RANDOM POLYTOPES WITH VERTICES ON THE BOUNDARY OF A CONVEX BODY POLYTOPES ALÉATOIRES À SOMMETS SUR LA FRONTIERE D'UN CORPS CONVEXE

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Abstract. Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}_+$ be a continuous, positive function with $\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1$ where $\mu_{\partial K}$ is the surface measure on ∂K. Let $\mathbb{P}_f$ be the probability measure on ∂K given by $d\mathbb{P}_f(x) = f(x) d\mu_{\partial K}(x)$. Let κ be the (generalized) Gauß-Kronecker curvature and $\mathbb{E}(f, N)$ the expected volume of the convex hull of N points chosen randomly on ∂K with respect to $\mathbb{P}_f$. Then, under some regularity conditions on the boundary of K

$$\lim_{N \to \infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu_{\partial K}(x),$$

where c_n is a constant depending on the dimension n only.

The minimum at the right-hand side is attained for the normalized affine surface area measure with density

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x)}.$$

Résumé. Soit K un corps convexe dans $\mathbb{R}^n$ et soit $f : \partial K \to \mathbb{R}_+$ une fonction continue positive telle que $\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1$ où $\mu_{\partial K}$ est la mesure de surface sur ∂K. Soit $\mathbb{P}_f$ la mesure de probabilité sur ∂K definie par $d\mathbb{P}_f(x) = f(x) d\mu_{\partial K}(x)$. Soit κ la courbure de Gauß-Kronecker (generalisée) et $\mathbb{E}(f, N)$ l'espérance du volume de l'enveloppe convexe de N points choisis aléatoirement sur ∂K par rapport à $\mathbb{P}_f$. Alors on a sous certaines conditions de regularité de ∂K

$$\lim_{N \to \infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu_{\partial K}(x),$$

où c_n est une constante qui ne depend que de la dimension n.

Le minimum du membre de droite est atteint pour la mesure normalisée de surface affine ayant pour densité

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x)}.$$

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Un corps convexe K dans $\mathbb{R}^n$ est un sous-ensemble compact, convexe dans $\mathbb{R}^n$ dont l'interieur est non vide. On dénote par $N_{\partial K}(x)$ la normale en un point $x \in \partial K$. La mesure de surface $\mu_{\partial K}$ est la restriction de la mesure de Hausdorff de dimension $n-1$ à ∂K. $B_2^n(x,r)$ est la boule Euclidienne de centre x et rayon r. Soit κ la courbure de Gauß-Kronecker (generalisée) [Lei, p. 1059].

Un polytope aléatoire est l'enveloppe convexe d'un nombre fini des points choisis dans K par rapport à un mesure de probabilité $\mathbb{P}$ sur K. L'espérance du volume d'un polytope aléatoire de N points est

$$\mathbb{E}(\mathbb{P},N)=\int_K \cdots \int_K \mathrm{vol}_n([x_1,\dots,x_N])d\mathbb{P}(x_1)\dots d\mathbb{P}(x_N),$$

où $[x_1,\dots,x_N]$ est l'enveloppe convexe de $x_1,\dots,x_N$. On demontre le resultat suivant.

Théorème. *Soit K un corps convexe dans $\mathbb{R}^n$. On suppose qu'il existe r et R dans $\mathbb{R}$, $0<r\le R<\infty$, telle qu'on a pour tout $x\in\partial K$*

$$B_2^n(x-rN_{\partial K}(x),r)\subseteq K\subseteq B_2^n(x-RN_{\partial K}(x),R).$$

Soit $f:\partial K\to\mathbb{R}_+$ une fonction continue positive telle que $\int_{\partial K} f(x)d\mu_{\partial K}(x)=1$. Soit $\mathbb{P}_f$ la mesure de probabilité sur ∂K definie par $d\mathbb{P}_f(x)=f(x)d\mu_{\partial K}(x)$. Alors on a

$$\lim_{N\to\infty}\frac{vol_n(K)-\mathbb{E}(f,N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}}=c_n\int_{\partial K}\frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}d\mu_{\partial K}(x)$$

où

$$c_n=\frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(vol_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}.$$

Le minimum du membre de droite est atteint pour la mesure normalisée de surface affine ayant pour densité

$$f_{as}(x)=\frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu_{\partial K}(x)}.$$

A convex body K in $\mathbb{R}^n$ is a compact, convex subset of $\mathbb{R}^n$ with non-empty interior. The normal at a point $x\in\partial K$ is denoted by $N_{\partial K}(x)$. The surface measure $\mu_{\partial K}$ is the restriction of the $n-1$-dimensional Hausdorff-measure to ∂K. $B_2^n(x,r)$ is the Euclidean ball with center x and radius r. One of the metrics measuring the distance between two convex bodies C and K is the symmetric difference metric $d_S(C,K)=\mathrm{vol}_n((C\setminus K)\cup(K\setminus C))$. Let κ denote the generalized Gauß-Kronecker curvature [Lei, p. 1059]. It coincides with the the Gauß-Kronecker curvature whenever the latter exists. By a result of Aleksandrov [Ale] the generalized curvature κ exists a.e. with respect to $\mu_{\partial K}$.

A random polytope is the convex hull of finitely many points that are chosen from K with respect to a probability measure $\mathbb{P}$ on K. The expected volume of a random polytope of N points is

$$\mathbb{E}(\mathbb{P},N)=\int_K\cdots\int_K \mathrm{vol}_n([x_1,\dots,x_N])d\mathbb{P}(x_1)\dots d\mathbb{P}(x_N)$$

where $[x_1,\dots,x_N]$ is the convex hull of the points $x_1,\dots,x_N$. Thus the expression $\mathrm{vol}_n(K)-\mathbb{E}(\mathbb{P},N)$ measures how close a random polytope and the convex body are in the symmetric difference metric.

In this paper we are considering convex bodies in arbitrary dimension and probability measures that are concentrated on the boundary of the convex body. In fact, we assume that the measure has a continuous density with respect to the surface measure on ∂K. Under some additional technical assumptions we prove an asymptotic formula. This formula shows that the optimal probability measure is the normalized affine surface measure. We prove the following result.

Theorem. *Let K be a convex body in $\mathbb{R}^n$ such that there are r and R in $\mathbb{R}$ with $0<r\le R<\infty$ so that we have for all $x\in\partial K$*

$$B_2^n(x-rN_{\partial K}(x),r)\subseteq K\subseteq B_2^n(x-RN_{\partial K}(x),R)$$

and let $f:\partial K\to\mathbb{R}_+$ be a continuous, positive function with $\int_{\partial K}f(x)d\mu_{\partial K}(x)=1$. Let $\mathbb{P}_f$ be the probability measure on ∂K given by $d\mathbb{P}_f(x)=f(x)d\mu_{\partial K}(x)$. Then we have

$$\lim_{N\to\infty}\frac{vol_n(K)-\mathbb{E}(f,N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}}=c_n\int_{\partial K}\frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}d\mu_{\partial K}(x)$$

where

$$c_n=\frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(vol_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}.$$

The minimum at the right-hand side is attained for the normalized affine surface area measure with density

$$f_{as}(x)=\frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K}\kappa(x)^{\frac{1}{n+1}}d\mu_{\partial K}(x)}.$$

J. Müller [Mü] proved this result for the case that the convex body is the Euclidean ball and the measure is the normalized surface measure, i.e. the function f is constant.

The condition: There are r and R in $\mathbb{R}$ with $0<r\le R<\infty$ so that we have for all $x\in\partial K$

$$B_2^n(x-rN_{\partial K}(x),r)\subseteq K\subseteq B_2^n(x-RN_{\partial K}(x),R)$$

is satisfied if K has a C^2-boundary with everywhere positive curvature. This follows from Blaschke's rolling theorem [Bla , p.118] and a generalization of it [Lei, Remark 2.3]. Indeed, we can choose

$$r=\min_{x\in\partial K}\min_{1\le i\le n-1}r_i(x)\qquad\qquad R=\max_{x\in\partial K}\max_{1\le i\le n-1}r_i(x)$$

where $r_i(x)$ denotes the i-th principal curvature radius.

It was shown in [SchüW] that $\kappa^{\frac{1}{n+1}}$ is an integrable function on ∂K with respect to $\mu_{\partial K}$. Therefore the density

$$f_{as}(x) = \frac{\kappa(x)^{\frac{1}{n+1}}}{\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x)}.$$

exists provided that $\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) > 0$. This is certainly assured by the above regularity assumption on the boundary of K. For the affine surface measure we get

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{as}, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}.$$

For an arbitrary function f with $\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1$, it follows from Hölder's inequality that

$$\int_{\partial K} \left| \frac{\kappa(x)}{f(x)^2} \right|^{\frac{1}{n-1}} d\mu_{\partial K}(x) \geq \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}}.$$

This shows that the minimum is attained for the affine surface measure.

The details of the proof of the theorem are given in a forthcoming paper.

It is interesting to compare the random approximation with the best approximation of a convex body K by a polytope that is contained in K and has a given number of vertices. McClure and Vitale [McVi] obtained an asymptotic formula for best approximation in the case $n = 2$. Gruber [Gr] generalized this to higher dimensions. Then these asymptotic best approximation results are:

If a convex body K in $\mathbb{R}^n$ has a C^2-boundary with everywhere positive curvature, then

$$\inf\{d_S(K, P_N) | P_N \subset K \text{ and } P_N \text{ is a polytope with at most N vertices}\}$$

is asymptotically the same as

$$c(n) \left(\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}} N^{-\frac{2}{n-1}}.$$

for some constant $c(n)$. This means that the quotient of these two expressions tends to 1 when $N \to \infty$. It was shown by Gordon, Reisner and Schütt in [GRS1, GRS2] that the constant $c(n)$ is of the order of n: There are constants a and b such that we have for all $n \in \mathbb{N}$

$$an \leq c(n) \leq bn$$

It is clear from the theorem that we get the best random approximation if we choose the affine surface area measure. The order of magnitude for random approximation is (choosing the points from the boundary with respect to the affine surface measure)

$$\frac{(n-1)^{\frac{n+1}{n-1}} \Gamma\left(n + 1 + \frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{n-1}} d\mu_{\partial K}(x) \right)^{\frac{n+1}{n-1}} \left(\frac{1}{N}\right)^{\frac{2}{n-1}}.$$

It follows that for some numerical constant $c > 0$ one has

$$c \lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{as}, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} \leq \lim_{N\to\infty} N^{\frac{2}{n-1}} \inf_{P\in\mathcal{P}_N} d_S(K,P)$$

where $\mathcal{P}_N$ is the set of all polytopes that are contained in K and that have at most N vertices. This shows that random and best approximation are equivalent.

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