,z_k)|\ z_{\tilde{s}}\in[z_1,\dots,z_k]\}\Big| < \epsilon.\end{aligned}$$

Therefore

$$\begin{aligned}\Big|\mathbb{P}^k_{f,\partial K\cap H^-}\{(x_1,\dots,x_k)|\quad & x_s\in[x_1,\dots,x_k]\} -\\ &\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{L}^-}\{(z_1,\dots,z_k)|\ z_s\in[z_1,\dots,z_k]\}\Big| < \epsilon\end{aligned}$$

with $\mathrm{vol}_{n-1}(\partial K\cap H^-) = cs$ and $\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap\tilde{L}^-) = csf(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})$. Introducing the constant $c' = cf(x_0)$

$$\mathrm{vol}_{n-1}(\partial K\cap H^-) = \frac{c's}{f(x_0)} \qquad\qquad \frac{\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap\tilde{L}^-)}{\mathrm{vol}_{n-1}(\partial\mathcal{E})} = c's.$$

Since

$$(1-\epsilon)\mathbb{P}_f(\partial K\cap H^-) \le f(x_0)\mathrm{vol}_{n-1}(\partial K\cap H^-) \le (1+\epsilon)\mathbb{P}_f(\partial K\cap H^-)$$

we get the result. □

Lemma 4.17. *Let K be a convex body in $\mathbb{R}^n$ and $x_0\in\partial K$. Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder with a base that is an ellipsoid). Let $\mathcal{E}$ be the standard approximating ellipsoid at x_0. Let $f:\partial K\to\mathbb{R}$ be a continuous, positive function with $\int_{\partial K} f\mathrm{d}\mu = 1$. Let*

K_s be the surface body with respect to the density f and $\mathcal{E}_s$ the surface body with respect to the measure with the constant density $(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$ on $\partial\mathcal{E}$. Let x_s and z_s be defined by

$$\{x_s\} = [x_T, x_0] \cap \partial K_s \qquad \text{and} \qquad \{z_s\} = [z_T, x_0] \cap \partial \mathcal{E}_s.$$

Then for all $\epsilon > 0$ there is s_ϵ such for all $s \in [0, s_\epsilon]$ and for all $N \in \mathbb{N}$

$$\left|\mathbb{P}_f^N\{(x_1,\dots,x_N)| \quad x_s \notin [x_1,\dots,x_N]\} - \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s \notin [z_1,\dots,z_N]\}\right| < \epsilon.$$

Moreover, for all $\epsilon > 0$ there is a $\delta > 0$ such that we have for all s and s' with $0 < s, s' \le s_\epsilon$ and $(1-\delta)s \le s' \le (1+\delta)s$

$$\left|\mathbb{P}_f^N\{(x_1,\dots,x_N)| \quad x_s \notin [x_1,\dots,x_N]\} - \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_{s'} \notin [z_1,\dots,z_N]\}\right| < \epsilon.$$

Proof. For all $\alpha \ge 1$, for all s with $0 < s \le T$ and all $N \in \mathbb{N}$ with

$$N \le \frac{1}{\alpha s}$$

we have

$$\begin{aligned} 1 &\ge \mathbb{P}_f^N\{(x_1,\dots,x_N)|x_s \notin [x_1,\dots,x_N]\} \\ &\ge \mathbb{P}_f^N\{(x_1,\dots,x_N)|x_1,\dots,x_N \in (H^-(x_s, N_{\partial K_s}(x_s)) \cap \partial K)^\circ\} \\ &\ge (1-s)^N \ge \left(1 - \frac{1}{\alpha N}\right)^N \ge 1 - \frac{1}{\alpha} \end{aligned}$$

and

$$\begin{aligned} 1 &\ge \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|z_s \notin [z_1,\dots,z_N]\} \\ &\ge \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|z_1,\dots,z_N \in (H^-(z_s, N_{\partial\mathcal{E}_s}(z_s)) \cap \partial\mathcal{E})^\circ\} \\ &\ge (1-s)^N \ge 1 - sN \ge 1 - \frac{1}{\alpha}. \end{aligned}$$

Therefore, if we choose $\alpha \ge \frac{1}{\epsilon}$ we get for all N with $N < \frac{1}{\alpha s}$

$$|\mathbb{P}_f^N\{(x_1,\dots,x_N)|x_s \notin [x_1,\dots,x_N]\} - \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|z_s \notin [z_1,\dots,z_N]\}| \le \epsilon.$$

By Lemma 4.8 for a given x_0 there are constants a, b with $0 \le a, b < 1$, and s_ϵ such that we have for all s with $0 < s \le s_\epsilon$

$$\begin{aligned}&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}\\ &\qquad\le 2^n(a-as+s)^N+2^n(1-s+bs)^N\\ &\qquad\le 2^n\exp(N(\ln a+s(\tfrac{1}{a}-1)))+2^n\exp(-Ns(1-b)).\end{aligned}$$

We choose s_ϵ so small that $|\ln a|\ge 2s_\epsilon(\frac{1}{a}-1)$. Thus

$$\begin{aligned}&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}\\ &\qquad\le 2^n\exp(-\tfrac{1}{2}sN|\ln a|)+2^n\exp(-Ns(1-b)).\end{aligned}$$

Now we choose β so big that

$$2^n e^{-\beta(1-b)}<\tfrac{1}{2}\epsilon \qquad\text{and}\qquad 2^n e^{-\frac{1}{2}\beta|\ln a|}<\tfrac{1}{2}\epsilon.$$

Thus, for sufficiently small s_ϵ and all N with $N\ge\frac{\beta}{s}$ we get

$$\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}\le\epsilon$$

and

$$\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[z_1,\dots,z_N]\}\le\epsilon.$$

Please note that β depends only on a, b, n and ϵ. This leaves us with the case $\frac{1}{\alpha\ s}\le N\le\frac{\beta}{s}$.

We put $\gamma=\alpha\ \mathrm{vol}_{n-1}(\partial K)$. By Lemma 4.15 for all c with $c\ge c_0$ and γ there is $s_{c,\gamma}$ such that for all s with $0<s\le s_{c,\gamma}$ and for all $N\in\mathbb{N}$ with

$$N\ge\tfrac{1}{\gamma\ s}\mathrm{vol}_{n-1}(\partial K)=\frac{1}{\alpha s}$$

that

$$\begin{aligned}&\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ \ x_s\notin[x_1,\dots,x_N]\}-\\ &\qquad\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-]\}\big|\\ &\qquad\qquad\le 2^{n-1}\exp(-\tfrac{c_1}{\gamma}\sqrt{c})=2^{n-1}\exp\left(-\frac{c_1\sqrt{c}}{\alpha\mathrm{vol}_{n-1}(\partial K)}\right)\end{aligned}$$

where $H=H(x_0-c\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))$ and $\Delta=\Delta(s)$ as in Lemma 4.15. We choose c so big that

$$2^{n-1}\exp(-\tfrac{c_1}{\gamma}\sqrt{c})<\epsilon.$$

Thus for all ϵ there are c and s_ϵ such that for all s with $0<s\le s_\epsilon$

$$\begin{aligned}&\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\quad x_s\notin[x_1,\dots,x_N]\}-\\ &\qquad\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-]\}\big|\le\epsilon\end{aligned}$$

and in the same way that

$$\left|\mathbb{P}^N_{\partial\mathcal{E}}\{(x_1,\dots,x_N)|\quad z_s\notin[x_1,\dots,x_N]\}-\right.$$
$$\left.\mathbb{P}^N_{\partial\mathcal{E}}\{(x_1,\dots,x_N)|\ z_s\notin[\{x_1,\dots,x_N\}\cap H^-]\}\right|\le\epsilon.$$

By Lemma 1.3 there are constants c_1 and c_2 such that

$$c_1\Delta^{\frac{n-1}{2}}\le \mathrm{vol}_{n-1}(H^-(x_0-c\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))\cap\partial\mathcal{E})\le c_2\Delta^{\frac{n-1}{2}}$$

where Δ is the height of the cap. Now we adjust the cap that will allow us to apply Lemma 4.16. There is $d>0$ such that for all s with $0<s\le s_\epsilon$ there are hyperplanes H_{ds} and $\tilde{H}_{ds}$ that are orthogonal to $N_{\partial K}(x_0)$ and that satisfy

$$\mathbb{P}_f(\partial K\cap H^-_{ds})=ds\qquad\qquad \frac{\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap\tilde{H}^-_{ds})}{\mathrm{vol}_{n-1}(\partial\mathcal{E})}=ds$$

and

$$\begin{aligned}\partial K\cap H^-(x_0-c\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))&\subseteq\partial K\cap H^-_{ds}\\ \partial\mathcal{E}\cap H^-(x_0-c\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))&\subseteq\partial\mathcal{E}\cap\tilde{H}^-_{ds}.\end{aligned}$$

Thus we have for all s with $0<s\le s_\epsilon$

$$\left|\mathbb{P}^N_f\{(x_1,\dots,x_N)|\quad x_s\notin[x_1,\dots,x_N]\}-\right.\qquad(130)$$
$$\left.\mathbb{P}^N_f\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H^-_{ds}]\}\right|\le\epsilon$$

and

$$\left|\mathbb{P}^N_{\partial\mathcal{E}}\{(x_1,\dots,x_N)|\quad z_s\notin[x_1,\dots,x_N]\}-\right.\qquad(131)$$
$$\left.\mathbb{P}^N_{\partial\mathcal{E}}\{(x_1,\dots,x_N)|\ z_s\notin[\{x_1,\dots,x_N\}\cap\tilde{H}^-_{ds}]\}\right|\le\epsilon.$$

We choose ℓ so big that

$$\sum_{k=\ell}^{\infty}\frac{(d\beta)^k}{k!}<\epsilon.$$

By Lemma 4.16.(vi) we can choose s_ϵ so small that we have for all k with $1\le k\le\ell$

$$\left|\mathbb{P}^k_{f,\partial K\cap H^-_{ds}}\{(x_1,\dots,x_k)|\quad x_s\in[x_1,\dots,x_k]\}-\right.\qquad(132)$$
$$\left.\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds}}\{(z_1,\dots,z_k)|\ z_s\in[z_1,\dots,z_k]\}\right|<\epsilon.$$

We have

$$
\begin{aligned}
&|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\} \\
&\qquad\qquad -\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[z_1,\dots,z_N]\}| \\
&\le |\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\} \\
&\qquad\qquad -\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H_{ds}^-]\}| \\
&+|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H_{ds}^-]\} \\
&\qquad\qquad -\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap \tilde{H}_{ds}^-]\}| \\
&+|\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[z_1,\dots,z_N]\} \\
&\qquad\qquad -\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap \tilde{H}_{ds}^-]\}|.
\end{aligned}
\tag{133}
$$

By (130) and (131) the first and third summand are smaller than ϵ. It remains to estimate the second summand. We do this now. We have

$$
\begin{aligned}
&\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H_{ds}^-]\} \\
&\quad=\sum_{k=0}^{N}\binom{N}{k}\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_k],\ x_1,\dots,x_k\in H_{ds}^-, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_{k+1},\dots,x_N\in H_{ds}^+\} \\
&\quad=\sum_{k=0}^{N}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{f,\partial K\cap H_{ds}^-}\{(x_1,\dots,x_k)|\ x_s\notin[x_1,\dots,x_k]\}.
\end{aligned}
$$

Moreover, since $N\le\frac{\beta}{s}$ we have

$$
\begin{aligned}
&\sum_{k=\ell}^{N}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{f,\partial K\cap H_{ds}^-}\{(x_1,\dots,x_k)|\ x_s\notin[x_1,\dots,x_k]\} \\
&\qquad\qquad\le\sum_{k=\ell}^{N}\binom{N}{k}\left(\frac{d\beta}{N}\right)^k\le\sum_{k=\ell}^{N}\frac{(d\beta)^k}{k!}<\epsilon.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\Bigg|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H_{ds}^-]\}- \\
&\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{f,\partial K\cap H_{ds}^-}\{(x_1,\dots,x_k)|\ x_s\notin[x_1,\dots,x_k]\}\Bigg|<\epsilon.
\end{aligned}
$$

In the same way we get

$$
\begin{aligned}
&\Bigg|\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap\tilde{H}_{ds}^-]\} \\
&-\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}_{ds}^-}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\Bigg|<\epsilon.
\end{aligned}
$$

From these two inequalities we get

$$\Big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[\{x_1,\dots,x_N\}\cap H_{ds}^-]\}$$
$$-\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap\tilde{H}_{ds}^-]\}\Big|$$
$$\le 2\epsilon+$$
$$\Big|\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{f,\partial K\cap H_{ds}^-}\{(x_1,\dots,x_k)|\ x_s\notin[x_1,\dots,x_k]\}$$
$$-\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}_{ds}^-}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\Big|$$
$$=2\epsilon+$$
$$\Big|\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\big[\mathbb{P}^k_{f,\partial K\cap H_{ds}^-}\{(x_1,\dots,x_k)|\ x_s\notin[x_1,\dots,x_k]\}$$
$$-\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}_{ds}^-}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\big]\Big|.$$

By (132) the last expression is less than

$$2\epsilon+\epsilon\sum_{k=0}^{\ell-1}\binom{N}{k}(1-ds)^{N-k}(ds)^k\le 3\epsilon.$$

Together with (133) this gives the first inequality of the lemma.

We show now that for all $\epsilon>0$ there is a $\delta>0$ such that we have for all s and s' with $0<s,s'\le s_\epsilon$ and $(1-\delta)s\le s'\le(1+\delta)s$

$$\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\quad x_s\notin[x_1,\dots,x_N]\}-$$
$$\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_{s'}\notin[z_1,\dots,z_N]\}\big|<\epsilon.$$

Using the first inequality we see that it is enough to show that for all $\epsilon>0$ there is a $\delta>0$ such that we have for all s and s' with $0<s,s'\le s_\epsilon$ and $(1-\delta)s\le s'\le(1+\delta)s$

$$\big|\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\quad z_s\notin[z_1,\dots,z_N]\}-$$
$$\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_{s'}\notin[z_1,\dots,z_N]\}\big|<\epsilon.$$

As in the proof of the first inequality we show that we just have to consider the case $\frac{1}{\alpha\ s}\le N\le\frac{\beta}{s}$. We choose $\delta=\frac{\epsilon}{\ell}$. Thus δ depends on ℓ, but ℓ depends only on β and c. In particular, ℓ does not depend on N. As above, we write

$$\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap\tilde{H}_{ds}^-]\}$$
$$=\sum_{k=0}^{N}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}_{ds}^-}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}.$$

We get as above

$$\begin{aligned}&\left|\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ z_s\notin[\{z_1,\dots,z_N\}\cap\tilde{H}^-_{ds}]\}\right.\\&\qquad\qquad\left.-\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ z_{s'}\notin[\{z_1,\dots,z_N\}\cap\tilde{H}^-_{ds'}]\}\right|\\&\le\left|\sum_{k=0}^{\ell}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds}}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\right.\\&\quad\left.-\sum_{k=0}^{\ell}\binom{N}{k}(1-ds')^{N-k}(ds')^k\,\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds'}}\{(z_1,\dots,z_k)|\ z_{s'}\notin[z_1,\dots,z_k]\}\right|.\end{aligned}$$

This expression is not greater than

$$\begin{aligned}&\sum_{k=0}^{\ell}\binom{N}{k}\left[(1-ds)^{N-k}(ds)^k-(1-ds')^{N-k}(ds')^k\right]\\&\qquad\qquad\qquad\qquad\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds}}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\\&+\sum_{k=0}^{\ell}\binom{N}{k}(1-ds')^{N-k}(ds')^k\left|\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds'}}\{(z_1,\dots,z_k)|\ z_{s'}\notin[z_1,\dots,z_k]\}\right.\\&\qquad\qquad\qquad\qquad\left.-\mathbb{P}^k_{\partial\mathcal{E}\cap\tilde{H}^-_{ds}}\{(z_1,\dots,z_k)|\ z_s\notin[z_1,\dots,z_k]\}\right|.\end{aligned}$$

By Lemma 4.16.(iv) the second summand is smaller than

$$\epsilon\sum_{k=0}^{\ell}\binom{N}{k}(1-ds')^{N-k}(ds')^k\le\epsilon.$$

The first summand can be estimated by (we may assume that $s>s'$)

$$\begin{aligned}&\sum_{k=0}^{\ell}\binom{N}{k}\left[(1-ds)^{N-k}(ds)^k-(1-ds')^{N-k}(ds')^k\right]\\&=\sum_{k=0}^{\ell}\binom{N}{k}(1-ds)^{N-k}(ds)^k\left[1-\left(\frac{1-ds'}{1-ds}\right)^{N-k}\left(\frac{s'}{s}\right)^k\right].\end{aligned}$$

Since $s>s'$ we have $1-ds'\ge 1-ds$ and the above expression is smaller than

$$\begin{aligned}&\sum_{k=0}^{\ell}\binom{N}{k}(1-ds)^{N-k}(ds)^k\left[1-(1-\delta)^k\right]\\&\qquad\qquad\le\sum_{k=0}^{\ell}\binom{N}{k}(1-ds)^{N-k}(ds)^k\,k\delta\le\ell\delta.\end{aligned}$$

□

4.2 Probabilistic Estimates for Ellipsoids

Lemma 4.18. *Let $x_0 \in \partial B_2^n$ and let $(B_2^n)_s$ be the surface body with respect to the measure $\mathbb{P}_f$ with constant density $f = (\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}$. We have*

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{\frac{1}{2}} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(B_2^n\cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n\cap H_s)}(y)} \mathrm{d}s$$
$$= (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}$$

where $H_s = H(x_s, N_{\partial(B_2^n)_s}(x_s))$ and $\{x_s\} = [0,x_0] \cap \partial(B_2^n)_s$. (Let us note that $N_{\partial(B_2^n)_s}(x_s) = x_0$ and $N_{\partial B_2^n}(y) = y$.)

Proof. Clearly, for all s with $0 \le s < \frac{1}{2}$ the surface body $(B_2^n)_s$ is homothetic to B_2^n. We have

$$\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N) = \int_{B_2^n} \mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|x \notin [x_1,\dots,x_N]\}\mathrm{d}x.$$

We pass to polar coordinates

$$\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N) = \int_0^1 \int_{\partial B_2^n} \mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|r\xi \notin [x_1,\dots,x_N]\} r^{n-1}\mathrm{d}\xi \mathrm{d}r$$

where $\mathrm{d}\xi$ is the surface measure on ∂B_2^n. Since B_2^n is rotationally invariant

$$\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|r\xi \notin [x_1,\dots,x_N]\}$$

is independent of ξ. We get that the last expression equals

$$\mathrm{vol}_{n-1}(\partial B_2^n) \int_0^1 \mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|r\xi \notin [x_1,\dots,x_N]\} r^{n-1}\mathrm{d}r$$

for all $\xi \in \partial B_2^n$. Now we perform a change of variable. We define the function $s : [0,1] \to [0,\frac{1}{2}]$ by

$$s(r) = \frac{\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-(r\xi,\xi))}{\mathrm{vol}_{n-1}(\partial B_2^n)}.$$

The function is continuous, strictly decreasing, and invertible. We have by Lemma 2.11.(iii)

$$\frac{\mathrm{d}s}{\mathrm{d}r} = -\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1- < N_{\partial(B_2^n)_s}(x_s), N_{\partial B_2^n}(y) >^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n \cap H_s)}(y).$$

We have $r(s)\xi = x_s$. Thus we get

$$\frac{\mathrm{vol}_n(B_2^n) - \mathbb{E}(\partial B_2^n, N)}{\mathrm{vol}_{n-1}(\partial B_2^n)} = \int_0^{\frac{1}{2}} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N) |\ x_s \notin [x_1,\dots,x_N]\}(r(s))^{n-1}\mathrm{d}s}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n \cap H_s)}(y)}.$$

Now we apply Proposition 3.1 and obtain

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{\frac{1}{2}} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N) |\ x_s \notin [x_1,\dots,x_N]\}(r(s))^{n-1}\mathrm{d}s}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n \cap H_s)}(y)} = (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.$$

By Lemma 4.13 it follows that we have for all s_0 with $0 < s_0 \le \frac{1}{2}$

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{s_0} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N) |\ x_s \notin [x_1,\dots,x_N]\}(r(s))^{n-1}\mathrm{d}s}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n \cap H_s)}(y)} = (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.$$

By this and since $r(s)$ is a continuous function with $\lim_{s\to 0} r(s) = 1$ we get

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{\frac{1}{2}} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N) |\ x_s \notin [x_1,\dots,x_N]\}\mathrm{d}s}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(x_s),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n \cap H_s)}(y)} = (n-1)^{\frac{n+1}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}.$$

□

Lemma 4.19. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder with a base that is an ellipsoid). Let $f, g : \partial K \to \mathbb{R}$ be continuous, strictly positive functions with*

$$\int_{\partial K} f \mathrm{d}\mu = \int_{\partial K} g \mathrm{d}\mu = 1.$$

Let

$$\mathbb{P}_f = f \mathrm{d}\mu_{\partial K} \quad \text{and} \quad \mathbb{P}_g = g \mathrm{d}\mu_{\partial K}.$$

Then for all $\epsilon > 0$ *there is* s_ϵ *such that we have for all* $0 < s < s_\epsilon$*, all* x_s *with* $\{x_s\} = [0, x_0] \cap \partial K_{f,s}$*, all* $\{y_s\} = [0, x_0] \cap \partial K_{g,s}$*, and all* $N \in \mathbb{N}$

$$|\mathbb{P}_f^N\{(x_1,\dots,x_N)|x_s \notin [x_1,\dots,x_N]\} - \mathbb{P}_g^N\{(x_1,\dots,x_N)|y_s \notin [x_1,\dots,x_N]\}| < \epsilon.$$

Proof. By Lemma 4.17

$$\begin{aligned} \big|\mathbb{P}_f^N\{(x_1,\dots,x_N)| \quad & x_s \notin [x_1,\dots,x_N]\} - \\ & \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s \notin [z_1,\dots,z_N]\}\big| < \epsilon, \end{aligned}$$

and

$$\begin{aligned} \big|\mathbb{P}_g^N\{(x_1,\dots,x_N)| \quad & y_s \notin [x_1,\dots,x_N]\} - \\ & \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ z_s \notin [z_1,\dots,z_N]\}\big| < \epsilon. \end{aligned}$$

The result follows by triangle-inequality. □

Lemma 4.20. *Let* $a_1, \dots, a_n > 0$ *and let* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be defined by* $Ax = (a_i x(i))_{i=1}^n$*. Let* $\mathcal{E} = A(B_2^n)$*, i.e.*

$$\mathcal{E} = \left\{ x \left| \sum_{i=1}^n \left| \frac{x(i)}{a_i} \right|^2 \le 1 \right. \right\}.$$

Let $f : \partial\mathcal{E} \to \mathbb{R}$ *be given by*

$$f(x) = \left(\left(\prod_{i=1}^n a_i \right) \sqrt{\sum_{i=1}^n \frac{x(i)^2}{a_i^4}} \operatorname{vol}_{n-1}(\partial B_2^n) \right)^{-1}.$$

Then we have $\int_{\partial\mathcal{E}} f \mathrm{d}\mu_{\partial\mathcal{E}} = 1$ *and for all* $x \in B_2^n$

$$\mathbb{P}_{\partial B_2^n}^N\{(x_1,\dots,x_N)|x \notin [x_1,\dots,x_N]\} = \mathbb{P}_f^N\{(z_1,\dots,z_N)|A(x) \notin [z_1,\dots,z_N]\}.$$

Proof. We have that

$$x \notin [x_1,\dots,x_N] \qquad \text{if and only if} \qquad Ax \notin [Ax_1,\dots,Ax_N].$$

For all subsets M of $\partial\mathcal{E}$ such that $A^{-1}(M)$ is measurable we put

$$\nu(M) = \mathbb{P}_{\partial B_2^n}(A^{-1}(M))$$

and get

$$\mathbb{P}_{\partial B_2^n}^N\{(x_1,\dots,x_N)|x \notin [x_1,\dots,x_N]\} = \nu^N\{(z_1,\dots,z_N)|Ax \notin [z_1,\dots,z_N]\}.$$

We want to apply the Theorem of Radon-Nikodym. ν is absolutely continuous with respect to the surface measure $\mu_{\partial\mathcal{E}}$. We check this.

$$\nu(M) = \mathbb{P}_{\partial B_2^n}(A^{-1}(M)) = \frac{h_{n-1}(A^{-1}(M))}{\mathrm{vol}_{n-1}(\partial B_2^n)}$$

where h_{n-1} is the $n-1$-dimensional Hausdorff-measure. By elementary properties of the Hausdorff-measure ([EvG], p. 75) we get

$$\nu(M) \le (\mathrm{Lip}(A))^{n-1}\frac{h_{n-1}(M)}{\mathrm{vol}_{n-1}(\partial B_2^n)} = (\mathrm{Lip}(A))^{n-1}\frac{1}{\mathrm{vol}_{n-1}(\partial B_2^n)}\mu_{\partial\mathcal{E}}(M)$$

where $\mathrm{Lip}(A)$ is the Lipschitz-constant of A. Thus $\nu(M) = 0$ whenever $\mu_{\partial\mathcal{E}}(M) = 0$.

Therefore, by the Theorem of Radon-Nikodym there is a density f such that $\mathrm{d}\nu = f\mathrm{d}\mu_{\partial\mathcal{E}}$. The density is given by

$$f(x) = \left(\left(\prod_{i=1}^n a_i\right)\sqrt{\sum_{i=1}^n \frac{x(i)^2}{a_i^4}}\mathrm{vol}_{n-1}(\partial B_2^n)\right)^{-1}.$$

We show this. We may assume that $x(n) \ge \frac{a_n}{\sqrt{n}}$ (there is at least one coordinate $x(i)$ with $|x(i)| \ge \frac{a_i}{\sqrt{n}}$). Let U be a small neighborhood of x in $\partial\mathcal{E}$. We may assume that for all $y \in U$ we have $y(n) \ge \frac{a_n}{2\sqrt{n}}$. Thus the orthogonal projection p_{e_n} onto the subspace orthogonal to e_n is injective on U. Since $x \in \partial\mathcal{E}$ we have $(\frac{x(i)}{a_i})_{i=1}^n \in \partial B_2^n$ and $N_{\partial B_2^n}(A^{-1}(x)) = (\frac{x(i)}{a_i})_{i=1}^n$. Then we have up to a small error

$$\begin{aligned}\nu(U) &= \mathbb{P}_{\partial B_2^n}(A^{-1}(U)) \\ &\sim \frac{\mathrm{vol}_{n-1}(p_{e_n}(A^{-1}(U)))}{< e_n, N_{\partial B_2^n}(A^{-1}(x)) > \ \mathrm{vol}_{n-1}(\partial B_2^n)} = \frac{a_n \mathrm{vol}_{n-1}(p_{e_n}(A^{-1}(U)))}{x(n)\ \mathrm{vol}_{n-1}(\partial B_2^n)}.\end{aligned}$$

Moreover, since

$$N_{\partial\mathcal{E}}(x) = \left(\sum_{i=1}^n \frac{x(i)^2}{a_i^4}\right)^{-\frac{1}{2}}\left(\frac{x(i)}{a_i^2}\right)_{i=1}^n$$

we have

$$\mu_{\partial\mathcal{E}}(U) \sim \frac{\mathrm{vol}_{n-1}(p_{e_n}(U))}{< e_n, N_{\partial\mathcal{E}}(x) >} = a_n^2\sqrt{\sum_{i=1}^n \frac{x(i)^2}{a_i^4}}\left(\frac{\mathrm{vol}_{n-1}(p_{e_n}(U))}{x(n)}\right).$$

We also have that

$$\mathrm{vol}_{n-1}(p_{e_n}(U)) = \left(\prod_{i=1}^{n-1} a_i\right) \mathrm{vol}_{n-1}(p_{e_n}(A^{-1}(U))).$$

Therefore we get

$$\begin{aligned}\mu_{\partial\mathcal{E}}(U) &\sim a_n \left(\prod_{i=1}^{n} a_i\right) \sqrt{\sum_{i=1}^{n} \frac{x(i)^2}{a_i^4}} \left(\frac{\mathrm{vol}_{n-1}(p_{e_n}(A^{-1}(U)))}{x(n)}\right) \\ &\sim \left(\prod_{i=1}^{n} a_i\right) \sqrt{\sum_{i=1}^{n} \frac{x(i)^2}{a_i^4}} \mathrm{vol}_{n-1}(\partial B_2^n)\nu(U).\end{aligned}$$

□

Lemma 4.21. *Let* $a_1, \dots, a_n > 0$ *and*

$$\mathcal{E} = \left\{ x \left| \sum_{i=1}^{n} \left|\frac{x(i)}{a_i}\right|^2 \le 1 \right.\right\}$$

Let $\mathcal{E}_s$, $0 < s \le \frac{1}{2}$, *be the surface body with respect to the measure* $\mathbb{P}_g$ *with constant density* $g = (\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$. *Moreover, let* $\lambda_{\mathcal{E}} : [0, \frac{1}{2}] \to [0, a_n]$ *be such that* $\lambda_{\mathcal{E}}(s)e_n \in \partial\mathcal{E}_s$. *Then we have for all* t *with* $0 \le t \le \frac{1}{2}$

$$\begin{aligned}&\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^t \frac{\mathbb{P}_{\partial\mathcal{E}}^N\{(x_1,\dots,x_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \\ &= a_n \left(\prod_{i=1}^{n-1} a_i\right)^{-\frac{2}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\end{aligned}$$

where $H_s = H(\lambda_{\mathcal{E}}(s)e_n, N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n))$. *(Please note that* $N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n) = e_n$.*)*

Proof. $(B_2^n)_t$, $0 < t \le \frac{1}{2}$, are the surface bodies with respect to the constant density $(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}$. $\lambda_B : [0, \frac{1}{2}] \to [0, 1]$ is the map defined by $\lambda_B(t)e_n \in \partial(B_2^n)_t$.

By Lemma 4.18

$$\begin{aligned}&\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{\frac{1}{2}} \frac{\mathbb{P}_{\partial B_2^n}^N\{(x_1,\dots,x_N)|\ \lambda_B(s)e_n \notin [x_1,\dots,x_N]\}}{\int_{\partial(B_2^n\cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(\lambda_B(s)e_n),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n\cap H_s)}(y)} \mathrm{d}s \\ &= \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\end{aligned}$$

where $\lambda_B(s)e_n \in \partial(B_2^n)_s$ and $H_s = H(\lambda_B(s)e_n, e_n)$. By Lemma 4.13 for c with $c_0 < c$ and N with $N_0 < N$

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|\ \lambda_B(s)e_n \notin [x_1,\dots,x_N]\}}{\int_{\partial B_2^n \cap H_s} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(\lambda_B(s)e_n),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n\cap H_s)}(y)} \mathrm{d}s \right.$$
$$\left. - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}} \right| \le c_1 \mathrm{e}^{-c} + c_2 \mathrm{e}^{-c_3 N}.$$

Let A be the diagonal operator with $A(x) = (a_i x_i)_{i=1}^n$ such that $A(B_2^n) = \mathcal{E}$. By Lemma 4.20 we have

$$\mathbb{P}^N_{\partial B_2^n}\{(x_1,\dots,x_N)|\quad A^{-1}(x) \notin [x_1,\dots,x_N]\}$$
$$= \mathbb{P}^N_f\{(z_1,\dots,z_N)|\ x \notin [z_1,\dots,z_N]\}$$

where $f : \partial\mathcal{E} \to (0,\infty)$

$$f(x) = \left(\left(\prod_{i=1}^n a_i\right) \sqrt{\sum_{i=1}^n \frac{x(i)^2}{a_i^4}} \mathrm{vol}_{n-1}(\partial B_2^n)\right)^{-1}.$$

For all c with $c_0 < c$ and N with $N_0 < N$

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_f\{(z_1,\dots,z_N)|\ A(\lambda_B(s)e_n) \notin [z_1,\dots,z_N]\}}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(\lambda_B(s)e_n),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n\cap H_s)}(y)} \mathrm{d}s \right.$$
$$\left. - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}} \right| \le c_1 \mathrm{e}^{-c} + c_2 \mathrm{e}^{-c_3 N}.$$

The functions λ_B and $\lambda_{\mathcal{E}}$ are strictly decreasing, bijective, continuous functions. Therefore, the function $s : [0, a_n] \to [0, 1]$

$$s(t) = \lambda_B^{-1}\left(\frac{\lambda_{\mathcal{E}}(t)}{a_n}\right)$$

exists, is continuous and has $t : [0, 1] \to [0, a_n]$

$$t(s) = \lambda_{\mathcal{E}}^{-1}(a_n \lambda_B(s))$$

as its inverse function. Clearly, $a_n \lambda_B(s(t)) = \lambda_{\mathcal{E}}(t)$ and $A(\lambda_B(s(t))e_n) = \lambda_{\mathcal{E}}(t)e_n$. Thus

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_f\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(t(s))e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(B_2^n \cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}}{(1-<N_{\partial(B_2^n)_s}(\lambda_B(s)e_n),N_{\partial B_2^n}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(B_2^n\cap H_s)}(y)} \mathrm{d}s \right.$$
$$\left. - \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}} \right| \le c_1 \mathrm{e}^{-c} + c_2 \mathrm{e}^{-c_3 N}.$$

Now we perform a change of variable. By Lemma 2.11.(iii) and $a_n\lambda_B(s(t)) = \lambda_{\mathcal{E}}(t)$

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{1}{a_n}\cdot\frac{\frac{\mathrm{d}\lambda_{\mathcal{E}}}{\mathrm{d}t}(t)}{\frac{\mathrm{d}\lambda_B}{\mathrm{d}s}(s(t))}\\ &= \frac{1}{a_n}\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-1}(\partial B_2^n)}\frac{\int_{\partial B_2^n\cap H(\lambda_B(s(t))e_n,e_n)}\frac{\mathrm{d}\mu_{\partial B_2^n\cap H(\lambda_B(s(t))e_n,e_n)}(y)}{\sqrt{1-<e_n,N(y)>^2}}}{\int_{\partial\mathcal{E}\cap H(\lambda_{\mathcal{E}}(t)e_n,e_n)}\frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H(\lambda_{\mathcal{E}}(t)e_n,e_n)}(y)}{\sqrt{1-<e_n,N(y)>^2}}}.\end{aligned}$$

Therefore we get for all c with $c_0 < c$ and N with $N_0 < N$

$$\begin{aligned}&\Bigg|N^{\frac{2}{n-1}}\int_0^{t(\frac{c}{N})}\frac{\mathbb{P}_f^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(t)e_n\notin[z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_t)}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t\\ &\qquad -a_n\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\Bigg|\\ &\le\frac{1}{a_n}\left[c_1\mathrm{e}^{-c}+c_2\mathrm{e}^{-c_3N}\right]\end{aligned}$$

where H_t now denotes $H(\lambda_{\mathcal{E}}(t)e_n, N(\lambda_{\mathcal{E}}(t)e_n))$. Since $a_n\lambda_B(s(t)) = \lambda_{\mathcal{E}}(t)$ we get that for sufficiently small t the quantities t and s are up to a small error directly proportional. We have

$$t(s)\sim s\frac{c_n a_n^{\frac{n-1}{2}}}{\kappa(a_ne_n)^{\frac{n-1}{4}}}.$$

Therefore, with a constant α and new constants c_1, c_2 we can substitute $t(\frac{c}{N})$ by $\frac{c}{N}$.

$$\begin{aligned}&\Bigg|N^{\frac{2}{n-1}}\int_0^{\frac{c}{N}}\frac{\mathbb{P}_f^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(t)e_n\notin[z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_t)}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t\\ &\qquad -a_n\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\Bigg|\\ &\le c_1\mathrm{e}^{-\alpha c}+c_2\mathrm{e}^{-c_3N}\end{aligned}$$

We have $\lambda_{\mathcal{E}}(tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E}))e_n\in\partial\mathcal{E}_{t'}$ with $t' = tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E})$. By Lemma 2.7.(i) for every $\delta > 0$ there is t'' with $\lambda_{\mathcal{E}}(t)e_n\in\partial\mathcal{E}_{f,t''}$ and

$$(1-\delta)tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E})\le t''\le(1+\delta)tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E})$$

i.e.

$$(1-\delta)t'\le t''\le(1+\delta)t'.$$

Applying Lemma 4.17 gives

$$\Big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ \ \lambda_{\mathcal{E}}(t)e_n\notin[x_1,\dots,x_N]\}-\\ \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E}))e_n\notin[z_1,\dots,z_N]\}\Big|<\epsilon.$$

Therefore

$$\left|N^{\frac{2}{n-1}}\int_0^{\frac{c}{N}}\frac{\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E}))e_n\notin[z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_t)}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac12}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t\right.$$
$$\left.-a_n\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\right|$$

$$\leq\left|N^{\frac{2}{n-1}}\int_0^{\frac{c}{N}}\frac{\epsilon}{\int_{\partial(\mathcal{E}\cap H_t)}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac12}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t\right|$$
$$+c_1\mathrm{e}^{-\alpha c}+c_2\mathrm{e}^{-c_3N}.$$

By Lemma 4.11

$$\int_{\partial\mathcal{E}\cap H_t}(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{-\frac12}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)\geq\gamma t^{\frac{n-3}{n-1}}.$$

Therefore we have

$$\int_0^{\frac{c}{N}}\frac{\epsilon}{\int_{\partial\mathcal{E}\cap H_t}(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{-\frac12}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t$$
$$\leq\frac{\epsilon}{\gamma}\int_0^{\frac{c}{N}}t^{-\frac{n-3}{n-1}}\mathrm{d}t=\frac{\epsilon}{\gamma}\frac{n-1}{2}\left(\frac{c}{N}\right)^{\frac{2}{n-1}}.$$

Therefore

$$\left|N^{\frac{2}{n-1}}\int_0^{\frac{c}{N}}\frac{\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E}))e_n\notin[z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_t)}\frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac12}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t\right.$$
$$\left.-a_n\left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}}\frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}\right|$$
$$\leq\frac{\epsilon}{\gamma}\frac{n-1}{2}\left(\frac{c}{N}\right)^{\frac{2}{n-1}}+c_1\mathrm{e}^{-\alpha c}+c_2\mathrm{e}^{-c_3N}.$$

We perform another transform, $u=tf(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E})$. With a new constant α

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(u)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_{t(u)})} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_{t(u)}}(\lambda_{\mathcal{E}}(t(u))e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_{t(u)})}(y)} \right.$$

$$\left. \times \frac{\mathrm{d}u}{f(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E})} - a_n \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(n-1)^{-\frac{n+1}{n-1}}} \right|$$

$$\leq \frac{\epsilon}{\gamma}\frac{n-1}{2}\left(\frac{c}{N}\right)^{\frac{2}{n-1}} + c_1\mathrm{e}^{-\alpha c} + c_2\mathrm{e}^{-c_3N}.$$

By Lemma 2.10.(iii)

$$\int_{\partial\mathcal{E}\cap H_u} \frac{1}{\sqrt{1-<N_{\partial\mathcal{E}_u}(x_u),N_{\partial\mathcal{E}}(y)>^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H_u}(y)$$
$$\leq (1+\epsilon)(\tfrac{u}{t})^{\frac{n-3}{n-1}} \int_{\partial\mathcal{E}\cap H_t} \frac{1}{\sqrt{1-<N_{\partial\mathcal{E}_t}(x_t),N_{\partial\mathcal{E}}(y)>^2}} \mathrm{d}\mu_{\partial\mathcal{E}\cap H_t}(y)$$

and the inverse inequality. Thus

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(u)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_u)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_u}(\lambda_{\mathcal{E}}(u)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_u)}(y)} \times \right.$$

$$\left. \frac{\mathrm{d}u}{(f(a_ne_n)\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{2}{n-1}}} - a_n \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(n-1)^{-\frac{n+1}{n-1}}} \right|$$

$$\leq \epsilon + \frac{\epsilon}{\gamma}\frac{n-1}{2}\left(\frac{c}{N}\right)^{\frac{2}{n-1}} + c_1\mathrm{e}^{-\alpha c} + c_2\mathrm{e}^{-c_3N}.$$

Since $f(a_ne_n) = ((\prod_{i=1}^{n-1} a_i)\mathrm{vol}_{n-1}(\partial B_2^n))^{-1}$

$$\left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(u)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_u)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_u}(\lambda_{\mathcal{E}}(u)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_u)}(y)} \mathrm{d}u \right.$$

$$\left. -a_n \left(\prod_{i=1}^{n-1} a_i\right)^{-\frac{2}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}} \right|$$

$$\leq \left(\frac{\mathrm{vol}_{n-1}(\partial B_2^n)}{\mathrm{vol}_{n-1}(\partial\mathcal{E})} \prod_{i=1}^{n-1} a_i\right)^{\frac{2}{n-1}} \left(\epsilon + \frac{\epsilon}{\gamma}\frac{n-1}{2}\left(\frac{c}{N}\right)^{\frac{2}{n-1}} + c_1\mathrm{e}^{-\alpha c} + c_2\mathrm{e}^{-c_3N}\right).$$

By choosing first c sufficiently big and then ϵ sufficiently small we get the above expression as small as possible provided that N is sufficiently large. By this and Lemma 4.13

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{t_0} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(t)e_n)\notin[z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_t)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_t}(\lambda_{\mathcal{E}}(t)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_t)}(y)}\mathrm{d}t$$

$$= a_n\left(\prod_{i=1}^{n-1} a_i\right)^{-\frac{2}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}.$$

□

5 Proof of the Theorem

Lemma 5.1. *Let K be a convex body in $\mathbb{R}^n$ such that the generalized Gauß-curvature exists at $x_0 \in \partial K$ and is not 0. Let $f : \partial K \to \mathbb{R}$ be a continuous, strictly positive function with $\int_{\partial K} f\mathrm{d}\mu = 1$. Let K_s be the surface body with respect to the measure $f\mathrm{d}\mu$. Let $\{x_s\} = [x_T, x_0] \cap K_s$ and $H_s = H(x_s, N_{\partial K_s}(x_s))$. Assume that there are r and R with $0 < r, R < \infty$ and*

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \subseteq K \subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R).$$

Then for all s_0 with $0 < s_0 \le T$

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{s_0} \frac{\mathbb{P}^N_f\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}\mathrm{d}s = c_n \frac{\kappa(x_0)^{\frac{1}{n-1}}}{f(x_0)^{\frac{2}{n-1}}}$$

where

$$c_n = \frac{(n-1)^{\frac{n+1}{n-1}}\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}.$$

We can recover Lemma 4.21 from Lemma 5.1 by choosing $K = \mathcal{E}$ and $f = (\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$.

Proof. Let $\mathcal{E}$ be the standard approximating ellipsoid at x_0 with principal axes having the lengths a_i, $i = 1,\dots,n-1$. Then we have (4)

$$\kappa(x_0) = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}.$$

Therefore, by Lemma 4.21 we get for all s_0 with $0 < s_0 \le \frac{1}{2}$

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{s_0} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s$$

$$= a_n \left(\prod_{i=1}^{n-1} a_i\right)^{-\frac{2}{n-1}} \left(\frac{\mathrm{vol}_{n-1}(\partial\mathcal{E})}{\mathrm{vol}_{n-2}(\partial B_2^{n-1})}\right)^{\frac{2}{n-1}} \frac{\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!}(n-1)^{\frac{n+1}{n-1}}$$

$$= c_n \kappa^{\frac{1}{n-1}}(x_0)(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{2}{n-1}}$$

where

$$c_n = \frac{(n-1)^{\frac{n+1}{n-1}}\Gamma(n+1+\frac{2}{n-1})}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}$$

and $H_s = H(\lambda_{\mathcal{E}}(s)e_n, e_n)$. H_s is a tangent hyperplane to the surface body $\mathcal{E}_s$ with respect to the constant density $(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}$.

By this for all $\epsilon > 0$ and sufficiently big N

$$\left| N^{\frac{2}{n-1}} \int_0^{s_0} \frac{\mathbb{P}^N_f\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}s - c_n \frac{\kappa(x_0)^{\frac{1}{n-1}}}{f(x_0)^{\frac{2}{n-1}}} \right|$$

$$\leq \epsilon +$$

$$\left| N^{\frac{2}{n-1}} \int_0^{s_0} \frac{\mathbb{P}^N_f\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}s \right.$$

$$- \left(\frac{N}{f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E})}\right)^{\frac{2}{n-1}} \times$$

$$\left. \int_0^{s_0} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-1}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \right|.$$

By Lemma 4.13 there are constants b_1, b_2, b_3 such that for all sufficiently big c the latter expression is smaller than

$$\epsilon + 2(b_1 e^{-c} + b_2 e^{-b_3 N})$$

$$+ \left| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_f\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}s \right.$$

$$\left. - N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}^N_{\partial\mathcal{E}}\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \right|.$$

By triangle-inequality this is smaller than

$$
\begin{aligned}
&\epsilon + 2(b_1 e^{-c} + b_2 e^{-b_3 N}) \\
&\quad + \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \\
&\qquad - \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \Bigg| \\
&\quad + \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [z_1,\dots,z_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \\
&\qquad - \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \Bigg| .
\end{aligned}
$$

By Lemma 4.17

$$
\begin{aligned}
\big|\mathbb{P}_f^N\{(x_1,\dots,x_N)|\quad & x_s \notin [x_1,\dots,x_N]\} - \\
& \mathbb{P}_{\partial\mathcal{E}}^N\{(z_1,\dots,z_N)|\ \lambda_{\mathcal{E}}(s)e_n \notin [z_1,\dots,z_N]\}\big| < \epsilon.
\end{aligned}
$$

Therefore, the above quantity is less than

$$
\begin{aligned}
&\epsilon + 2(b_1 e^{-c} + b_2 e^{-b_3 N}) \\
&\quad + \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \\
&\qquad - \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \Bigg| \\
&\quad + \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\epsilon}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \Bigg| .
\end{aligned}
$$

By Lemma 4.11 we have

$$
\begin{aligned}
&\int_0^{\frac{c}{N}} \frac{1}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{\mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}s \\
&\quad \le c_0^n \frac{R^{n-1}}{r^n} (\mathrm{vol}_{n-1}(B_2^{n-1}))^{-\frac{2}{n-1}} (\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}} \int_0^{\frac{c}{N}} s^{-\frac{n-3}{n-1}} \mathrm{d}s \\
&\quad = c_0^n \frac{R^{n-1}}{r^n} (\mathrm{vol}_{n-1}(B_2^{n-1}))^{-\frac{2}{n-1}} (\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}} \frac{n-1}{2} \left(\frac{c}{N}\right)^{\frac{2}{n-1}} .
\end{aligned}
$$

Therefore, the above expression is not greater than

$$\epsilon + b_1 \mathrm{e}^{-c} + b_2 \mathrm{e}^{-b_3 N} + b_4 \epsilon\, c^{\frac{2}{n-1}}$$
$$+ \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H(x_s,N_{\partial K_s}(x_s)))}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}$$
$$- \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H_s)} \frac{f(x_0)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{-\frac{n-3}{n-1}}}{(1-<N_{\partial\mathcal{E}_s}(\lambda_{\mathcal{E}}(s)e_n),N_{\partial\mathcal{E}}(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H_s)}(y)} \mathrm{d}s \Bigg|$$

for some constant b_4. Let z_s be defined by

$$\{z_s\} = \{x_0 + tN_{\partial K}(x_0) | t \in \mathbb{R}\} \cap H(x_s, N_{\partial K}(x_0)).$$

By Lemma 2.7 there is a sufficiently small s_ϵ such that we have for all s with $0 < s \le s_\epsilon$

$$s \le \mathbb{P}_f(\partial K \cap H^-(z_s, N_{\partial K}(x_0))) \le (1+\epsilon)s.$$

Because f is continuous at x_0 and because $\mathcal{E}$ is the standard approximating ellipsoid at x_0 we have for all s with $0 < s \le s_\epsilon$

$$(1-\epsilon)s \le f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E} \cap H^-(z_s, N_{\partial K}(x_0))) \le (1+\epsilon)s.$$

Since $s = \frac{\mathrm{vol}_{n-1}(\partial\mathcal{E}\cap H_s^-)}{\mathrm{vol}_{n-1}(\partial\mathcal{E})}$ we get by Lemma 2.10.(iii) for a new s_ϵ that for all s with $0 < s \le s_\epsilon$

$$\frac{(1-\epsilon)}{(f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{n-3}{n-1}}} \int_{\partial\mathcal{E}\cap H_s} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H_s}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_s}(x_0),N_{\partial\mathcal{E}}(y)>^2}}$$
$$\le \int_{\partial\mathcal{E}\cap H^-(z_s,N_{\partial K}(x_0))} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H^-(z_s,N_{\partial K}(x_0))}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_t}(x_0),N_{\partial\mathcal{E}}(y)>^2}}$$
$$\le \frac{(1+\epsilon)}{(f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{n-3}{n-1}}} \int_{\partial\mathcal{E}\cap H_s} \frac{\mathrm{d}\mu_{\partial\mathcal{E}\cap H_s}(y)}{\sqrt{1-<N_{\partial\mathcal{E}_s}(x_0),N_{\partial\mathcal{E}}(y)>^2}}$$

where $t \sim s(f(x_0)\mathrm{vol}_{n-1}(\partial\mathcal{E}))^{\frac{n-3}{n-1}}$. Please note that $N_{\partial K}(x_0) = N_{\partial\mathcal{E}_s}(z_s)$. Therefore, if we pass to another s_ϵ the above expression is not greater than

$$\epsilon + b_1 e^{-c} + b_2 e^{-b_3 N} + b_4 \epsilon c^{\frac{2}{n-1}}$$
$$+ \Bigg| N^{\frac{2}{n-1}} \int_0^{\frac{c}{N}} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H(x_s,N(x_s)))} \frac{f(y)}{(1-<N(x_s),N(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(K\cap H(x_s,N(x_s)))}(y)}$$
$$- \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(\mathcal{E}\cap H(z_s,N_{\partial K}(x_0)))} \frac{f(x_0)}{(1-<N(\lambda_{\mathcal{E}}(s)e_n),N(y)>^2)^{\frac{1}{2}}} \mathrm{d}\mu_{\partial(\mathcal{E}\cap H(z_s,N_{\partial K}(x_0)))}(y)} \mathrm{d}s \Bigg|.$$

Now we apply Lemma 2.10.(i). Choosing another s_ϵ the above expression is less than $\epsilon + b_1 \mathrm{e}^{-c} + b_2 \mathrm{e}^{-b_3 N} + b_4 \epsilon c^{\frac{2}{n-1}}$. We choose c and N sufficiently big and ϵ sufficiently small. □

Proof. (Proof of Theorem 1.1) We assume here that $x_T = 0$. For $x_0 \in \partial K$ the point x_s is given by $\{x_s\} = [x_T, x_0] \cap \partial K_s$.

$$\mathrm{vol}_n(K) - \mathbb{E}(f,N) = \int_K \mathbb{P}_f^N\{(x_1,\dots,x_N) | x \notin [x_1,\dots,x_N]\} \mathrm{d}x.$$

By Lemma 2.1.(iv) we have that $K_0 = K$ and by Lemma 2.4.(iii) that K_T consists of one point only. Since $\mathbb{P}_f^N\{(x_1,\dots,x_N)|x_s \notin [x_1,\dots,x_N]\}$ is a continuous functions of the variable x_s we get by Lemma 2.12

$$\begin{aligned}&\mathrm{vol}_n(K) - \mathbb{E}(f,N)\\ &\quad= \int_0^T \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)} \frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}} \mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s\end{aligned}$$

where $H_s = H(x_s, N_{\partial K_s}(x_s))$. By Lemma 4.9 for all s_0 with $0 < s_0 \le T$

$$\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_{s_0}^T \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}\mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)} = 0.$$

We get for all s_0 with $0 < s_0 \le T$

$$\begin{aligned}&\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f,N)}{N^{-\frac{2}{n-1}}} =\\ &\quad\lim_{N\to\infty} N^{\frac{2}{n-1}} \int_0^{s_0} \int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}\mathrm{d}\mu_{\partial K_s}(x_s)\mathrm{d}s}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}.\end{aligned}$$

We apply now the bijection between ∂K and ∂K_s mapping an element $x \in \partial K$ to x_s given by $\{x_s\} = [x_T, x_0] \cap \partial K_s$. The ratio of the volumes of a surface element in ∂K and its image in ∂K_s is

$$\frac{\|x_s\|^n < x_0, N_{\partial K}(x_0) >}{\|x_0\|^n < x_s, N_{\partial K_s}(x_s) >}.$$

Thus we get

$$\begin{aligned}&\int_{\partial K_s} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\mathrm{d}\mu_{\partial K_s}(x_s)\\ &\quad= \int_{\partial K} \frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s \notin [x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s} \frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)} \times\\ &\qquad\qquad\qquad \frac{\|x_s\|^n < x, N_{\partial K}(x) >}{\|x\|^n < x_s, N_{\partial K_s}(x_s) >}\mathrm{d}\mu_{\partial K}(x).\end{aligned}$$

We get for all s_0 with $0 < s_0 \le T$

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{N^{-\frac{2}{n-1}}}=$$
$$\lim_{N\to\infty}N^{\frac{2}{n-1}}\int_0^{s_0}\int_{\partial K}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\times$$
$$\frac{\|x_s\|^n<x,N_{\partial K}(x)>}{\|x\|^n<x_s,N_{\partial K_s}(x_s)>}\mathrm{d}\mu_{\partial K}(x)\mathrm{d}s.$$

By the theorem of Tonelli

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{N^{-\frac{2}{n-1}}}=$$
$$\lim_{N\to\infty}N^{\frac{2}{n-1}}\int_{\partial K}\int_0^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial K\cap H_s}\frac{f(y)}{\sqrt{1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2}}\mathrm{d}\mu_{\partial K\cap H_s}(y)}\times$$
$$\frac{\|x_s\|^n<x_0,N_{\partial K}(x_0)>}{\|x_0\|^n<x_s,N_{\partial K_s}(x_s)>}\mathrm{d}s\mathrm{d}\mu_{\partial K}(x).$$

Now we want to apply the dominated convergence theorem in order to change the limit and the integral over ∂K. By Lemma 5.1 for all s_0 with $0<s_0\le T$

$$\lim_{N\to\infty}N^{\frac{2}{n-1}}\int_0^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)}\frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}\mathrm{d}s=c_n\frac{\kappa(x_0)^{\frac{1}{n-1}}}{f(x_0)^{\frac{2}{n-1}}}.$$

Clearly, we have $\lim_{s\to 0}\|x_s\|=\|x\|$ and by Lemma 2.5

$$\lim_{s\to 0}<x_s,N_{\partial K_s}(x_s)>=<x,N_{\partial K}(x)>.$$

By this and since the above formula holds for all s_0 with $0<s_0\le T$

$$\lim_{N\to\infty}N^{\frac{2}{n-1}}\int_0^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)}\frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}\frac{\|x_s\|^n<x,N(x)>}{\|x\|^n<x_s,N(x_s)>}\mathrm{d}s$$
$$=c_n\frac{\kappa(x_0)^{\frac{1}{n-1}}}{f(x_0)^{\frac{2}{n-1}}}.$$

By Lemma 4.12 the functions with variable $x_0\in\partial K$

$$N^{\frac{2}{n-1}}\int_0^{s_0}\frac{\mathbb{P}_f^N\{(x_1,\dots,x_N)|\ x_s\notin[x_1,\dots,x_N]\}}{\int_{\partial(K\cap H_s)}\frac{f(y)\mathrm{d}\mu_{\partial(K\cap H_s)}(y)}{(1-<N_{\partial K_s}(x_s),N_{\partial K}(y)>^2)^{\frac{1}{2}}}}\frac{\|x_s\|^n<x_0,N(x_0)>}{\|x_0\|^n<x_s,N(x_s)>}\mathrm{d}s$$

are uniformly bounded. Thus we can apply the dominated convergence theorem.

$$\lim_{N\to\infty}\frac{\mathrm{vol}_n(K)-\mathbb{E}(f,N)}{N^{-\frac{2}{n-1}}}=c_n\int_{\partial K}\frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\mathrm{d}\mu_{\partial K}(x)$$

□

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