On the p-affine surface area

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There are many links between Differential Geometry and Convexity Theory. An example of such a link is the affine surface area which has attracted increased attention in recent years.

Originally a concept of Affine Differential Geometry, it was introduced by Blaschke [Bl] for convex bodies in $\mathbf{R}^3$ with sufficiently smooth boundary and extended by Leichtweiss [L 1] to convex bodies in $\mathbf{R}^n$ with sufficiently smooth boundary. Its definition involves the Gauss curvature of the boundary points of a convex body.

This explains why it has become important in Convexity Theory: it provides a tool to "measure" the boundary structure of a convex body. This is one of the reasons for the renewed interest in the affine surface area in recent years. In many applications, so for instance in the approximation of convex bodies by polytopes, one needs to have information about the boundary structure of a convex body. Therefore it is not surprising that the affine surface area occurs naturally in many such approximation results -random as well as non-random- of the last few years by e.g. Bárány [Ba 1], [Ba 2], Gruber [G 1], [G 2], Gruber and Glasauer [G-G], Ludwig [Lud], Schneider [Sch] and Schütt [S 2], to name only a few.

Another reason is that various isoperimetric inequalities involving the affine surface area are very closely related to other important affine isoperimetric inequalities (e.g., the curvature image inequality, the Blaschke-Santaló inequality, and Petty 's geominimal surface area inequality) (see e.g., [Lu 1], [P 1], [P 2]). As an application, it has been proved by Lutwak and Oliker [Lu-O] that some of these inequalities lead to a priori estimates for a certain class of non-linear PDE's.

From the point of view of Convexity Theory as well as the applications it is a drawback to have the affine surface area only defined for convex bodies with sufficiently smooth boundary. To find extension of the affine surface area to arbitrary convex bodies without any smoothness assumptions on the boundary had been a problem that was open for a long time. Fortunately within the last decade several such extensions to arbitrary convex bodies have been given, namely by Leichtweiss [L 2], Lutwak [Lu 2], Schmuckenschläger [Schm] (for symmetric convex bodies), Schütt & Werner [S-W] and Werner [W 1]. These extensions all coincide as was shown by Dolzmann and Hug [D-H] (for the ones given by Leichtweiss and Lutwak), Schmuckenschläger [Schm] (for the ones given by Schmuckenschläger and Schütt & Werner), Schütt [S 1] (for the ones given by Leichtweiss and Schütt & Werner) and Werner [W 1] (for the ones given by Schütt & Werner and Werner).

In [M-W] we investigated a new class of convex bodies which Lutwak called the Santaló-bodies because of their connection with the Blaschke-Santaló inequality. It came as a surprise to us that these bodies provide yet another (completely different) extension of the affine surface area to arbitrary convex bodies which also coincides with the existing ones. All these extensions have a common feature which illustrates nicely the fact that the affine surface area is a link between Convexity Theory and Differential Geometry: geometric features of the affine surface area are used to give the desired extensions. In this way we also obtain a geometric characterization of the affine surface area. From these geometric characterizations we get further insight into the nature of the affine surface area and thus the boundary structure of a convex body. In a recent paper this idea has been used in [W 2] to give completely general geometric constructions for the affine surface area for arbitrary convex bodies which as special cases give all the (so far) known definitions.

Lutwak [Lu 3] introduced a generalization of the affine surface area, the p-affine surface area. For $p = 1$, the p-affine surface area is just the affine surface area. Lutwak also showed in [Lu 3] that the p-affine surface area satisfies extensions of the known inequalities involving affine surface area. As one of the big varieties of results the following p-extension of the affine isoperimetric inequality may be stated:

among all convex bodies in $\mathbf{R}^n$ with fixed volume the p-affine surface area is maximal if and oly if the body is an ellipsoid.

Therefore we expect that the p-affine surface area will turn out to be an equally useful tool as the affine surface area.

Hug [H 1] gave new definitions of the p-affine surface area. He also proved that this new definitions give the same p-affine surface area as that defined by Lutwak. (In particular, Hug generalized the work of Dolzmann-Hug from the $p = 1$ case to arbitrary p). Hug showed that for the case $p = n$, the p-affine surface area is the well-known centro-affine surface area. Thus the notion of p-affine surface area connects two important affine geometric functionals.

The purpose of this paper is to give a geometric interpretation of the p-affine surface area comparable to the ones given for the affine surface area. This is done in terms of the generalized Santaló-bodies which are of interest in their own right. In the first part of the paper we provide the necessary background and definitions and introduce the generalized Santaló-bodies and study some of their properties. In the second part of the paper we give geometric interpretations of the p-affine surface area.

The analytic expression of p-affine surface area in [Lu 3] for convex bodies with positive continuous curvature function makes sense not only for positive p, but also for $-n < p \leq 0$ and our geometric interpretation holds for those p too. We also introduce a definition of the p-affine surface area for $p = -n$ together with its geometric interpretation.

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Let K be a convex body in $\mathbf{R}^n$ and $t \in \mathbf{R}$, $t > 0$. In [M-W] the Santaló-bodies $S(K,t)$ were defined as

$$S(K,t) = \{x \in K : \frac{|K||K^x|}{v_n^2} \leq t\},$$

where $|K|$ denotes the n-dimensional volume of the convex body K and v_n is the volume of the n-dimensional Euclidean unit ball $B(0,1)$.

For a convex body K with sufficiently smooth boundary the affine surface area $O_1(K)$ is

$$O_1(K) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}} d\sigma(u) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu_K(x),$$

where $f_K(u)$ is the Gauss curvature function, that is the reciprocal of the Gauss curvature $\kappa(x)$ at this point $x \in \partial K$ that has u as outer normal. μ_K is the usual surface measure on the boundary ∂K of K and σ is the spherical Lebesgue measure.

The connection between $O_1(K)$ and the Santaló-bodies is as follows

$$\lim_{t \to \infty} t^{\frac{2}{n+1}} (|K| - |S(K,t)|) = \frac{1}{2} (\frac{|K|}{v_n})^{\frac{2}{n+1}} \quad O_1(K), \tag{1}$$

and thus the left hand side provides an extension of the affine surface area to arbitrary convex bodies without any smoothness assumptions on the boundary of K. The one given by (1) coincides with the ones given earlier by [L 2], [Lu 2], [S-W] and [W 1].

In [Lu 3] Lutwak introduced the p-affine surface area $O_p(K)$. For a convex body K in $\mathbf{R}^n$ with positive continuous curvature function it can be written as

$$O_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N(x) >^{\frac{n(p-1)}{n+p}}} d\mu_K(x),$$

where h_K is the support function of K and $N(x)$ is the outer normal in $x \in \partial K$.

By $||.||$ we denote the standard Euclidean norm on $\mathbf{R}^n$, $<\cdot,\cdot>$ is the usual inner product on $\mathbf{R}^n$. $B(a,r)$ is the n-dimensional Euclidean ball with radius r centered at a.

For two non-empty sets A and B in $\mathbf{R}^n$, $\text{co}[A,B]$ is the convex hull of $A \cup B$.

Unless stated otherwise we will always assume that a convex body K in $\mathbf{R}^n$ has its Santaló-point at the origin. Then 0 is the center of mass of the polar body K^0 which may be written as

$$\int_{K^0} < x, y > dy = 0 \quad \text{for every} \quad x \in \mathbf{R}^n.$$

Let int(K) be the interior of K and for $x \in \text{int}(K)$, $K^x = (K-x)^0 = \{y \in \mathbf{R}^n :< y, z-x > \leq 1 \ \text{ for all z} \in K\}$ is the polar body of K with respect to x; K^0 denotes the polar body with respect to the Santaló-point. Moreover for $u \in S^{n-1}$ we will denote by $g_{K,u}(s)$ the $(n-1)$ - dimensional volume of the sections of K orthogonal to u, that is

$$g_{K,u}(s) = |\{z \in K :< z, u >= s\}|.$$

Let $\phi : (-1,1) \to [0,\infty)$ be a continuous function such that

$$\lim_{s \to 1} \phi(s) = \infty.$$

For $x \in \text{int}(K)$ we put

$$\Phi_K(x) = \int_{K^0} \phi(< x, y >) dy. \tag{2}$$

If $u \in S^{n-1}$ and $\lambda \in \mathbf{R}$ are such that $0 \leq \lambda < \frac{1}{h_{K^0}(u)}$, then $x = \lambda u \ \in \text{int}(K)$ and (2) can be written as

$$\Phi_K(x) = \int_{-h_{K^0}(-u)}^{h_{K^0}(u)} g_{K^0,u}(s) \phi(\lambda s) ds. \tag{3}$$

Remark 1

The motivation to introduce Φ_K comes from [M-W] where it was observed that for $u \in S^{n-1}$ and $\lambda \in \mathbf{R}$ such that $0 \leq \lambda < \frac{1}{h_{K^0}(u)}$ we have for $x = \lambda u$,

$$|K^x| = \int_{K^0} \frac{dy}{(1- < x, y >)^{n+1}} = \int_{-h_{K^0}(-u)}^{h_{K^0}(u)} \frac{g_{K^0,u}(s)}{(1-\lambda s)^{n+1}} ds.$$

The above expressions (2) and (3) generalize this.

We will eventually be interested in more specific functions ϕ. But first let us state some general Lemmas.

Lemma 2

Let K be a convex body in $\mathbf{R}^n$ and ϕ and Φ_K as above.
If ϕ is convex, then Φ_K is a convex function on int(K).
If ϕ is strictly convex, then so is Φ_K.

Proof

Let x_1 and x_2 be in $\text{int}(K)$, $0 \leq \lambda \leq 1$. Then

$$\Phi_K(\lambda x_1 + (1-\lambda)\ x_2) = \int_{K^0} \phi(< \lambda x_1 + (1-\lambda)\ x_2, y >)\ dy,$$

which, by convexity of ϕ is

$$\leq \int_{K^0} \Big(\lambda \phi(< x_1, y >) + (1-\lambda)\phi(< x_2, y >) \Big)\ dy$$

$$= \lambda \Phi_K(x_1) + (1-\lambda)\Phi_K(x_2).$$

If ϕ is strictly convex, then Φ_K is strictly convex, as for x_1 and x_2 in $\text{int}(K)$, $x_1 \neq x_2$, the n-dimensional volume of the set

$$\{y \in K^0 :< x_1, y >=< x_2, y >\}$$

is equal to 0.

We define now for a function ϕ with above properties and for $t \in \mathbf{R}, t > 0$

$$S_\phi(K,t) = \{x \in K : \Phi_K(x) \leq t\}.$$

In the sequel we consider only those t for which the $S_\phi(K,t)$ are non-empty.

Then we have

Lemma 3
Let $\phi : (-1,1) \to [0,\infty)$ be a continuous convex function such that $lim_{s \to 1}\phi(s) = \infty$. Then

(i) For all $t > 0$, $S_\phi(K,t)$ is a convex set.
If ϕ is strictly convex, then so is $S_\phi(K,t)$.

(ii) For every affine transformation A with det $A \neq 0$, for all $t > 0$

$$S_\phi(A(K),t) = A(S_\phi(K, |detA|t)).$$

Proof

(i) follows immediately from Lemma 2.

(ii) Let A be a affine transformation with $\det A \neq 0$. We can write $A = L + a$, where L is a linear transformation with $\det L \neq 0$ and a is a vector in $\mathbf{R}^n$. Then, as 0 is the Santaló-point of K, a is the Santaló-point of $A(K)$ and thus

$$(A(K))^a = \{z \in \mathbf{R}^n :< z, Ay - a >\leq 1 \text{ for all } y \in K\}$$

$$= \{z \in \mathbf{R}^n :< L^* z, y >\leq 1 \text{ for all } y \in K\}$$

$$= \{(L^*)^{-1} w :< w, y >\leq 1 \text{ for all } y \in K\}$$

$$= (L^*)^{-1}(K^0).$$

Hence

$$S_\phi(A(K), t) = \{z \in A(K) : \Phi_{A(K)}(z) \leq t\}$$

$$= \{Ax : x \in K, \int_{(A(K))^a} \phi(< Ax - a, y >)\ dy \leq t\}$$

$$= \{Ax : x \in K, \int_{(L^*)^{-1}(K^0)} \phi(< Lx, y >)\ dy \leq t\}$$

$$= \{Ax : x \in K, |\det(L^*)^{-1}| \int_{K^0} \phi(< Lx, (L^*)^{-1} y >)\ dy \leq t\}$$

$$= A(S_\phi(K, |\det A| t)).$$

Now we consider special functions ϕ. In view of Remark 1 a natural class of functions ϕ to consider are

$$\phi_\beta(s) = \frac{1}{(1-s)^\beta},$$

for $\frac{n+1}{2} \leq \beta$.
For such ϕ_β we denote $S_{\phi_\beta}(K, t)$ by $S_\beta(K, t)$, that is

$$S_\beta(K, t) = \{x \in K : \int_{K^0} \frac{dy}{(1- < x, y >)^\beta} \leq t\},$$

or with $x = \lambda u$, $u \in S^{n-1}$, $0 \leq \lambda < \frac{1}{h_{K^0}(u)}$,

$$S_\beta(K, t) = \{x = \lambda u \in K : \int_{-h_{K^0}(-u)}^{h_{K^0}(u)} \frac{g_{K^0,u}(s)}{(1-\lambda s)^\beta} ds \leq t\}.$$

In particular for $\beta = n + 1$ we get the Santaló-bodies $S(K, t)$ of [M-W].

In the same way for $y \in \partial K^0$, $y = \lambda u$, $u \in S^{n-1}$, $0 \le \lambda < \frac{1}{h_K(u)}$

$$S_\beta(K^0, t) = \{y \in K^0 : \int_K \frac{dx}{(1 - < x, y >)^\beta} \le t\} =$$

$$\{y = \lambda u \in K^0 : \int_{-h_K(-u)}^{h_K(u)} \frac{g_{K,u}(s)}{(1 - \lambda s)^\beta} ds \le t\}.$$

Remark

Instead of the functions ϕ_β, a slightly more general class of functions ϕ can be considered for which the conclusions of Theorem 6 and Proposition 7 will still hold; namely for functions $\phi : (-1, 1) \to [0, \infty)$ that are convex, continuous and such that

$$lim_{s \to 1} (1 - s)^\beta \phi(s) = c,$$

where c is a constant. This is easy to see from Lemma 5 and the proofs of Theorem 6 and Proposition 7.

As for the Santaló-bodies $S(K, t)$ one can give estimates on the "size" of $S_\beta(K, t)$ in terms of ellipsoids. Recall that for a convex body K the Binet ellipsoid $E(K)$ is defined by (see for instance [Mi-P])

$$||u||^2_{E(K)} = \frac{1}{|K|} \int_K < x, u >^2 dx, \qquad \text{for all u} \in \mathbf{R}^n.$$

Proposition 4

Let K be a convex body in $\mathbf{R}^n$. Then for $\frac{n+1}{2} \le \beta \le n + 1$,

$$d_n(t, \beta) \;\; E(K^0) \subseteq S_\beta(K, t) \subseteq c_n(t, \beta) \;\; E(K^0),$$

where

$$d_n(t, \beta) = \frac{1}{\sqrt{3} \;\; n} \;\; (1 - \frac{n|K^0|}{t(\beta - 1)})^{\frac{1}{2}}$$

and

$$c_n(t, \beta) = \frac{2\sqrt{2}}{((e - 2)\beta(\beta + 1))^{\frac{1}{2}}} (\frac{t}{|K^0|})^{\frac{1}{2}} (1 - \frac{|K^0|}{t})^{\frac{1}{2}}.$$

If K is in addition symmetric, then $c_n(t, \beta)$ can be chosen as follows:

$$c_n(t, \beta) = min\{\frac{2\sqrt{2}}{((e - 2)\beta(\beta + 1))^{\frac{1}{2}}} (\frac{t}{|K^0|})^{\frac{1}{2}} (1 - \frac{|K^0|}{t})^{\frac{1}{2}}, \sqrt{2}(1 - (\frac{|K^0|}{t})^{\frac{1}{\beta - 1}})^{\frac{1}{2}}\}.$$

Remark

Observe that for $\frac{n+1}{2} \le \beta \le n+1$, $\frac{c_n(t,\beta)}{d_n(t,\beta)}$ *is equal to a constant that depends only on* $\frac{|K^0|}{t}$. *This means that if* $t \le c|K^0|$, *c a constant, then* $S_\beta(K,t)$ *has bounded (in terms of c) Banach Mazur distance to the ellipsoid* $E(K^0)$.

Proof of Proposition 4

To get the right hand side inclusions we proceed exactly as in [M-W], Theorems 6 and 7. To get the left hand side inclusion, it is enough to consider the symmetric case by [F]. Then, as in the proof of Theorem 6 of [M-W],

$$t \le g_{K^0,u}(0) \int_0^a \Big(\frac{(1-\frac{s}{a})^{n-1}}{(1-\lambda s)^\beta} + \frac{(1-\frac{s}{a})^{n-1}}{(1+\lambda s)^\beta} \Big) ds,$$

where $a = \frac{n|K^0|}{2g_{K^0,u}(0)}$ and for $u \in S^{n-1}$, $\lambda u \in \partial S_\beta(K,t)$.
In the case $\lambda a \ge 1$, we get immediately as in [M-W]

$$S_\beta(K,t) \supseteq \frac{1}{\sqrt{3} \;\; n} E(K^0).$$

In the case $\lambda a < 1$, we estimate

$$g_{K^0,u}(0) \int_0^a \Big(\frac{(1-\frac{s}{a})^{n-1}}{(1-\lambda s)^\beta} + \frac{(1-\frac{s}{a})^{n-1}}{(1+\lambda s)^\beta} \Big) ds \le$$

$$g_{K^0,u}(0) \int_0^a \Big(\frac{(1-\frac{s}{a})^{\beta-2}}{(1-\lambda s)^\beta} + \frac{(1-\frac{s}{a})^{\beta-2}}{(1+\lambda s)^\beta} \Big) ds =$$

$$\frac{g_{K^0,u}(0) \;\; a}{\beta - 1} \left(\frac{1}{(1-\lambda a)} + \frac{1}{(1+\lambda a)} \right),$$

from which it follows, as in [M-W], that

$$S_\beta(K,t) \supseteq \frac{1}{\sqrt{3} \;\; n} \left(1 - \frac{n|K^0|}{t(\beta-1)} \right)^{\frac{1}{2}} E(K^0).$$

To relate the convex bodies $S_\beta(K,t)$ and $S_\beta(K^0,t)$ to the p-affine surface area we need the following Lemma.

Lemma 5

(i) Let $\gamma > -1$ and $\beta > \gamma+1$. For $\alpha \in (0,1)$ let

$$I(\alpha) = \frac{\alpha^{\gamma+1}(1-\alpha)^{\beta-(\gamma+1)}}{2^\gamma B(\gamma+1,\beta-(\gamma+1))} \int_0^1 \frac{(1-x^2)^\gamma dx}{(1-\alpha x)^\beta},$$

where for $x,y>0$, $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Betafunction. Then

$$I(\alpha) \le 1 \quad \text{and} \quad lim_{\alpha\to 1} I(\alpha) = 1.$$

(ii) Let $\gamma > 0$ and for $\alpha \in (0,1)$ let

$$J(\alpha) = \frac{1}{2^\gamma ln\frac{1}{1-\alpha}} \int_0^1 \frac{(1-x^2)^\gamma dx}{(1-\alpha x)^{\gamma+1}}.$$

Then

$$J(\alpha) \le 1 + \frac{1}{(\gamma+1)ln\frac{1}{1-\alpha}} \quad \text{and} \quad lim_{\alpha\to 1} J(\alpha) = 1$$

Proof

(i) Put $x = 1 - w\frac{1-\alpha}{\alpha}$. Then

$$I(\alpha) = \frac{1}{2^\gamma B(\gamma+1,\beta-(\gamma+1))} \int_0^{\frac{\alpha}{1-\alpha}} \frac{w^\gamma (2-\frac{1-\alpha}{\alpha}w)^\gamma dw}{(1+w)^\beta}.$$

The upper estimate for (i) follows immediately from this last expression. And by the Monotone Convergence Theorem this last expression tends to

$$\frac{1}{B(\gamma+1,\beta-(\gamma+1))} \int_0^\infty \frac{w^\gamma dw}{(1+w)^\beta},$$

which is equal to 1.

(ii) Put $\alpha = 1-e^{-q}$ and $x = 1-e^{-qs}$. Then

$$\int_0^1 \frac{(1-x^2)^\gamma dx}{(1-\alpha x)^{\gamma+1}} = q\int_0^\infty \frac{(2-e^{-qs})^\gamma ds}{(1+e^{-q(1-s)}-e^{-q})^{\gamma+1}}.$$

Hence

$$J(\alpha) = \frac{1}{2^\gamma}\int_0^\infty \frac{(2-e^{-qs})^\gamma ds}{(1+e^{-q(1-s)}-e^{-q})^{\gamma+1}}$$

$$\leq \int_0^1 \frac{ds}{(1+e^{-q(1-s)}-e^{-q})^{\gamma+1}} + \int_1^\infty \frac{ds}{e^{q(s-1)(\gamma+1)}}$$

$$\leq 1 + \frac{1}{q(\gamma+1)}.$$

The Monotone Convergence Theorem again implies that $\lim_{\alpha\to 1} J(\alpha) = 1$.

The following theorem gives a geometric interpretation of the p-affine surface area for all $p > -n$.

Theorem 6

Let K be a convex body in $\mathbf{R}^n$ such that ∂K is C^3 and has strictly positive Gaussian curvature everywhere. Then for $\frac{n+1}{2} < \beta$

$$lim_{t\to\infty} \left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} (|K| - |S_\beta(K,t)|) = \int_{S^{n-1}} \frac{f_{K^0}(u)^{\frac{1}{2\beta-(n+1)}}}{h_{K^0}(u)^{n-\frac{n+1}{2\beta-(n+1)}}} d\sigma(u),$$

where $c_{n,\beta} = 2^{\frac{n-1}{2}} v_{n-1} B(\frac{n+1}{2}, \beta - \frac{n+1}{2})$.

Remarks

(i) Thus we have that

$$lim_{t\to\infty} \left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} (|K| - |S_\beta(K,t)|) = O_{n(2\beta-n-2)}(K^0).$$

Especially for $\beta = n+1$ we get

$$O_{n^2}(K^0) = lim_{t\to\infty} \left(\frac{t}{c_{n,n+1}}\right)^{\frac{2}{n+1}} (|K| - |S_{n+1}(K,t)|).$$

$S_{n+1}(K,t)$ however is the Santaló-body $S(K,t)$ introduced in [M-W] and it was shown there that

$$lim_{t\to\infty} \left(\frac{t}{c_{n,n+1}}\right)^{\frac{2}{n+1}} (|K| - |S(K,t)|) = O_1(K).$$

Thus we get again a special case of a general formula of Hug [H 2] who showed that for $p > 0$

$$O_p(K) = O_{\frac{n^2}{p}}(K^0).$$

Therefore we also get by Hug's formula and by Theorem 6 that for $\beta > \frac{n+1}{2}$

$$lim_{t\to\infty} \left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} (|K| - |S_\beta(K,t)|) = O_{\frac{n}{2\beta-n-2}}(K).$$

(ii) As $(K^0)^0 = K$, it follows immediately from Theorem 6 under the same hypothesis on K that

$$lim_{t\to\infty}(\frac{t}{c_{n,\beta}})^{\frac{1}{\beta-\frac{n+1}{2}}}(|K^0| - |S_\beta(K^0,t)|) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{1}{2\beta-(n+1)}}}{h_K(u)^{n-\frac{n+1}{2\beta-(n+1)}}} d\sigma(u)$$

$$= O_{n(2\beta-n-2)}(K).$$

(iii) Using a compactness argument , the method of our proof shows that the statement of Theorem 6 holds if we only suppose that ∂K is C^2 and has strictly positive Gaussian curvature everywhere.

Proof of Theorem 6

Note first that if K is such that ∂K is C^3 and has strictly positive Gaussian curvature everywhere, then the same holds for K^0.

For $u \in S^{n-1}$ let $y \in \partial K^0$ be such that $N(y) = u$.

By assumption the indicatrix of Dupin at y exists and is an ellipsoid. We can assume that it is a sphere (see for instance [S-W]). Let $\sqrt{\rho} = \sqrt{\rho(u)}$ be the radius of this sphere.

We introduce a coordinate system such that $y = 0$ and $u = N(y) = (0, \dots 0, -1)$. H_0 is the tangent hyperplane to ∂K^0 in $y = 0$ and $\{H_s : s \geq 0\}$ is the family of hyperplanes parallel to H_0 that have non-empty intersection with K^0 and are at distance s from H_0. For $s > 0$, H_s^+ is the halfspace generated by H_s that contains $y = 0$. For $a \in \mathbf{R}$, let $z_a = (0, \dots 0, a)$ and $B_a = B(z_a, a)$ be the Euclidean ball with center z_a and radius a. As in [W 1], for $\varepsilon > 0$ there exists $s_\varepsilon = s_\varepsilon(u)$ so that for all $s \leq s_\varepsilon$

$$B_{\rho-\varepsilon} \cap H_s^+ \subseteq K^0 \cap H_s^+ \subseteq B_{\rho+\varepsilon} \cap H_s^+.$$

We choose $s_0 = \min\{s_\varepsilon, \frac{\rho-\varepsilon}{2}\}$.

Define C_1 to be the cone tangent to $B_{\rho+\varepsilon}$ at $H_{s_0} \cap B_{\rho+\varepsilon}$ and choose the minimal s_1 so that

$$K^0 \cap H_{s_0}^- \subseteq D = C_1 \cap \{z : s_0 \leq < z, -u > \leq s_1\}.$$

Then K^0 is contained in the union of the truncated cone D and the cap $B_{\rho+\varepsilon} \cap H_{s_0}^+ = \{z \in B_{\rho+\varepsilon} : < z, -u > \leq s_0\}$

$$K^0 \subseteq D \cup (B_{\rho+\varepsilon} \cap H_{s_0}^+). \tag{4}$$

Let P be the point of intersection of ∂K^0 with the positive x_n-axis. Let C_2 be the spherical cone $C_2 = \mathrm{co}[P, B_{\rho-\varepsilon} \cap H_{s_0}]$ and let h be the height of this cone. Then

$$K^0 \supseteq C_2 \cup (B_{\rho-\varepsilon} \cap H_{s_0}^+). \tag{5}$$

figure6.pdf

Now we want to estimate $g_{K^0,u}(s)$. To do so we switch the origin of the coordinate system such that the u-coordinate of the centroid of K^0 is at 0 and the positive u-direction coincides with the positive s-direction (see the figure for the body containing K^0).

Thus, because of (4)

$$g_{K^0,u}(s) \leq v_{n-1}\left[(\rho+\varepsilon)^2 - \left(s-(h_{K^0}(u)-(\rho+\varepsilon))\right)^2\right]^{\frac{n-1}{2}},$$

if

$$h_{K^0}(u) - s_0 \leq s \leq h_{K^0}(u),$$

and

$$g_{K^0,u}(s) \leq v_{n-1}\left[\frac{s_0(\rho+\varepsilon)+(\rho+\varepsilon-s_0)(h_{K^0}(u)-s)}{\left(2(\rho+\varepsilon)s_0 - s_0^2\right)^{\frac{1}{2}}}\right]^{n-1},$$

if

$$-h_{K^0}(-u) \leq s \leq h_{K^0}(u) - s_0.$$

And because of (5)

$$g_{K^0,u}(s) \geq v_{n-1}\left[(\rho-\varepsilon)^2 - \left(s-(h_{K^0}(u)-(\rho-\varepsilon))\right)^2\right]^{\frac{n-1}{2}},$$

if

$$h_{K^0}(u) - s_0 \leq s \leq h_{K^0}(u),$$

and

$$g_{K^0,u}(s) \geq \frac{v_{n-1}}{h^{n-1}}\left[(2s_0(\rho-\varepsilon)-s_0^2)^{\frac{1}{2}}(s-h_{K^0}(u)+s_0+h)\right]^{n-1},$$

if

$$h_{K^0}(u) - s_0 - h \leq s \leq h_{K^0}(u) - s_0.$$

Let $\lambda \in \mathbf{R},\ 0 \leq \lambda < \frac{1}{h_{K^0}(u)}$. Then we have for $x = \lambda u \in \partial S_\beta(K,t)$

$$\begin{aligned} t &= \int_{-h_{K^0}(-u)}^{h_{K^0}(u)} \frac{g_{K^0,u}(s)ds}{(1-\lambda s)^\beta} \\ &\leq v_{n-1}(I_1+I_2), \qquad (6) \end{aligned}$$

where

$$I_1 = \int_{h_{K^0}(u)-s_0}^{h_{K^0}(u)} \frac{\left((\rho+\varepsilon)^2 - \left(s - (h_{K^0}(u) - (\rho+\varepsilon))\right)^2\right)^{\frac{n-1}{2}} ds}{(1-\lambda s)^{\beta}} \Bigg\}$$

and

$$I_2 = \int_{-h_{K^0}(-u)}^{h_{K^0}(u)-s_0} \frac{\left(s_0(\rho+\varepsilon) + (\rho+\varepsilon-s_0)(h_{K^0}(u)-s)\right)^{n-1} ds}{\left(2(\rho+\varepsilon)s_0 - s_0^2\right)^{\frac{n-1}{2}} (1-\lambda s)^{\beta}}.$$

We consider first I_1.

$$I_1 = (\rho+\varepsilon)^{n-1} \int_{h_{K^0}(u)-s_0}^{h_{K^0}(u)} \frac{\left[1 - \left(1 + \frac{s}{\rho+\varepsilon} - \frac{h_{K^0}(u)}{\rho+\varepsilon}\right)^2\right]^{\frac{n-1}{2}} ds}{(1-\lambda s)^{\beta}}.$$

We put $v = 1 + \frac{s}{\rho+\varepsilon} - \frac{h_{K^0}(u)}{\rho+\varepsilon}$ and get

$$I_1 \leq \frac{(\rho+\varepsilon)^n}{\left(1 - \lambda(h_{K^0}(u) - (\rho+\varepsilon))\right)^{\beta}} \int_0^1 \frac{(1-v^2)^{\frac{n-1}{2}} dv}{\left(1 - \frac{\lambda(\rho+\varepsilon)v}{1-\lambda(h_{K^0}(u)-(\rho+\varepsilon))}\right)^{\beta}},$$

which, by Lemma 5 (i) is

$$\leq \frac{2^{\frac{n-1}{2}} (\rho+\varepsilon)^{\frac{n-1}{2}} B(\frac{n+1}{2}, \beta - \frac{n+1}{2})}{\lambda^{\frac{n+1}{2}} (1 - \lambda h_{K^0}(u))^{\beta - \frac{n+1}{2}}}.$$

Using $\rho+\varepsilon \geq s_0$ and $\lambda(s+s_0) \leq \lambda h_{K^0}(u) < 1$, we have

$$I_2 \leq$$

$$\frac{\left(s_0(\rho+\varepsilon) + (\rho+\varepsilon-s_0)(h_{K^0}(u) + h_{K^0}(-u))\right)^n - \left(s_0(\rho+\varepsilon) + (\rho+\varepsilon-s_0)s_0\right)^n}{n(\lambda s_0)^{\beta} (2(\rho+\varepsilon)s_0 - s_0^2)^{\frac{n-1}{2}} (\rho+\varepsilon-s_0)}$$

$$\leq \frac{(\rho+\varepsilon)^{\frac{n+1}{2}} \left(s_0 + h_{K^0}(u) + h_{K^0}(-u)\right)^n}{n\lambda^{\beta}(\rho+\varepsilon-s_0) s_0^{\beta+\frac{n-1}{2}}}.$$

Thus, putting $c_{n,\beta} = 2^{\frac{n-1}{2}} v_{n-1} B(\frac{n+1}{2}, \beta - \frac{n+1}{2})$, we get

$$t \leq \frac{c_{n,\beta}(\rho+\varepsilon)^{\frac{n-1}{2}}}{\lambda^{\frac{n+1}{2}}(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}} \left\{ 1 + \frac{v_{n-1}(\rho+\varepsilon)(1-\lambda h_{K^0}(u))^{\beta-\frac{n-1}{2}}(s_0 + h_{K^0}(u) + h_{K^0}(-u))^n}{n c_{n,\beta}(\rho+\varepsilon-s_0) s_0^{\beta+\frac{n+1}{2}} \lambda^{\beta-\frac{n+1}{2}}} \right\}.$$

We choose λ so big that

$$\lambda \geq \frac{1 - \min\{\varepsilon^{\frac{\beta+1+\frac{n+1}{2}}{\beta-\frac{n+1}{2}}}, s_0^{\frac{\beta+1+\frac{n+1}{2}}{\beta-\frac{n+1}{2}}}\}}{\min\{h_{K^0}(u), \rho-\varepsilon\}}. \tag{7}$$

Then

$$t \leq \frac{c_{n,\beta}(\rho+\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}}}{(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}}(1+c_1\varepsilon), \tag{8}$$

where c_1 is a constant.

On the other hand

$$t \geq v_{n-1} \int_{h_{K^0}(u)-s_0}^{h_{K^0}(u)} \frac{\left((\rho-\varepsilon)^2 - \left(s - (h_{K^0}(u) - (\rho-\varepsilon))\right)^2\right)^{\frac{n-1}{2}} ds}{(1-\lambda s)^\beta},$$

which with $v = 1 + \frac{s}{\rho-\varepsilon} - \frac{h_{K^0}(u)}{\rho-\varepsilon}$ is equal to

$$\frac{v_{n-1}(\rho-\varepsilon)^n}{\left(1-\lambda(h_{K^0}(u)-(\rho-\varepsilon))\right)^\beta} \left\{ \int_0^1 \frac{(1-v^2)^{\frac{n-1}{2}} dv}{\left(1 - \frac{\lambda(\rho-\varepsilon)v}{1-\lambda(h_{K^0}(u)-(\rho-\varepsilon))}\right)^\beta} - \int_0^{1-\frac{s_0}{\rho-\varepsilon}} \frac{(1-v^2)^{\frac{n-1}{2}} dv}{\left(1 - \frac{\lambda(\rho-\varepsilon)v}{1-\lambda(h_{K^0}(u)-(\rho-\varepsilon))}\right)^\beta} \right\}. \tag{9}$$

As (7) holds we get with Lemma 5 (i)

$$t \geq \frac{c_{n,\beta}(\rho-\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}}}{(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}} \left(1 - \varepsilon - \frac{2^{\frac{n+3}{2}} v_{n-1}(\rho-\varepsilon)^{\frac{n+1}{2}}(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}}{(n+1)c_{n,\beta} s_0^\beta \lambda^{\beta-\frac{n+1}{2}}}\right)$$

or, again using (7)

$$t \geq \frac{c_{n,\beta}(\rho-\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}}}{(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}}(1-c_2\varepsilon), \tag{10}$$

where c_2 is a constant.

Thus we get for $x = \lambda u \in \partial S_\beta(K,t)$

$$\lambda = \frac{1}{h_{S_\beta(K,t)^0}(u)}$$

and from (8)

$$\lambda \geq \frac{1}{h_{K^0}(u)} \left\{ 1 - \left(\frac{c_{n,\beta}(\rho+\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}} (1+c_1\varepsilon)}{t} \right)^{\frac{1}{\beta - \frac{n+1}{2}}} \right\}$$

respectively from (10)

$$\lambda \leq \frac{1}{h_{K^0}(u)} \left\{ 1 - \left(\frac{c_{n,\beta}(\rho-\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}} (1-c_2\varepsilon)}{t} \right)^{\frac{1}{\beta - \frac{n+1}{2}}} \right\}.$$

Therefore for t big enough

$$(1-d_1\varepsilon)^{\frac{1}{\beta-\frac{n+1}{2}}} (\rho(u)-\varepsilon)^{\frac{n-1}{2\beta-(n+1)}} h_{K^0}(u)^{-n+\frac{n+1}{2\beta-(n+1)}} \leq$$

$$\frac{1}{n}\left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} \left(\left(\frac{1}{h_{K^0}(u)}\right)^n - \left(\frac{1}{h_{S_\beta(K,t)^0}(u)}\right)^n \right) \leq$$

$$(1+d_2\varepsilon)^{\frac{1}{\beta-\frac{n+1}{2}}} (\rho(u)+\varepsilon)^{\frac{n-1}{2\beta-(n+1)}} h_{K^0}(u)^{-n+\frac{n+1}{2\beta-(n+1)}},$$

where d_1 and d_2 are constants. This means that for every $u \in S^{n-1}$

$$\lim_{t\to\infty} \frac{1}{n}\left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} \left(\left(\frac{1}{h_{K^0}(u)}\right)^n - \left(\frac{1}{h_{S_\beta(K,t)^0}(u)}\right)^n \right) = \frac{f_{K^0}(u)^{\frac{1}{2\beta-(n+1)}}}{h_{K^0}(u)^{n-\frac{n+1}{2\beta-(n+1)}}}.$$

Hence

$$\lim_{t\to\infty} \left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} \Big(|K| - |S_\beta(K,t)| \Big) =$$

$$\lim_{t\to\infty} \int_{S^{n-1}} \frac{1}{n}\left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} \left(\left(\frac{1}{h_{K^0}(u)}\right)^n - \left(\frac{1}{h_{S_\beta(K,t)^0}(u)}\right)^n \right) d\sigma(u) =$$

$$\int_{S^{n-1}} \lim_{t\to\infty} \frac{1}{n}\left(\frac{t}{c_{n,\beta}}\right)^{\frac{1}{\beta-\frac{n+1}{2}}} \left(\left(\frac{1}{h_{K^0}(u)}\right)^n - \left(\frac{1}{h_{S_\beta(K,t)^0}(u)}\right)^n \right) d\sigma(u)$$

$$= \int_{S^{n-1}} \frac{f_{K^0}(u)^{\frac{1}{2\beta-(n+1)}}}{h_{K^0}(u)^{n-\frac{n+1}{2\beta-(n+1)}}} d\sigma(u).$$

We still have to justify that we can interchange integration and limit. This follows from Lebesgue's Theorem and the
Claim
for every $u \in S^{n-1}$

$$t^{\frac{1}{\beta-\frac{n+1}{2}}}\left((\frac{1}{h_{K^0}(u)})^n - (\frac{1}{h_{S_\beta(K,t)^0}(u)})^n \right) \leq l(u),$$

where l is a function independent of t and integrable on S^{n-1}.
Proof of the Claim

For $u \in S^{n-1}$ let again $y \in \partial K^0$ be such that $N(y) = u$. We assume again that the indicatrix of Dupin at y is a Euclidean ball with radius $\sqrt{\rho} = \sqrt{\rho(u)}$. We choose t_0 so big that $S_\beta(K, t_0)$ is a convex body. Then $S_\beta(K, t_0)$ has non-empty interior and there is $\alpha > 0$ such that

$$B(0, \alpha) \subseteq S_\beta(K, t_0) \subseteq B(0, \frac{1}{\alpha}).$$

and the same holds for all $t \geq t_0$. Thus for $x = \lambda u \in \partial S_\beta(K, t)$

$$\alpha \leq \lambda = \frac{1}{h_{S_\beta(K,t)^0}(u)} \leq \frac{1}{\alpha}. \tag{11}$$

As before, we estimate

$$t = \int_{-h_{K^0}(-u)}^{h_{K^0}(u)} \frac{g_{K^0,u}(s)ds}{(1-\lambda s)^\beta} \leq v_{n-1}(I_1 + I_2),$$

where I_1 and I_2 are as above. From above we get for all t

$$I_1 \leq \frac{2^{\frac{n-1}{2}}(\rho+\varepsilon)^{\frac{n-1}{2}} B(\frac{n+1}{2}, \beta - \frac{n+1}{2})}{\lambda^{\frac{n+1}{2}}(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}},$$

which, using (11), can be estimated for all $t \geq t_0$ by

$$\leq k_1 \frac{(\rho+\varepsilon)^{\frac{n-1}{2}}}{(1-\lambda h_{K^0}(u))^{\beta-\frac{n+1}{2}}},$$

where k_1 is a constant independent of t and u. Also from above we get for all t

$$I_2 \leq \frac{(\rho+\varepsilon)^{\frac{n+1}{2}}\left(s_0 + h_{K^0}(u) + h_{K^0}(-u)\right)^n}{n\lambda^\beta(\rho+\varepsilon-s_0)s_0^{\beta+\frac{n-1}{2}}}.$$

As $s_0 = \min\{s_\varepsilon, \frac{\rho-\varepsilon}{2}\} \leq \frac{\rho-\varepsilon}{2}$ and as K^0 is bounded, hence contained in some ball, for $t \geq t_0$ the last expression can be estimated with (11) by

$$I_2 \leq k_2 \frac{(\rho+\varepsilon)^{\frac{n-1}{2}}}{s_0^{\beta+\frac{n-1}{2}}},$$

where k_2 is a constant. As ∂K^0 is $C^3$and has strictly positive Gaussian curvature everywhere, $\sigma_\varepsilon = \min_{u\in S^{n-1}} s_\varepsilon(u) > 0$. Let $R_0 = \min_{z\in\partial K^0, 1\leq i\leq n-1} R_i(z)$, where $R_i(z)$ is the i-th principal radius of curvature at $z \in \partial K^0$. By the assumptions on ∂K, $R_0 > 0$ (see [L 1]) and $\rho \geq R_0$ for all ρ. Thus

$$s_0 \geq \min\{\frac{\rho - \varepsilon}{2}, \sigma_\varepsilon\} \geq \min\{\frac{R_0 - \varepsilon}{2}, \sigma_\varepsilon\},$$

which is a strictly positive number independent of u and t. Thus for $t \geq t_0$

$$t \leq k \; \frac{\rho^{\frac{n-1}{2}}}{(1-\lambda h_{K^0}(u))^{\beta - \frac{n+1}{2}}},$$

where k is a (new) constant independent of t and u. Therefore

$$\lambda \geq \frac{1}{h_{K^0}(u)}\left(1 - k\frac{\rho^{\frac{n-1}{2\beta-n-1}}}{t^{\frac{1}{\beta-\frac{n+1}{2}}}}\right)$$

and thus for all $t \geq t_0$

$$t^{\frac{1}{\beta-\frac{n+1}{2}}}\left((\frac{1}{h_{K^0}(u)})^n - (\frac{1}{h_{S_\beta(K,t)^0}(u)})^n \right) \leq c\frac{(\rho(u))^{\frac{n-1}{2\beta-n-1}}}{(h_{K^0}(u))^n},$$

where c is a (new) constant independent of t and u.
And the function $l(u) = \frac{(\rho(u))^{\frac{n-1}{2\beta-n-1}}}{(h_{K^0}(u))^n}$ is integrable on S^{n-1}.

This proves the Claim and thus the Theorem.

The next Proposition deals with the case $\beta = \frac{n+1}{2}$. First we want to give a definition for $O_{-n}(K)$ and the motivation for this definition. This definition is probably well known though we did not find a reference.
For $\lambda \in \mathbf{R}$, $\lambda \geq 0$ ([Lu 3], [H 1])

$$O_p(\lambda K) = \lambda^{\frac{n(n-p)}{n+p}} O_p(K).$$

Therefore $\tilde{O}_p(K) = O_p(K)^{\frac{n+p}{n-p}}$ is homogeneous of degree n and affine invariant.

$$\tilde{O}_p(K) = \left(\int_{S^{n-1}} (\frac{f_K(u)^n}{h_K(u)^{n(p-1)}})^{\frac{1}{n+p}} d\sigma(u) \right)^{\frac{n+p}{n-p}}$$

$$= || \frac{f_K^n}{h_K^{n(p-1)}} ||_{\frac{1}{n+p}}^{\frac{1}{n-p}},$$

where for a function $g \in L^q(S^{n-1})$, $||g||_q = (\int_{S^{n-1}} g(u)^q d\sigma(u))^{\frac{1}{q}}$. Thus, if $p \to -n$,

$$\tilde{O}_p(K) \to \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}} = ||f_K^{\frac{1}{2}} h_K^{\frac{n+1}{2}}||_\infty.$$

The latter is also an affine invariant and it is therefore natural to put

$$\tilde{O}_{-n}(K) = \max_{u \in S^{n-1}} f_K(u)^{\frac{1}{2}} h_K(u)^{\frac{n+1}{2}}.$$

Proposition 7 respectively Remark (ii) following Proposition 7 gives a geometric interpretation of this affine invariant.

Proposition 7

Let K be a convex body in $\mathbf{R}^n$ such that ∂K is C^3 and has strictly positive Gaussian curvature everywhere. Then

$$n lim_{t \to \infty} \frac{|K| - |S_{\frac{n+1}{2}}(K,t)|}{\int_{S^{n-1}} \frac{f_{K^0}(u)^{\frac{1}{n-1}}}{h_{K^0}(u)^{n+1}} \quad exp\left(\frac{-t}{2^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}} f_{K^0}(u)^{\frac{1}{2}}}\right) d\sigma(u)} = 1.$$

Remarks

(i) As $(K^0)^0 = K$, it follows immediately from Proposition 7 that

$$n lim_{t \to \infty} \frac{|K^0| - |S_{\frac{n+1}{2}}(K^0,t)|}{\int_{S^{n-1}} \frac{f_K(u)^{\frac{1}{n-1}}}{h_K(u)^{n+1}} \quad exp\left(\frac{-t}{2^{\frac{n-1}{2}} h_K(u)^{\frac{n+1}{2}} f_K(u)^{\frac{1}{2}}}\right) d\sigma(u)} = 1.$$

(ii) It also follows from Proposition 7 that

$$lim_{t \to \infty} \frac{1}{t} ln\left(\frac{1}{|K| - |S_{\frac{n+1}{2}}(K,t)|}\right) = \frac{2^{-\frac{n-1}{2}}}{max_{u \in S^{n-1}} (h_{K^0}(u)^{\frac{n+1}{2}} f_{K^0}(u)^{\frac{1}{2}})}.$$

Indeed, if we put $B(t) = \int_{S^{n-1}} g(u) \left(\exp(-\frac{1}{\Phi(u)})\right)^t d\sigma(u)$, where $g(u) = \frac{f_{K^0}(u)^{\frac{1}{n-1}}}{h_{K^0}(u)^{n+1}}$ and $\Phi(u) = 2^{\frac{n-1}{2}} f_{K^0}(u)^{\frac{1}{2}} h_{K^0}(u)^{\frac{n+1}{2}}$ and $A(t) = n(|K| - |S_{\frac{n+1}{2}}(K,t)|)$, then by Proposition 7, $\frac{A(t)}{B(t)} \to 1$, as $t \to \infty$. Hence

$$\ln\left(\frac{A(t)}{B(t)}\right) = \ln(A(t)) - \ln(B(t)) = \ln(A(t))\left(1 - \frac{\ln(B(t))}{\ln(A(t))}\right) \to 0.$$

As $A(t) \to 0$ and $B(t) \to 0$ as $t \to \infty$, it follows that $\frac{\ln(B(t))}{\ln(A(t))} \to 1$ as $t \to \infty$ and thus

$$\ln(B(t))^{\frac{1}{t}} \frac{t}{\ln(A(t))} = \frac{\ln(B(t))}{t} \frac{t}{\ln(A(t))} \to 1.$$

But

$$(B(t))^{\frac{1}{t}} = \left(\int_{S^{n-1}} g(u) \left(\exp(-\frac{1}{\Phi(u)}) \right)^t d\sigma(u) \right)^{\frac{1}{t}}$$

$$= ||\exp(-\frac{1}{\Phi})||_{L^t(S^{n-1}, g d\sigma)} \to ||\exp(-\frac{1}{\Phi})||_{L^\infty(S^{n-1})},$$

as $t \to \infty$. From this (ii) follows.

Proof of Proposition 7

We use the same notations as in the proof of Theorem 6. In fact up to the estimates (6) and (9) for t both proofs are identical. Then we apply Lemma 5 (ii) instead of Lemma 5 (i) to estimate I_1 and get again with $v = 1 + \frac{s}{\rho+\varepsilon} - \frac{h_{K^0}(u)}{\rho+\varepsilon}$

$$I_1 \le \frac{(\rho+\varepsilon)^n}{\left(1 - \lambda(h_{K^0}(u) - (\rho+\varepsilon))\right)^{\frac{n+1}{2}}} \int_0^1 \frac{(1-v^2)^{\frac{n-1}{2}} dv}{\left(1 - \frac{\lambda(\rho+\varepsilon)v}{1-\lambda(h_{K^0}(u)-(\rho+\varepsilon))}\right)^{\frac{n+1}{2}}}$$

$$\le \frac{2^{\frac{n-1}{2}} (\rho+\varepsilon)^{\frac{n-1}{2}}}{\lambda^{\frac{n+1}{2}}} \ln(1 + \frac{\lambda(\rho+\varepsilon)}{1-\lambda h_{K^0}(u)}) \left(1 + \frac{2}{(n+1)\ln(1+\frac{\lambda(\rho+\varepsilon)}{1-\lambda h_{K^0}(u)})}\right).$$

We use the same estimate as in the proof of Theorem 6 for I_2 and get with $\beta = \frac{n+1}{2}$

$$I_2 \le \frac{(\rho+\varepsilon)^{\frac{n+1}{2}} (s_0 + h_{K^0}(u) + h_{K^0}(-u))^n}{n(\rho+\varepsilon-s_0)\lambda^{\frac{n+1}{2}} s_0^n}.$$

Thus for

$$\lambda \ge \frac{\max\{1-\frac{\varepsilon}{2}, 1 - \frac{e^{\frac{-1}{\varepsilon s_0^n}}}{2}\}}{\min\{h_{K^0}(u), \rho-\varepsilon\}}$$

$$t \le (1+c\varepsilon) v_{n-1} 2^{\frac{n-1}{2}} (\rho+\varepsilon)^{\frac{n-1}{2}} h_{K^0}(u)^{\frac{n+1}{2}} \ln(1 + \frac{\lambda(\rho+\varepsilon)}{1-\lambda h_{K^0}(u)}),$$

where c is a constant. This implies that

$$\lambda = \frac{1}{h_{(S_{\frac{n+1}{2}}(K,t))^0}(u)}$$

$$\geq \frac{1}{h_{K^0}(u)}\left[1-\frac{\rho+\varepsilon}{h_{K^0}(u)}\left(\exp(\frac{t}{(1+c\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho+\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}})-1\right)^{-1}\right].$$

Consequently

$$|K|-|S_{\frac{n+1}{2}}(K,t)|=\frac{1}{n}\int_{S^{n-1}}\left(\frac{1}{h_{K^0}(u)^n}-\frac{1}{h_{(S_{\frac{n+1}{2}}(K,t))^0}(u)^n}\right)d\sigma(u)$$

$$\leq$$

$$\int_{S^{n-1}}\left\{\frac{\rho(u)+\varepsilon}{h_{K^0}(u)^{n+1}}\exp\left(\frac{-t}{(1+c\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho+\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}}\right)\right.$$
$$\left.\left(1-\frac{1}{\exp(\frac{t}{(1+c\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho+\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}})}\right)^{-1}\right\}d\sigma(u).$$

For t sufficiently big, this gives the estimate from above.

In the same way

$$t\geq(1-d\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho-\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}\ln(1+\frac{\lambda(\rho-\varepsilon)}{1-\lambda h_{K^0}(u)}),$$

where d is a constant. Therefore

$$\lambda=\frac{1}{h_{(S_{\frac{n+1}{2}}(K,t))^0}(u)}\leq$$

$$\frac{1}{h_{K^0}(u)}\left[1-\frac{(1-\varepsilon)(\rho-\varepsilon)}{h_{K^0}(u)}\exp\left(\frac{-t}{(1-d\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho-\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}}\right)\right]$$

and thus for t sufficiently big

$$|K|-|S_{\frac{n+1}{2}}(K,t)|\geq$$

$$\int_{S^{n-1}}\frac{(1-\varepsilon)^2(\rho(u)-\varepsilon)}{h_{K^0}(u)^{n+1}}\exp\left(\frac{-t}{(1-d\varepsilon)v_{n-1}2^{\frac{n-1}{2}}(\rho-\varepsilon)^{\frac{n-1}{2}}h_{K^0}(u)^{\frac{n+1}{2}}}\right)d\sigma(u).$$

The floating body can also be used to give a geometric interpretation of the p-affine surface area for certain p.

Recall that for $\delta>0$, δ small enough, K_δ is said to be a (convex) floating body of K, if it is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume δ of K ([S-W]). More precisely, for $u \in S^{n-1}$ and for $0 < \delta$ let a_δ^u be defined by

$$|\{x \in K :< x, u >\geq a_\delta^u\}| = \delta.$$

Then $K_\delta = \cap_{u \in S^{n-1}} \{x \in K :< x, u >\leq a_\delta^u\}$. Observe that one has always $h_{K_\delta}(u) \leq a_\delta^u$, with generally strict inequality. There is equality for every u and every $\delta \leq \frac{|K|}{2}$ whenever K is centrally symmetric (see [M-R]).

Then we have

Theorem 8

Let K be a convex body in $\mathbf{R}^n$ such that ∂K is C^2 and has strictly positive Gaussian curvature everywhere. Then

$$lim_{\delta \to 0} c_n \frac{|(K_{\delta|K|})^0| - |K^0|}{(\delta |K|)^{\frac{2}{n+1}}} = \int_{S^{n-1}} \frac{d\sigma(u)}{f_K(u)^{\frac{1}{n+1}} h_K(u)^{n+1}},$$

where $c_n = 2 \left(\frac{v_{n-1}}{n+1} \right)^{\frac{2}{n+1}}$.

Remark

Thus

$$lim_{\delta \to 0} c_n \frac{|(K_{\delta|K|})^0| - |K^0|}{(\delta |K|)^{\frac{2}{n+1}}} = O_{-n(n+2)}(K).$$

Proof of Theorem 8

For $u \in S^{n-1}$ let $x \in \partial K$ be such that $N(x) = u$. By assumption the indicatrix of Dupin at x exists and is an ellipsoid. We can again assume that it is a Euclidean ball with radius $\sqrt{\rho} = \sqrt{\rho(u)}$. With the notations and the coordinate system introduced in the proof of Theorem 6 (with x instead of y), for $\varepsilon > 0$ there exists s_0 such that

$$B_{\rho-\varepsilon} \cap H_{s_0}^+ \subseteq K^0 \cap H_{s_0}^+ \subseteq B_{\rho+\varepsilon} \cap H_{s_0}^+.$$

We choose δ so small that $s_0 > h_{K_{\delta|K|}}(u)$. Now, as in the proof of Theorem 6, we change the coordinate system so that the Santaló-point of K is at the origin and the positive u-direction is the positive s-direction. By construction of the floating body

$$\delta|K| \leq \int_{h_{K_{\delta|K|}}(u)}^{h_K(u)} g_{K,u}(s) ds$$

$$\leq v_{n-1} \int_{h_{K_{\delta|K|}}(u)}^{h_K(u)} \left((\rho+\varepsilon)^2 - (s+\rho+\varepsilon-h_K(u))^2\right)^{\frac{n-1}{2}} ds$$

$$\leq \frac{2^{\frac{n+1}{2}} v_{n-1}}{n+1} (\rho+\varepsilon)^{\frac{n-1}{2}} \left(h_K(u) - h_{K_{\delta|K|}}(u)\right)^{\frac{n+1}{2}}.$$

Thus

$$\frac{1}{h_{K_{\delta|K|}}(u)^n} \geq \frac{1}{h_K(u)^n} \left(1 + \frac{n(\delta|K|)^{\frac{2}{n+1}}}{c_n h_K(u)(\rho+\varepsilon)^{\frac{n-1}{n+1}}}\right)$$

and hence

$$\frac{c_n}{n(\delta|K|)^{\frac{2}{n+1}}} \left[\frac{1}{h_{K_{\delta|K|}}(u)^n} - \frac{1}{h_K(u)^n}\right] \geq \frac{1}{h_K(u)^{n+1}(\rho+\varepsilon)^{\frac{n-1}{n+1}}}. \qquad (12)$$

On the other hand by the fact that for each $z \in \partial K_{\delta|K|}$ there exists a tangent hyperplane that cuts off exactly $\delta|K|$ of K and by [S-W], Lemma 11, for $\varepsilon > 0$ there exists δ_0 such that for all $\delta \leq \delta_0$

$$\delta|K| \geq (1-\varepsilon) v_{n-1} \int_{h_{K_{\delta|K|}}(u)+\varepsilon}^{h_K(u)} \left((\rho-\varepsilon)^2 - (s+\rho-\varepsilon-h_K(u))^2\right)^{\frac{n-1}{2}} ds$$

$$\geq \frac{(1-\varepsilon)2^{\frac{n+1}{2}} v_{n-1}}{n+1} (\rho-\varepsilon)^{\frac{n-1}{2}} \left(h_K(u) - h_{K_{\delta|K|}}(u) - \varepsilon\right)^{\frac{n+1}{2}}$$

$$\left(1 - \frac{1}{\rho-\varepsilon}(h_K(u) - h_{K_{\delta|K|}}(u) - \varepsilon)\right)^{\frac{n-1}{2}}$$

$$\geq \frac{(1-c\varepsilon)2^{\frac{n+1}{2}}v_{n-1}}{n+1}(\rho-\varepsilon)^{\frac{n-1}{2}}\Big(h_K(u)-h_{K_{\delta|K|}}(u)\Big)^{\frac{n+1}{2}},$$

for some constant c. Thus for δ small enough

$$\frac{1}{h_{K_{\delta|K|}}(u)^n} \leq \frac{1}{h_K(u)^n}\Bigg\{1+\frac{n(\delta|K|)^{\frac{2}{n+1}}(1+dA(\delta))}{c_n(1-c\varepsilon)^{\frac{2}{n+1}}h_K(u)(\rho-\varepsilon)^{\frac{n-1}{n+1}}}$$

$$\Bigg(1+\frac{1}{n}\binom{n}{2}A(\delta)(1+dA(\delta))+\frac{1}{n}\binom{n}{3}(A(\delta))^2(1+dA(\delta))^2+\cdots$$

$$\cdots+\frac{1}{n}(A(\delta)^n(1+dA(\delta))^n\Bigg)\Bigg\},$$

where d is a constant and $A(\delta)=\frac{n(\delta|K|)^{\frac{2}{n+1}}}{c_n(1-c\varepsilon)^{\frac{2}{n+1}}h_K(u)(\rho-\varepsilon)^{\frac{n-1}{n+1}}}$.

Thus

$$\frac{c_n}{n(\delta|K|)^{\frac{2}{n+1}}}\left[\frac{1}{h_{K_{\delta|K|}}(u)^n}-\frac{1}{h_K(u)^n}\right] \leq \frac{(1+dA(\delta))}{(1-c\varepsilon)^{\frac{2}{n-1}}h_K(u)^{n+1}(\rho-\varepsilon)^{\frac{n-1}{n+1}}}$$

$$\Bigg\{1+\frac{1}{n}\binom{n}{2}A(\delta)(1+dA(\delta))+\frac{1}{n}\binom{n}{3}(A(\delta))^2(1+dA(\delta))^2+\cdots$$

$$\cdots+\frac{1}{n}(A(\delta)^n(1+dA(\delta))^n\Bigg\}. \qquad (13)$$

(12) and (13) show that for $u\in S^{n-1}$

$$\lim_{\delta\to 0}\frac{c_n}{n(\delta|K|)^{\frac{2}{n+1}}}\left[\frac{1}{h_{K_{\delta|K|}}(u)^n}-\frac{1}{h_K(u)^n}\right]=\frac{\rho(u)^{-\frac{n-1}{n+1}}}{h_K(u)^{n+1}}.$$

Therefore

$$\lim_{\delta\to 0}c_n\frac{|(K_{\delta|K|})^0|-|K^0|}{(\delta|K|)^{\frac{2}{n+1}}}=$$

$$\lim_{\delta\to 0}\int_{S^{n-1}}\frac{c_n}{n(\delta|K|)^{\frac{2}{n+1}}}\left[\frac{1}{h_{K_{\delta|K|}}(u)^n}-\frac{1}{h_K(u)^n}\right]d\sigma(u)=$$

$$\int_{S^{n-1}}\lim_{\delta\to 0}\frac{c_n}{n(\delta|K|)^{\frac{2}{n+1}}}\left[\frac{1}{h_{K_{\delta|K|}}(u)^n}-\frac{1}{h_K(u)^n}\right]d\sigma(u)=$$

$$\int_{S^{n-1}}\frac{d\sigma(u)}{f_K(u)^{\frac{1}{n+1}}h_K(u)^{n+1}}.$$

We still have to justify that we can interchange integration and limit. This follows from Lebesgue's Theorem and the
Claim
for every $u \in S^{n-1}$

$$\frac{1}{n(\delta|K|)^{\frac{2}{n+1}}}\left[\frac{1}{h_{K_{\delta|K|}}(u)^n} - \frac{1}{h_K(u)^n}\right] \leq g(u),$$

where g is a function independent of δ and integrable on S^{n-1}.
Proof of the Claim

For $u \in S^{n-1}$ let $x \in \partial K$ be such that $N(x) = u$. Moreover we can suppose that $0 \in \text{int}(K)$ and choose $\alpha > 0$ such that

$$B(0,\alpha) \subseteq K \subseteq B(0, \frac{1}{\alpha}). \tag{14}$$

Let again $R_0 = \min_{x\in\partial K, 1\leq i\leq n-1} R_i(x)$, where $R_i(x)$ is the i-th principal radius of curvature at $x \in \partial K$. $R_0 > 0$ (see [L 1]).
By the Blaschke Rolling Theorem (see [L 1]), we have for all $x \in \partial K$

$$B(x - R_0 N(x), R_0) \subseteq K. \tag{15}$$

By [L 1] there exists δ_0 such that for all $\delta < \delta_0$ $\partial K_{\delta|K|}$ is C^2.
We put $\delta_1 = \min\{\delta_0, (\frac{3}{4}\alpha)^{\frac{n+1}{2}} \frac{c_n R_0^{\frac{n-1}{2}}}{|K|}\}$. Then we have for all $\delta < \delta_1$ that $\partial K_{\delta|K|}$ is C^2. Consequently for $u \in S^{n-1}$ there is $z \in \partial K_{\delta|K|}$ such that $N(z) = u$ and the tangent-hyperplane to $K_{\delta|K|}$ in z orthogonal to u cuts off exactly $\delta|K|$ from K.

By (15) we can estimate $\delta|K|$ from below.
If $h_K(u) - h_{K_{\delta|K|}}(u) \geq R_0$, then $\delta|K| \geq \frac{1}{2}R_0^n v_n$. Otherwise $\delta|K|$ is not less than the volume of the cap of height $h_K(u) - h_{K_{\delta|K|}}(u)$ of a Euclidean ball with radius R_0. Thus

$$\delta|K| \geq \frac{2v_{n-1}}{n+1} R_0^{\frac{n-1}{2}} \left(h_K(u) - h_{K_{\delta|K|}}(u)\right)^{\frac{n+1}{2}}.$$

Now, the Claim follows easliy from these bounds.
This finishes the proof of the Claim and thus of the Theorem.

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