Dropping a Vertex or a Facet from a Convex Polytope *

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Abstract

There exist positive constants c_0 and $c_1 = c_1(n)$ such that for every $0 < \epsilon < 1/2$ the following holds: Let P be a convex polytope in $\mathbb{R}^n$ having $N \geq c_0^n/\epsilon$ vertices $x_1, \ldots, x_N$. Then there exists a subset $A \subset \{1, \ldots, N\}$, card $(A) \geq (1-2\epsilon)N$, such that for all $i \in A$

$$\frac{\mathrm{vol}_n(P) - \mathrm{vol}_n(\mathrm{conv}(\mathrm{vert}\,(P) \setminus \{x_i\}))}{\mathrm{vol}_n(P)} \leq c_1(n)\epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}} .$$

Also, if P is a convex polytope in $\mathbb{R}^n$ having $N \geq c_0^n/\epsilon$ facets. Let H_i^+ be the half space determined by the facet F_i, which contains P $(i = 1, \ldots, N)$. Then there exists a subset $A \subset \{1, \ldots, N\}$, card $(A) \geq (1-2\epsilon)N$, such that for all $i \in A$

$$\frac{\mathrm{vol}_n\left(\bigcap_{j \neq i} H_j^+\right) - \mathrm{vol}_n(P)}{\mathrm{vol}_n(P)} \leq c_1(n)\epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}} .$$

1 Introduction

In the last three decades, a number of papers have been published that deal with approximation of convex bodies in the Euclidean space $\mathbb{R}^n$ by convex polytopes. The precision of the approximation is being measured using various metrics. A fairly recent survey covering the essential results on the subject is [7]. The main results of this paper use the *symmetric distance* metric in the space of convex bodies to estimate the precision of the approximation. We concentrate here on the case that the approximated convex body is itself a polytope, which we wish to approximate by a polytope with fewer vertices or fewer facets. The essential claims of the main results, Theorems 3.2 and 4.2, are: If P is a convex polytope in $\mathbb{R}^n$ with N vertices or M facets (N or M sufficiently large), then there exists a vertex x of P, such that the polytope $Q =$ (convex hull of the vertices of P except x) satisfies

$$\frac{\mathrm{vol}_n(P) - \mathrm{vol}_n(Q)}{\mathrm{vol}_n(P)} \leq \alpha(n) N^{-\frac{n+1}{n-1}} . \tag{1.1}$$

Also, there exists a facet F of P, such that the polytope R which is the intersection of all the "facet half spaces containing P", except the one of F, satisfies

$$\frac{\mathrm{vol}_n(R) - \mathrm{vol}_n(P)}{\mathrm{vol}_n(P)} \leq \alpha(n) M^{-\frac{n+1}{n-1}} . \tag{1.2}$$

where $\alpha(n)$ is a constant which depends only on the dimension. These results are best possible in general, for the dependence on the numbers of vertices or facets of the approximating polytope, as shown in Remark 5.3.

In dimension 2, (1.1) is a consequence of a result of Rényi and Sulanke [11] and (1.2) was proved in [9]. One difficulty in higher dimensions is that the sets $P \setminus Q$ and $R \setminus P$ are of a more complicated nature, and, in particular, $P \setminus Q$ does not have to be convex.

In Section 5 we make a repeated application of the inequalities (1.1) and (1.2), to provide a method of constructing an inner approximating polytope of P, with $K < N$

vertices or an outer one with $K < M$ facets, with precision in terms of the symmetric distance whose dependence on K is best in general (Theorem 5.1). Such construction has been suggested and analysed in [9] for the dimension $n = 2$.

We suspect though, that the constants which depend on the dimension n that we obtain in Sections 3, 4 and 5, are not the best possible.

For the proof of the main results we use two results concerning approximation of convex bodies by polytopes in the Hausdorff-distance sense. Results of this type were proved in [3] and [4], among them one of the results that we use. The proof of the other one uses similar methods. For the sake of completeness, and also in order to have some control over the size of the constants involved, we bring detailed proofs of both results in Section 2.

It had been conjectured by Bárány [2] that (1.1) is always true for some vertex of P (if N is big enough). Clearly, the results of this paper confirm this conjecture. Bárány also pointed out there, that this result implies a theorem of G.E. Andrews [1] which states that for a lattice polytope P in $\mathbb{R}^n$ with N vertices and positive volume

$$N^{\frac{n+1}{n-1}} \leq const(n)\mathrm{vol}_n(P).$$

Finally, we wish to thank A. Giannopoulos for a helpful comment which helped to improve the dimension-dependent constants.

2 Notations and Preliminary Results

The notations used here are standard. We mention some of them in particular. A *convex body* is a convex, compact set in $\mathbb{R}^n$, with non-empty interior. The boundary of a convex body C is ∂C. A *polytope* (here meant always to be convex) in $\mathbb{R}^n$ is the convex hull of a finite set of points, or equivalently, the intersection of a finite set of closed half spaces, provided that this intersection is bounded. A *facet* of a convex polytope P in $\mathbb{R}^n$ is an $(n-1)$-dimensional face of P.

The canonical scalar product in $\mathbb{R}^n$ is $\langle x, y\rangle = \sum_{i=1}^n x(i)y(i)$ for $x = (x(1), \dots, x(n))$, $y = (y(1), \dots, y(n))$ in $\mathbb{R}^n$. The *Euclidean norm* is $|x| = \sqrt{\langle x, x\rangle}$. The *Euclidean ball* is $B_2^n = \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$. The notation $\mathrm{vol}_k(A)$ stands for the k-dimensional volume of the set $A \subset \mathbb{R}^n$, that is, for its k-dimensional Hausdorff measure.

The *Hausdorff distance* $d_H(C, D)$ between the convex bodies C and D, is

$$d_H(C, D) = \inf\{\varepsilon \,|\, C \subset D + \varepsilon B_2^n \text{ and } D \subset C + \varepsilon B_2^n\}.$$

Algebraic operations on sets are meant in the Minkowski sense. The *symmetric distance* is

$$d_S(C, D) = \mathrm{vol}_n((C \setminus D) \cup (D \setminus C)).$$

If $G_1, \dots, G_k$ are subsets of $\mathbb{R}^n$, we denote the convex hull of their union by $[G_1, \dots, G_k]$. We omit the braces in case some of these sets are singletons, thus $[x_1, \dots, x_N]$ is the convex hull of the finite set $\{x_1, \dots, x_N\}$.

Lemma 2.1 (F. John, [8]) *Let C be a convex body in $\mathbb{R}^n$ and let T be the affine transformation which maps the ellipsoid of maximal volume contained in C onto the Euclidean ball $\frac{1}{n}B_2^n$. Then*

$$\frac{1}{n}B_2^n \subseteq T(C) \subseteq B_2^n\,.$$

The remaining results in this section concern the approximation of a convex body by polytopes with a given number of vertices or of facets, in the Hausdorff-distance sense. The method which is used to get these results is well known. In fact, Proposition 2.7 - the result dealing with the *external* approximation with a given number of *facets* has been proved by Dudley [4] (the proof there is somewhat hidden inside the proof of another theorem). The same method has been used also by Bronshteyn and Ivanov [3] to prove an *external* approximation result with a given number of *vertices*. We did not find in the literature an explicit proof of the analogous result on *internal* approximation with a given number of vertices (Proposition 2.5 here). All in all, we have found it useful for the sake of completeness to bring here full proofs of these results. We also made some effort to control the size of the constants involved in the estimates below. We concentrate on the cases $n \geq 3$, since the main results have been proved in [9] for the case $n = 2$.

The following lemma supplies good estimates for the constants involved in a well known asymptotic behaviour.

Lemma 2.2

$$\frac{\sqrt{2\pi}}{\sqrt{n+2}} \leq \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \leq \frac{\sqrt{2\pi}}{\sqrt{n}}\,.$$

Proof. This follows from [12], where it is proved that

$$\frac{\sqrt{2}}{\sqrt{n+2}} \leq \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} \leq \frac{\sqrt{2}}{\sqrt{n}}\,.$$

■

Lemma 2.3 *Let $\Gamma(\Delta, r) = \{x \in \partial(rB_2^n) \,|\, x(n) \geq r - \Delta\}$ be the cap with height Δ of the Euclidean ball with radius r. Then (for $n \geq 3$)*

$$\mathrm{vol}_{n-1}(B_2^{n-1})\left(1-\frac{\Delta}{2r}\right)^{\frac{n-3}{2}}(2r\Delta)^{\frac{n-1}{2}} \leq \mathrm{vol}_{n-1}(\Gamma(\Delta,r)) \leq \mathrm{vol}_{n-1}(B_2^{n-1})(2r\Delta)^{\frac{n-1}{2}}\,. \tag{2.1}$$

It follows that for $0 < \delta < 1$ there exists a δ-net (with respect to the Euclidean distance) on the boundary of $2B_2^n$, whose cardinality is not greater than

$$f(n,\delta) = \frac{\sqrt{2\pi\, n}\, 4^{n-1}}{\left(1-\frac{\delta^2}{2^6}\right)^{\frac{n-3}{2}}\delta^{n-1}}\,. \tag{2.2}$$

Proof. We have

$$\mathrm{vol}_{n-1}(\Gamma(\Delta,r)) = \mathrm{vol}_{n-2}(\partial B_2^{n-1}) r \int_{x(n)=r-\Delta}^{r} (r^2 - x(n)^2)^{\frac{n-3}{2}}\, dx(n) =$$

$$(n-1)\mathrm{vol}_{n-1}(B_2^{n-1}) r^{n-1} 2^{\frac{n-3}{2}} \int_0^{\frac{\Delta}{r}} s^{\frac{n-3}{2}} \left(1 - \frac{s}{2}\right)^{\frac{n-3}{2}} ds\,.$$

Now, the fact that $s \le \Delta/r$ in the last integral implies the lower bound, and $1 - s/2 \le 1$ implies the upper bound in (2.1).

The resulting upper bound (2.2) is obtained using a standard volume argument: fix a maximal set $\{y_1, \ldots, y_k\}$ in $\partial(2B_2^n)$, such that $(\frac{\delta}{2}B_2^n + y_i) \cap (\frac{\delta}{2}B_2^n + y_j) = \emptyset$ if $i \ne j$. Note that $(\frac{\delta}{2}B_2^n + y_i) \cap \partial(2B_2^n) = \Gamma(\frac{\delta^2}{16}, 2)$, so that

$$k\,\mathrm{vol}_{n-1}\left(\Gamma\left(\frac{\delta^2}{16}, 2\right)\right) \le 2^{n-1} n \mathrm{vol}_n(B_2^n)\,.$$

Now use (2.1) and Lemma 2.2. Clearly $\{y_1, \ldots, y_k\}$ is a δ-net in $\partial(2B_2^n)$. ■

From here on, the notation $f(\cdot,\cdot)$ is always the one defined by (2.2).

The proof of Lemma 2.4 which follows, is taken essentially from [3]. For a convex body C with boundary of class $\mathcal{C}^1$ and $x \in \partial C$, denote by $\nu(x)$ the unique external unit normal of C at x.

Lemma 2.4 *Let C be a convex body in $\mathbb{R}^n$ such that $C \subset B_2^n$ and ∂C is of class $\mathcal{C}^1$. Then for $0 < \delta < 1$ there exists a finite subset $\{x_1, \ldots, x_k\}$ of ∂C, which is a δ-net for ∂C in the metric $d(x,y) = \max\{|x-y|, |\nu(x) - \nu(y)|\}$, and $k \le f(n,\delta)$.*

Proof. For $x \in \partial C$ let $\nu_0(x) = x + \nu(x)$. Then ν_0 is a homeomorphism of ∂C onto ∂C_0 where $C_0 = C + B_2^n$. Of course, $C_0 \subset 2B_2^n$. By Lemma 2.3, there exists a δ-net $\{y_1, \ldots, y_k\}$ on $\partial(2B_2^n)$ with $k \le f(n,\delta)$. Let now $\{z_1, \ldots, z_k\}$ be the images of $\{y_1, \ldots, y_k\}$ under the metric projection of $\mathbb{R}^n$ onto C_0. As the metric projection is contractive (cf. e.g. [13]), $\{z_1, \ldots, z_k\}$ form a δ-net on ∂C_0. We now let $x_i = \nu_0^{-1}(z_i)$, $i = 1, \ldots, k$. Then $x_i \in \partial C$ and the fact that they constitute a δ-net in ∂C in the metric mentioned in the lemma, is a consequence of the inequality $|\nu(x) - \nu(y)|^2 + |x-y|^2 \le |\nu_0(x) - \nu_0(y)|^2$, which holds for $x, y \in \partial C$ since $|\nu_0(x) - \nu_0(y)|^2 - |\nu(x) - \nu(y)|^2 - |x-y|^2 = 2\langle x-y, \nu(x) - \nu(y)\rangle = 2\langle x-y, \nu(x)\rangle + 2\langle y-x, \nu(y)\rangle \ge 0$ by convexity (in fact, each of the two summands in the last expression is non-negative). ■

Proposition 2.5 *Let C be a convex body in $\mathbb{R}^n$ such that $C \subset B_2^n$. Then for $0 < \varepsilon < 1$ there exists a convex polytope P_ε such that $P_\varepsilon \subset C$, $d_H(C, P_\varepsilon) \le \varepsilon$ and the number of vertices of P_ε is not greater than $f(n, \sqrt{\frac{\varepsilon}{d}})$, where $d = 2\sqrt{6}(\sqrt{6} - \sqrt{5})$.*

Proof. We may assume that ∂C is of class $\mathcal{C}^1$. Let $\delta = \sqrt{\varepsilon/d}$, construct a δ-net $\{x_1, \ldots, x_k\}$ on ∂C as in Lemma 2.4, with respect to the metric of that lemma and with $k \le f(n,\delta)$. Let $P_\varepsilon = [x_1, \ldots, x_k]$. Assume that $d_H(C, P_\varepsilon) > \varepsilon$. Then there is a point

$w \in \partial C$ such that the ball $w + \varepsilon B_2^n$ is disjoint of P_ε. Let H be a supporting hyperplane of $w + \varepsilon B_2^n$ which separates the interior of this ball from P_ε and let x_0 be the point of $\partial C \cap H^-$ which is furthest away from H (H^- being the half space bounded by H, which contains w). Then distance $(x_0, H) \geq \varepsilon$ and $\nu(x_0)$ is orthogonal to H.

We shall show that $\max\{|x_0 - x_i|, |\nu(x_0) - \nu(x_i)|\} > \delta$ for $i = 1, \ldots, k$, contradicting $\{x_1, \ldots, x_k\}$ being a δ-net.

If there exists i, $1 \leq i \leq k$, with $\alpha(\nu(x_0), \nu(x_i)) > d\delta$, where $\alpha(\cdot, \cdot)$ is the (small) angle between vectors, then $|\nu(x_0) - \nu(x_i)| \geq 2 \sin \frac{d}{2}\delta \geq 2(\frac{d}{2}\delta - \frac{(d\delta)^3}{48}) > \delta$ as $\delta < 1/\sqrt{d}$. So we may assume that for all i, $\alpha(\nu(x_0), \nu(x_i)) \leq d\delta < \pi/2$. Let K be the 2-dimensional plane through x_0 and x_i, which is parallel to $\nu(x_0)$, $L = K \cap C$ and ∂L the relative boundary of L. Let $\tilde{x}_i$ be the point that lies on $\partial L \cap H$, between x_0 and x_i (x_i, being in P_ε, is separated from x_0 by H). For $x \in \partial L$ denote by $\nu_1(x)$ the unit vector parallel to the plane K, which is (within K) an external normal of L at x. Then $\nu_1(x_i)$ has the direction of the orthogonal projection of $\nu(x_i)$ on K. We get

$$\alpha(\nu(x_0), \nu_1(\tilde{x}_i)) \leq \alpha(\nu(x_0), \nu_1(x_i)) \leq \alpha(\nu(x_0), \nu(x_i)) \leq d\delta < \pi/2\,.$$

Now

$$|x_0 - x_i| \geq |x_0 - \tilde{x}_i| \geq \frac{\varepsilon}{\sin \alpha(\nu(x_0), \nu_1(\tilde{x}_i))} \geq \frac{d\delta^2}{d\delta} = \delta,$$

which is the required contradiction. ∎

Corollary 2.6 *There exists a constant c_0 such that for all n, for every convex body C in $\mathbb{R}^n$ which is contained in B_2^n and for $N > c_0^{\frac{n-1}{2}}$, there exists a convex polytope $P \subset C$ with no more than N vertices, such that*

$$d_H(P, C) \leq \frac{c_0}{N^{\frac{2}{n-1}}}\,.$$

(An n-dependent, but bounded with n, upper bound for c_0 is given in (2.6) below. This bound is asymptotically $16.9826\ldots$.)

Proof. Fix an a-priori constant c_0 and assume that

$$N > c_0^{\frac{n-1}{2}}\,. \tag{2.3}$$

Using $\varepsilon = c_0/N^{\frac{2}{n-1}}$ in Proposition 2.5, we get a polytope $P \subset C$ that satisfies

$$d_H(P, C) \leq \frac{c_0}{N^{\frac{2}{n-1}}}$$

and the number k of vertices of P is not greater than $f(n, \sqrt{\varepsilon/d})$. That is:

$$k \leq \frac{\sqrt{2\pi n}\, 4^{n-1}}{\left(1 - \frac{\varepsilon}{2^6 d}\right)^{\frac{n-3}{2}} \left(\sqrt{\frac{\varepsilon}{d}}\right)^{n-1}} = N\, \frac{(4\sqrt{d})^{n-1}\sqrt{2\pi n}}{c_0^{\frac{n-1}{2}} \left(1 - \frac{c_0}{2^6 d N^{\frac{2}{n-1}}}\right)^{\frac{n-3}{2}}}\,. \tag{2.4}$$

In order that the coefficient of N in (2.4) will not be greater than 1, we need

$$\left(1 - \frac{c_0}{2^6 d N^{\frac{2}{n-1}}}\right) \geq \frac{g(n)}{c_0^{\frac{n-1}{n-3}}}$$

where $g(n) = (2\pi n)^{1/(n-3)}(16d)^{\frac{n-1}{n-3}}$. That is,

$$N \geq \left[\frac{c_0^{1+\frac{n-1}{n-3}}}{2^6 d(c_0^{\frac{n-1}{n-3}} - g(n))}\right]^{\frac{n-1}{2}} . \tag{2.5}$$

Thus, if $c_0^{\frac{n-1}{n-3}} > g(n)$ and if

$$\frac{c_0^{1+\frac{n-1}{n-3}}}{2^6 d(c_0^{\frac{n-1}{n-3}} - g(n))} = c_0 \, ,$$

then (2.3) and (2.5) become the same condition and we are done. The solution of this last equation is

$$c_0 = \left(\frac{2^6 d}{2^6 d - 1}\right)^{\frac{n-3}{n-1}} (2\pi n)^{\frac{1}{n-1}} 16d \tag{2.6}$$

(and this c_0 *does* satisfy $c_0^{(n-1)/(n-3)} > g(n)$). ■

The proof of Proposition 2.7 that we present here follows the lines of Dudley's proof in [4] where we pay careful attention to how the constants depend on the dimension.

Proposition 2.7 ([4]) *Let C be a convex body in $\mathbb{R}^n$, such that $C \subset B_2^n$. Then for $0 < \varepsilon < 1$ there exists a convex polytope P_ε, such that $C \subset P_\varepsilon$, $d_H(C, P_\varepsilon) < \varepsilon$ and the number of facets of P_ε is not greater than $f(n, \sqrt{\varepsilon}/3^{1/4})$.*

Proof. We may assume that ∂C is of class C^1. Let $\delta = \sqrt{\varepsilon}/3^{1/4}$ and construct a δ-net $\{x_1, \ldots, x_k\}$ on ∂C, as in Lemma 2.4, with respect to the metric of that lemma and with $k \leq f(n, \delta)$. For $x \in \partial C$, let $H(x)$, $H(x)^+$, $H(x)^-$ be, respectively, the supporting hyperplane of C at x and the half spaces bounded by it: the one containing C and the one which does not.

We define $P_\varepsilon = \bigcap_{i=1}^k H(x_i)^+$. P_ε is a convex polytope which contains C and has at most k facets. We claim that $d_H(P_\varepsilon, C) < \varepsilon$. Given $x \in \partial C$, let i, $1 \leq i \leq k$, be such that $0 < d(x, x_i) < \delta$ (d being the metric from Lemma 2.4). Let K be the 2-dimensional plane through x and x_i, which is parallel to $\nu(x_i)$, $L = K \cap C$ and ∂L the relative boundary of L. Let $\nu_1(x)$ be the external unit normal of L at x (relative to the plane K). Then $\nu_1(x)$ has the direction of the orthogonal projection of $\nu(x)$ on K. Hence

$$\alpha := \alpha(\nu(x_i), \nu_1(x)) \leq \alpha(\nu(x_i), \nu(x)) \, .$$

Let $\gamma \geq 0$ be such that $x + \gamma\nu_1(x) \in H(x_i)$. Clearly distance $(x, P_\varepsilon) \leq \gamma$.

$d(x,x_i)<\delta$ implies that $2\sin\frac{\alpha(\nu(x_i),\nu(x))}{2}<\delta<1$. Observing Figure 1, we see that

$$\frac{\gamma}{\delta}<\frac{\gamma}{|x-x_i|}\le\tan\alpha\le\sqrt{3}\,\delta\,.$$

Hence $\gamma<\sqrt{3}\,\delta^2=\varepsilon$. ■

The argument of the proof of Corollary 2.6, applied to Proposition 2.7, yields the following corollary (with the constant c_1 bounded above by $\left(\frac{2^6\sqrt{3}}{2^6\sqrt{3}-1}\right)^{\frac{n-3}{n-1}}(2\pi n)^{\frac{1}{n-1}}2^4\sqrt{3}\sim 27.9651\dots$).

Corollary 2.8 *There exists a constant c_1 such that for all n, for every convex body C in $\mathbb{R}^n$ which is contained in B_2^n and for $N>c_1^{\frac{n-1}{2}}$ there exists a polytope $P\supset C$ with at most N facets, such that*

$$d_H(P,C)\le\frac{c_1}{N^{\frac{2}{n-1}}}\,.$$

3 Dropping a Vertex

Theorem 3.1 *There exists a positive constant c_2, such that for every $0<\epsilon<1/2$ the following holds: Let P be a polytope in $\mathbb{R}^n$ having N vertices $x_1,\dots,x_N$, $N>c_2^n/\epsilon$, which is contained in the Euclidean unit ball B_2^n. Then there exists a subset $A\subset\{1,\dots,N\}$, with $\operatorname{card}(A)\ge(1-2\epsilon)N$, such that for all $i\in A$:*

$$\mathrm{vol}_n(P)-\mathrm{vol}_n([x_1,\dots,x_{i-1},x_{i+1},\dots,x_N])\le c_0\,\mathrm{vol}_{n-1}(\partial P)\,\epsilon^{-\frac{n+1}{n-1}}N^{-\frac{n+1}{n-1}},$$

where c_0 is the constant from Corollary 2.6.

Theorem 3.2 *For c_0 and c_2 as in Theorem 3.1 and for every $0<\epsilon<1/2$ the following holds: Let P be a polytope in $\mathbb{R}^n$ having N vertices $x_1,\dots,x_N$, $N>c_2^n/\epsilon$. Then there exists a subset $A\subset\{1,\dots,N\}$, with $\operatorname{card}(A)\ge(1-2\epsilon)N$, such that for all $i\in A$:*

$$\frac{\mathrm{vol}_n(P)-\mathrm{vol}_n([x_1,\dots,x_{i-1},x_{i+1},\dots,x_N])}{\mathrm{vol}_n(P)}\le c_0\,n^2\epsilon^{-\frac{n+1}{n-1}}N^{-\frac{n+1}{n-1}}\,.$$

If P is centrally symmetric, then we may replace n^2 by $n^{3/2}$ on the right hand side of the above inequality.

Lemma 3.3 *Let $\epsilon>0$ and $N>c_2^n/\epsilon$, where $c_2=(nc_0^{\frac{n-1}{2}})^{\frac{1}{n}}$ and c_0 is the constant of Corollary 2.6. Let P be a simplicial polytope in $\mathbb{R}^n$, which is contained in B_2^n, with vertices $x_1,\dots,x_N$. Let h_{x_i} be the distance from x_i to the polytope $[x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]$. Then the set*

$$\left\{i\,\middle|\,h_{x_i}\le\frac{c_0}{(\epsilon N)^{\frac{2}{n-1}}}\right\}$$

has at least $(1-\epsilon)N$ elements.

Proof. We apply Corollary 2.6 to $C = P$ and $M = \epsilon\frac{N}{n}$ instead of N (assuming w.l.o.g. that this M is an integer). To be able to do this we need $N > nc_0^{\frac{n-1}{2}}/\epsilon$. We get a polytope $Q \subset P$ with at most M vertices, such that

$$d_H(P,Q) \le \frac{c_0}{M^{\frac{2}{n-1}}}\,.$$

We have

$$h_{x_i} \le \frac{c_0}{M^{\frac{2}{n-1}}}$$

provided that

$$Q \subseteq [x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]\,.$$

Let $z_1,\dots,z_M$ be the extreme points of Q. We may assume that the extreme points are elements of the boundary of P. Moreover, we may assume that the extreme points are in the relative interior of the facets of P. We have

$$Q \not\subseteq [x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]$$

if and only if an extreme point z_j of Q is contained in a facet of ∂P that contains x_i. We have

$$\text{card}\,\{i\,|\,\exists z_j:\ z_j \text{ is in a facet containing } x_i\} \le$$
$$\sum_{j=1}^{M} \text{card}\,\{i\,|\,x_i \text{ is in a facet containing } z_j\} = nM = \epsilon N\,.$$

■

Proof of Theorem 3.1. Using a small perturbation, we may assume that P is a simplicial polytope. Let F_{x_i} denote the union of all the facets of P that meet at the vertex x_i, $i = 1,\dots,N$. We have

$$\sum_{i=1}^{N} \text{vol}_{n-1}(F_{x_i}) = n\,\text{vol}_{n-1}(\partial P)\,.$$

This implies that for every ϵ we have

$$\text{card}\left\{i \,\middle|\, \text{vol}_{n-1}(F_{x_i}) \le \frac{n\ \text{vol}_{n-1}(\partial P)}{\epsilon N}\right\} \ge (1-\epsilon)N\,. \tag{3.1}$$

Let $z \in [x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]$ be such that $|x_i - z| = h_{x_i}$. Then, for every facet G of P which contains x_i, we have distance $(z,G) \le h_{x_i}$. Also,

$$P \setminus [x_1,\dots,x_{i-1},x_{i+1},\dots,x_N] \subset \bigcup \{[z,G]\,|\,G \text{ a facet of } P \text{ and } x_i \in G\}\,.$$

Hence

$$\text{vol}_n(P \setminus [x_1,\dots,x_{i-1},x_{i+1},\dots,x_N]) \le \sum_{G \text{ a facet of } P \text{ and } x_i \in G} \frac{\text{vol}_{n-1}(G)\cdot h_{x_i}}{n}\,.$$

That is

$$\mathrm{vol}_n(P) - \mathrm{vol}_n([x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N]) \le \frac{h_{x_i}\mathrm{vol}_{n-1}(F_{x_i})}{n}\,.$$

By Lemma 3.3 and (3.1), for $\epsilon > 0$ there exists a set $A \subseteq \{1, \dots, N\}$ with $\mathrm{card}\,(A) \ge (1-2\epsilon)N$, such that we have for all $i \in A$

$$h_{x_i} \le \frac{c_0}{(\epsilon N)^{\frac{2}{n-1}}} \qquad \text{and} \qquad \mathrm{vol}_{n-1}(F_{x_i}) \le \frac{n\ \mathrm{vol}_{n-1}(\partial P)}{\epsilon\, N}\,.$$

It follows that for $\epsilon < \frac{1}{2}$ we have for $i \in A$

$$\mathrm{vol}_n(P) - \mathrm{vol}_n([x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N]) \le c_0\,\epsilon^{-\frac{n+1}{n-1}}\mathrm{vol}_{n-1}(\partial P) N^{-\frac{n+1}{n-1}}\,.$$

■

Proof of Theorem 3.2. Let T be the transformation associated with P by Lemma 2.1. Apply Theorem 3.1 to $T(P)$. Then for i in the resulting set $A \subset \{1, \dots, N\}$ we have

$$\frac{\mathrm{vol}_n(T(P)) - \mathrm{vol}_n([Tx_1, \dots, Tx_{i-1}, Tx_{i+1}, \dots, Tx_N])}{\mathrm{vol}_n(T(P))} \le \tag{3.2}$$

$$c_0 \epsilon^{-\frac{n+1}{n-1}}\, N^{-\frac{n+1}{n-1}} \frac{\mathrm{vol}_{n-1}(\partial T(P))}{\mathrm{vol}_n(T(P))}\,.$$

As the left hand side of (3.2) is affine-invariant, we may drop all the T's on this side. On the right hand side of (3.2) we have to estimate

$$\frac{\mathrm{vol}_{n-1}(\partial T(P))}{\mathrm{vol}_n(T(P))} = \lim_{t\to 0+} \frac{\mathrm{vol}_n(T(P) + tB_2^n) - \mathrm{vol}_n(T(P))}{t\,\mathrm{vol}_n(T(P))}\,.$$

Since $\frac{1}{n}B_2^n \subset T(P)$ we have

$$\frac{\mathrm{vol}_n(T(P) + tB_2^n) - \mathrm{vol}_n(T(P))}{t\,\mathrm{vol}_n(T(P))} \le \frac{\mathrm{vol}_n(T(P) + tnT(P)) - \mathrm{vol}_n(T(P))}{t\,\mathrm{vol}_n(T(P))}\,.$$

Hence

$$\frac{\mathrm{vol}_{n-1}(\partial T(P))}{\mathrm{vol}_n(T(P))} \le \lim_{t\to 0+} \frac{(1+tn)^n - 1}{t} = n^2\,.$$

This completes the proof. The case of a centrally symmetric P is handled using $\frac{1}{\sqrt{n}}B_2^n$ instead of $\frac{1}{n}B_2^n$, which is possible due to the form of F. John's theorem in this case. ■

4 Dropping a Facet

Let P be a polytope in $\mathbb{R}^n$, with facets $F_1, \dots, F_N$. For $1 \le i \le N$ let H_i, H_i^+, H_i^- be, respectively, the hyperplane containing the facet F_i and the half-spaces bounded by it: the one containing P and the one which does not. For each $1 \le j \le N$ let

$$P(j) = \bigcap_{i=1,\dots,N,\ \ i\ne j} H_i^+\,.$$

$P(j)$ is a (possibly unbounded) polyhedron containing P. Let $h_j = \sup_{x\in P(j)} \mathrm{distance}(x, P)$.

Theorem 4.1 *For every $0 < \epsilon < 1/2$ the following holds: Let P be a polytope in $\mathbb{R}^n$ having N facets, $N > (\sqrt{c_1}n)^{n-1}/\epsilon$, which is contained in the Euclidean unit ball B_2^n and contains $\frac{1}{n}B_2^n$. Then there exists a subset $A \subset \{1, \dots, N\}$, with $\mathrm{card}(A) \geq (1-2\epsilon)N$, such that for all $i \in A$:*

$$\mathrm{vol}_n(P(i)) - \mathrm{vol}_n(P) \leq c_1 \, \mathrm{e} \, \mathrm{vol}_{n-1}(\partial P) \, \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}}.$$

c_1 is the constant from Corollary 2.8.

Theorem 4.2 *There exists a positive constant c_4, such that for every $0 < \epsilon < 1/2$ the following holds: Let P be a polytope in $\mathbb{R}^n$ having N facets, $N > c_4^n/\epsilon$. Then there exists a subset $A \subset \{1, \dots, N\}$, with $\mathrm{card}(A) \geq (1-2\epsilon)N$, such that for all $i \in A$:*

$$\frac{\mathrm{vol}_n(P(i)) - \mathrm{vol}_n(P)}{\mathrm{vol}_n(P)} \leq c_1 \, n^2 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}}.$$

If P is centrally symmetric then we may replace n^2 by $n^{3/2}$ on the right hand side of the above inequality.

Lemma 4.3 *Let $\epsilon > 0$ and $N \geq c_1^{(n-1)/2} n/\epsilon$ where c_1 is the constant from Corollary 2.8. Let P be a simple polytope in $\mathbb{R}^n$ with facets $F_1, \dots, F_N$. Then the set*

$$\left\{ i \,\middle|\, h_i \leq \frac{c_1}{(\epsilon N)^{\frac{2}{n-1}}} \right\}$$

has at least $(1-\epsilon)N$ elements.

Proof. We apply Corollary 2.8 to $C = P$ and $M = \epsilon \frac{N}{n}$, assuming w.l.o.g. that this M is an integer. To use it we need $N > n c_1^{\frac{n-1}{2}}/\epsilon$. We get a polytope $Q \supset P$ with at most M facets, such that

$$d_H(P, Q) \leq \frac{c_1}{M^{\frac{2}{n-1}}}.$$

We have

$$h_i \leq \frac{c_1}{M^{\frac{2}{n-1}}}$$

provided that

$$Q \supseteq P(i).$$

Let $G_1, \dots, G_M$ be the facets of Q. We may assume that each G_i meets exactly one vertex of P in its interior. Now, if $Q \not\supseteq P(i)$ then there is a facet G_j of Q which meets a vertex of F_i. We have

$$\mathrm{card}\{i | \exists G_j : \ G_j \text{ meets a vertex of } F_i\} \leq$$

$$\sum_{j=1}^{M} \mathrm{card}\{i | \text{ a vertex of } F_i \text{ is in } G_j\} = nM = \epsilon N.$$

∎

Lemma 4.4 *Let $0 < \epsilon < 1$ and let c_1 be as in Corollary 2.8. Assume that $N \geq c_1^{\frac{n-1}{2}} n^{n-1}/\epsilon$ and let P be as in the statement of Theorem 4.1. If F_i is a facet of P with $h_i \leq c_1/(\epsilon N)^{\frac{2}{n-1}}$ then*

$$\mathrm{vol}_n(P(i) \setminus P) \leq \mathrm{e} \cdot \mathrm{vol}_{n-1}(F_i)\, h_i\,.$$

Proof. Let C_i be the one-sided cone with vertex 0, spanned by F_i. For $h = \text{distance}\,(0, H_i)$ we have $\frac{1}{n} \leq h \leq 1$. Also, since $0 \in P$, we have

$$P(i) \setminus P \subseteq C_i \cap H_i^- \cap (H_i^+ + h_i \nu(H_i))\,,$$

where $\nu(H_i)$ is the external unit normal of the facet F_i.

Hence

$$\mathrm{vol}_n(P(i) \setminus P) \leq \frac{1}{n}\mathrm{vol}_{n-1}(F_i)\left[\left(\frac{h+h_i}{h}\right)^{n-1}(h+h_i) - h\right] =$$

$$\frac{h}{n}\mathrm{vol}_{n-1}(F_i)\left[\left(1+\frac{h_i}{h}\right)^{n} - 1\right] \leq \mathrm{vol}_{n-1}(F_i)\left(1+\frac{h_i}{h}\right)^{n-1} h_i \leq$$

$$\mathrm{vol}_{n-1}(F_i)(1 + n \cdot h_i)^{n-1} h_i\,.$$

The upper bound for h_i and the lower bound for N which are assumed in this lemma imply now that

$$(1 + h_i \cdot n)^{n-1} \leq \left(1 + \frac{1}{n}\right)^{n-1} \leq \mathrm{e}$$

which completes the proof of the lemma. ■

Proof of Theorem 4.1. Using a small perturbation, we may assume that P is a simple polytope. We have

$$\sum_{i=1}^{N} \mathrm{vol}_{n-1}(F_i) = \mathrm{vol}_{n-1}(\partial P)\,.$$

This implies that for every ϵ we have

$$\mathrm{card}\left\{ i \,\middle|\, \mathrm{vol}_{n-1}(F_i) \leq \frac{\mathrm{vol}_{n-1}(\partial P)}{\epsilon N} \right\} \geq (1-\epsilon)N \tag{4.1}$$

By Lemma 4.3 and (4.1), for $\epsilon > 0$ there exists a set $A \subseteq \{1, \ldots, N\}$ with $\mathrm{card}(A) \geq (1 - 2\epsilon)N$, such that for all $i \in A$

$$h_i \leq \frac{c_1}{(\epsilon N)^{\frac{2}{n-1}}} \qquad \text{and} \qquad \mathrm{vol}_{n-1}(F_i) \leq \frac{\mathrm{vol}_{n-1}(\partial P)}{\epsilon\, N}\,.$$

It now follows from Lemma 4.4 that for $\epsilon < \frac{1}{2}$ and for all $i \in A$

$$\mathrm{vol}_n(P(i)) - \mathrm{vol}_n(P) \leq c_1\, \mathrm{e}\, \mathrm{vol}_{n-1}(\partial P) \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}}\,.$$

■

We say that a convex body C in $\mathbb{R}^n$ is *in distinguished position* if the ellipsoid of maximal volume contained in C is $\frac{1}{n} B_2^n$.

Lemma 4.5 *Let P be a convex polytope in $\mathbb{R}^n$ having N facets. Assume further, that for some* $0 < \epsilon < 1$

$$N > \frac{\mathrm{vol}_{n-1}(\partial P) \cdot (\text{the maximal width of } P)}{\epsilon \cdot \mathrm{vol}_n(P)} \,. \tag{4.2}$$

Then, if F_i is a facet of P for which

$$\mathrm{vol}_{n-1}(F_i) \le \frac{\mathrm{vol}_{n-1}(\partial P)}{\epsilon \, N} \,,$$

the $(n-1)$-dimensional volume of a section of $P(i)$ which is parallel to F_i and cuts $P(i)$ outside of P, forms a decreasing function of the section's distance from F_i. Therefore, for such F_i we have:

$$\mathrm{vol}_n(P(i) \setminus P) \le \mathrm{vol}_{n-1}(F_i) \cdot h_i \,.$$

If we assume that P is in distinguished position, then the condition $N > 2n^2/\epsilon$ guarantees that (4.2) is satisfied and hence the result holds.

Proof. Due to concavity (by Brunn-Minkowski theorem) of $(\mathrm{vol}_{n-1}(F))^{1/(n-1)}$ as a function of the distance from F_i of the parallel section F, if we assume (to the contrary of Lemma 4.5) that this function is increasing or stationary at F_i in the direction external to P, then it must be non- increasing in the direction internal to P all the way. We get:

$$\mathrm{vol}_n(P) \le (\text{width of } P, \text{ orthogonal to } F_i) \cdot \mathrm{vol}_{n-1}(F_i) \,.$$

This implies

$$N \le \frac{(\text{maximal width of } P)}{\epsilon} \cdot \frac{\mathrm{vol}_{n-1}(\partial P)}{\mathrm{vol}_n(P)} \,, \tag{4.3}$$

contradicting the assumption (4.2).

If P is in distinguished position, then, by Lemma 2.1, $\frac{1}{n} B_2^n \subset P \subset B_2^n$, hence an argument identical to the one in the proof of Theorem 3.2 implies:

$$\frac{(\text{maximal width of } P)}{\epsilon} \cdot \frac{\mathrm{vol}_{n-1}(\partial P)}{\mathrm{vol}_n(P)} \le \frac{2n^2}{\epsilon} \tag{4.4}$$

Now, if $N > 2n^2/\epsilon$ then, by (4.4), (4.2) is satisfied. ■

Proof of Theorem 4.2. As the left hand side of the inequality in Theorem 4.2 is affine-invariant, we may assume that P is in distinguished position. Therefore, by Lemma 2.1,

$$\frac{1}{n} B_2^n \subset P \subset B_2^n \,.$$

We repeat the argument of the proof of Theorem 4.1, but we replace the estimate of Lemma 4.4 by the one of Lemma 4.5. Doing this, we may decrease the lower bound on N to c_4^n/ϵ, for a constant c_4 that would guarantee the requirements of both Lemma 4.3 and Lemma 4.5. We get for i in the resulting set $A \subset \{1, \dots, N\}$:

$$\frac{\mathrm{vol}_n(P(i)) - \mathrm{vol}_n(P)}{\mathrm{vol}_n(P)} \le c_1 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}} \frac{\mathrm{vol}_{n-1}(\partial P)}{\mathrm{vol}_n(P)} \,. \tag{4.5}$$

We conclude as in the proof of Theorem 3.2.

5 Approximating Convex Polytopes by Polytopes with Fewer Vertices (or Facets)

Using the results of Section 2 and [10], or those of [5], one can prove that any convex body C in $\mathbb{R}^n$ can be approximated by a polytope $P \subset C$ with at most N vertices and by a polytope $Q \supset C$ with at most N facets, so that

$$\text{(5.1)} \qquad \frac{\mathrm{vol}_n(C) - \mathrm{vol}_n(P)}{\mathrm{vol}_n(C)} \leq \frac{c \cdot n}{N^{\frac{2}{n-1}}} \quad \text{and} \quad \frac{\mathrm{vol}_n(Q) - \mathrm{vol}_n(C)}{\mathrm{vol}_n(C)} \leq \frac{c \cdot n^2}{N^{\frac{2}{n-1}}}\,.$$

The above estimates are known to be best possible in general for the dependence on N. Moreover, the one for the inner approximation is known to be best possible, up to the constant c involved, also with respect to its numerator (see [6]).

Using these results one can easily construct a polytope $R \subset C$ with $2N$ vertices such that R approximates C and the polar R^* approximates the polar C^* with the same rate, depending on N, as in (5.1).

In this section we show how a successive application of the results of Sections 3 and 4 provides, in the case that C is a polytope (implicitly assumed to have many vertices or facets), a method to construct inner or outer approximating polytopes with a given, smaller, number N of vertices, respectively facets, that approximates C with the same rate, depending on N as in (5.1). In fact, as above, the construction can be carried out in such a way that we get mutually polar polytopes, approximating simultaneously C and its polar polytope C^*. We shall describe such algorithm for constructing an inner polytope in C and and outer one for C^*. Clearly an analogous dual algorithm exists.

If the origin is an interior point of a convex body C in $\mathbb{R}^n$ we define the polar (with respect to the origin) body C^* of C to be

$$C^* = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \leq 1 \text{ for all } y \in C\}\,.$$

Let P be a convex polytope in $\mathbb{R}^n$, with vertices $x_1, \ldots, x_N$. *A successive minimizing choice of vertices* of P is a sequence $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ of mutually different vertices of P, such that for all $k = 1, \ldots, m$

$$\mathrm{vol}_n([\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{k-1}}\}]) - \mathrm{vol}_n([\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_k}\}])$$

is minimal over all choices of $x_{i_k} \in \{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{k-1}}\}$ (with the obvious meaning for $k = 1$). *A successive minimizing choice of facets* of P, $\{F_{i_1}, \ldots, F_{i_m}\}$, is defined in an analogous way by dropping a "minimal volume facet" at a time. *A successive dual minimizing choice of vertices and facets* of P and P^* is a sequence $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ of mutually different vertices of P, such that for all $k = 1, \ldots, m$ the maximum between

$$\frac{\mathrm{vol}_n([\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{k-1}}\}]) - \mathrm{vol}_n([\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_k}\}])}{\mathrm{vol}_n([\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{k-1}}\}])}$$

and the analogous quantity obtained for P^* by removing its facets F_{i_j} whose normal vectors are x_{i_j}, $j = 1, \ldots, k$, is minimal over all choices of $x_{i_k} \in \{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{k-1}}\}$.

Theorem 5.1 *There exist positive constants a_0, a_1, such that for $0 < \epsilon < 1/2$ the following holds: Let $N, k \in \mathbb{N}$ be such that $N > k \geq a_0^n/\epsilon$ and let P be a convex polytope in $\mathbb{R}^n$ with the origin as an interior point and with N vertices $x_1, \ldots, x_N$. Then for all $K \in \mathbb{N}$, $N > K \geq k$ there exists a successive minimizing choice of vertices $x_{i_1}, \ldots, x_{i_{N-K}}$ of P such that the polytope $Q = [\{x_1, \ldots, x_N\} \setminus \{x_{i_1}, \ldots, x_{i_{N-K}}\}]$ with K vertices, satisfies*

$$\text{vol}_n(P) - \text{vol}_n(Q) \leq a_1 n^3 \epsilon^{-\frac{n+1}{n-1}} \text{vol}_n(P) \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}} \right) . \tag{5.2}$$

Moreover, there exist constants b_0, b_1, such that if, in the previous situation, the additional restrictions: $0 < \epsilon < 1/4$ and $k \geq (b_0 n)^{\frac{3(n-1)}{2}}/\epsilon^{\frac{n+1}{2}}$ are assumed, then the above choice can be made a successive dual minimizing choice of vertices and facets of P and P^* so that in addition to (5.2) we have

$$\text{vol}_n(Q^*) - \text{vol}_n(P^*) \leq b_1 n^3 \epsilon^{-\frac{n+1}{n-1}} \text{vol}_n(P^*) \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}} \right) . \tag{5.3}$$

Proof. We shall prove both parts of Theorem 5.1 together, assuming the stronger restrictions needed. Fix $0 < \epsilon < 1/4$. By Theorems 3.2 and 4.2, if $a_0 = \max(c_2, c_4)$ and $N > a_0^n/\epsilon$, there exists a subset $A_N \subset \{1, \ldots, N\}$ with card $A_N \geq (1 - 4\epsilon)N$, such that for all $i \in A_N$

$$\frac{\text{vol}_n(P) - \text{vol}_n([x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N])}{\text{vol}_n(P)} \leq b \cdot n^2 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}} \tag{5.4}$$

and

$$\frac{\text{vol}_n(P^*(i)) - \text{vol}_n(P^*)}{\text{vol}_n(P^*)} \leq b \cdot n^2 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}} , \tag{5.5}$$

for $b = \max(c_0, c_1)$. Here the notation $P^*(i)$ is defined as in Section 4. The facets of P^* are enumerated according to their normal vectors $\{x_1, \ldots, x_N\}$. Note also that

$$P^*(i) = ([x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N])^*.$$

We choose $i_0 \in A_N$ and apply the same argument to

$$P_{N-1} = [x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N]$$

instead of P. Continuing in this manner we get a sequence of polytopes $P_N = P$, $P_{N-1}, \ldots, P_K$. We set $Q = P_K$. It remains to show that Q satisfies (5.2) and (5.3).

Now, the repeated use of (5.4) and the fact that $P_j \subset P$ for $j = K, \ldots, N$, imply

$$\frac{\text{vol}_n(P) - \text{vol}_n(Q)}{\text{vol}_n(P)} \leq b\, n^2 \epsilon^{-\frac{n+1}{n-1}} \sum_{j=K+1}^{N} j^{-\frac{n+1}{n-1}} \leq$$

$$b\, n^2 \epsilon^{-\frac{n+1}{n-1}} \int_K^N \frac{dx}{x^{-\frac{n+1}{n-1}}} = b\, n^2 \epsilon^{-\frac{n+1}{n-1}} \cdot \frac{n-1}{2} \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}} \right) ,$$

which proves(5.2) with $a_1 = b/2$.

In proving (5.3)we encounter a technical obstacle: P_j^* is bigger then P^*. The repeated application of (5.5) imply

$$\frac{\mathrm{vol}_n(Q^*)}{\mathrm{vol}_n(P^*)} \le \prod_{j=K+1}^{N} \left(1 + b\, n^2 \epsilon^{-\frac{n+1}{n-1}}\, j^{-\frac{n+1}{n-1}}\right) . \tag{5.6}$$

Denoting the right hand side of (5.6) by S, we have (using the inequality $\log(1+x) < x$ for $x > 0$)

$$\log S \le b\, n^2 \epsilon^{-\frac{n+1}{n-1}} \sum_{j=K+1}^{N} j^{-\frac{n+1}{n-1}} \le a_1 n^3 \epsilon^{-\frac{n+1}{n-1}} \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}}\right) .$$

We now use the inequality $\mathrm{e}^x < 1 + x(\mathrm{e}-1)$ for $0 < x < 1$, to conclude that

$$S < 1 + (\mathrm{e}-1) a_1 n^3 \epsilon^{-\frac{n+1}{n-1}} \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}}\right) , \tag{5.7}$$

provided that $a_1 n^3 \epsilon^{-\frac{n+1}{n-1}} \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}}\right) < 1$, which holds if

$$K > (\sqrt{a_1})^{(n-1)} \epsilon^{-\frac{n+1}{2}} n^{\frac{3(n-1)}{2}} .$$

We choose $b_0 = \max(a_0^{\frac{2n}{3(n-1)}}, a_1^{\frac{1}{3}})$ and $b_1 = \frac{(\mathrm{e}-1)\cdot b}{2}$. This proves (5.3) and the second part of Theorem 5.1. We remark that at each step of removing one vertex we did not ‘expose the origin’, the origin remains in the interior of the successive polytopes; in fact if at some step the origin remains outside, then the resulting polar polyhedron at this step would be unbounded. Yet it is not, by the dual choice of vertices. ∎

Having a second look at the proof of Theorem 5.1, one may observe that (with the notations of that proof) we have

$$\left(1 - b \cdot n^2 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}}\right) \mathrm{vol}_n(P_N) \le \mathrm{vol}_n(P_{N-1}) \le \mathrm{vol}_n(P_N)$$

and

$$\mathrm{vol}_n(P_N^*) \le \mathrm{vol}_n(P_{N-1}^*) \le \left(1 + b \cdot n^2 \epsilon^{-\frac{n+1}{n-1}} N^{-\frac{n+1}{n-1}}\right) \mathrm{vol}_n(P_N^*) .$$

Using this observation and its counterparts in the next steps, we can follow the steps of the above proof and prove the following result concerning the *volume product* of P: $\Pi(P) = \mathrm{vol}_n(P)\,\mathrm{vol}_n(P^*)$.

Proposition 5.2 *Let the constants be as in Theorem 5.1 and let $0 < \epsilon < 1/4$ and $N > k > (2b_0 n)^{\frac{3(n-1)}{2}} / \epsilon^{\frac{n+1}{2}}$. If P is a convex polytope in $\mathbb{R}^n$ with N vertices, then for all $K \in \mathbb{N}$, $N > K \ge k$, there exists a successive dual minimizing choice of $N-K$ vertices of P, so that the resulting polytope with K vertices $Q \subset P$ satisfies*

$$|\Pi(P) - \Pi(Q)| \le c n^3 \epsilon^{-\frac{n+1}{n-1}} \Pi(P) \left(K^{-\frac{2}{n-1}} - N^{-\frac{2}{n-1}}\right) . \tag{5.8}$$

where $c = \max(a_1, b_1)$.

Remark 5.3 The proof of Theorem 5.1 shows that in Theorems 3.2 and 4.2 the dependence of the form $N^{-\frac{n+1}{n-1}}$ on the number of vertices, or of facets, is best possible in the following sense: assume that there exists a function $f(K,n)$ which satisfies for fixed n

$$\frac{f(K,n)}{K^{-\frac{n+1}{n-1}}} \searrow 0 \tag{5.9}$$

as K tends to infinity, and such that the inequality (5.4) holds for all sufficiently large N, with the right hand side replaced by $f(N,n)$. Take large K and $N > K$ so large as to approximate B_2^n to any desired precision by a polytope P with N vertices. Using the argument of the proof of Theorem 5.1, we would get a polytope $Q \subset P$, with K vertices, such that

$$\frac{\mathrm{vol}_n(P) - \mathrm{vol}_n(Q)}{\mathrm{vol}_n(P)} \le \frac{n-1}{2} \left(\frac{f(K,n)}{K^{-\frac{n+1}{n-1}}} \right) \frac{1}{K^{\frac{2}{n-1}}} . \tag{5.10}$$

Taking N sufficiently large, we may replace B_2^n for P in (5.10). But it is well known that we can not get an approximation like (5.10) for B_2^n if (5.9) holds. A similar argument works for the outer approximation.

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