Available at www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

ELSEVIER

Advances in Mathematics ∎ (∎∎∎∎) ∎∎∎–∎∎∎

ADVANCES IN Mathematics

http://www.elsevier.com/locate/aim

Surface bodies and p-affine surface area

Carsten Schütt[a,b,*] and Elisabeth Werner[b,c,1]

[a] *Mathematisches Seminar, Christian Albrechts Universität, D-24098 Kiel, Germany*
[b] *Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, USA*
[c] *Université de Lille 1, Ufr de Mathematique, 59655 Villeneuve d'Ascq, France*

Received 9 January 2003; accepted 9 July 2003

Communicated by Erwin Lutwak

Abstract

The surface body is a generalization of the floating body. Its relation to p-affine surface area is studied.

MSC: 52A20

Keywords: ∎; ∎; ∎

1. Introduction

1.1. Background

The affine surface area was originally introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficiently smooth boundary. Its definition involves the Gauss curvature of the boundary points of a convex body. Hence, it provides a tool to "measure" the boundary structure of a convex body. Therefore, it is not surprising that the affine surface area occurs naturally in problems related to the boundary of a convex body, so for instance in the approximation of convex bodies by polytopes. For more information about this subject and the role the affine surface area plays

*Corresponding author. Fax: +431-880-4091.
E-mail addresses: schuett@math.uni-kiel.de (C. Schütt), emw2@po.cwru.edu (E. Werner).
[1] Partially supported by a grant from the National Science Foundation.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2003.07.018

there, we refer to the works by Bárány, [Ba1,Ba2], Gruber [Gr1,Gr2,Gr3], Schütt [Sch1,Sch2] and Schütt and Werner [SchW2].

Extensions of the affine surface area to higher dimensions and arbitrary convex bodies were only found much later than Blaschke's times by Leichtweiss [L1,L2], Lutwak [Lu1], Schütt and Werner [SchW1], Schmuckenschläger [Schm], Meyer and Werner [MW1] and Werner [W1]. Additional references to the affine surface area as well as further applications can also be found in those papers as well as in Leichtweiss [L3], Ludwig and Reitzner [LudR], Lutwak and Oliker [LuO] and [W2].

Here we want to concentrate on the p-affine surface area which, for $p>0$, was introduced in 1996 by Lutwak [Lu2]. For $p=1$, the p-affine surface area is just the affine surface area. Hug [H] gave new definitions of the p-affine surface area. He also proved that these new definitions give the same p-affine surface area as that defined by Lutwak.

Meyer and Werner [MW2] found a geometric interpretation of the p-affine surface area in terms of the (generalized) Santaló bodies. They also observed that the definition of Lutwak for the p-affine surface area makes sense for $-n<p\leqslant 0$ and their geometric interpretation in terms of the Santaló bodies also holds for this range of p. They also gave a definition of the p-affine surface area for $p=-n$ together with its geometric interpretation.

In [SchW2,W3] it was suggested to extend the p-range even further, namely to $-\infty\leqslant p\leqslant\infty$. This extension was motivated in [SchW2] by the fact that there is a characterization of the p-affine surface area in terms of random polytopes and this characterization holds for $-\infty\leqslant p\leqslant\infty$. In [W3] a characterization of the p-affine surface area for all p is given using weighted floating bodies.

In this paper we give a new characterization of the p-affine surface area using surface bodies. The paper is organized as follows: In Section 2 we define the surface bodies and discuss some of their properties. The surface bodies were introduced in [SchW2] in connection with approximating convex bodies by random polytopes. Many of the properties mentioned here have already been stated and proved in [SchW2]. We include them here for completeness.

In Section 3 we introduce the p-affine surface area for $-\infty\leqslant p\leqslant\infty$ and discuss some of the properties of the p-affine surface area. For a given probability density f on the boundary of a convex body K and a positive number s the surface body $K_{f,s}$ is the intersection of all half-spaces H^+ such that $\int_{\partial K\cap H^-} f d\mu_{\partial K}\leqslant s$. Our main theorem is that under certain assumptions on the density f and the boundary ∂K

$$d_n \lim_{s\to 0}\frac{\mathrm{vol}_n(K)-\mathrm{vol}_n(K_{f,s})}{s^{\frac{2}{n-1}}}=\int_{\partial K}\frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}}\,d\mu_{\partial K},$$

where d_n is a constant depending only on the dimension n and κ the generalized Gauß–Kronecker curvature. As a consequence, for the p-affine surface area O_p there is $q=\frac{n-p(n-2)}{p+1}$ and a function f_q such that

$$d_n \lim_{s\to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{f_q,s})}{(sO_q(K))^{\frac{2}{n-1}}} = O_p(K).$$

1.2. Notation

Throughout the paper we shall use the following notations. $B_2^n(a,r)$ is an n-dimensional Euclidean ball with radius r centered at a. We put $B_2^n = B_2^n(0,1)$. By $||.||$ we denote the standard Euclidean norm on $\mathbb{R}^n$, by $\langle \cdot,\cdot \rangle$ the standard inner product on $\mathbb{R}^n$. For two points x and y in $\mathbb{R}^n$ $[x,y] = \{\alpha x + (1-\alpha)y : 0 \leqslant \alpha \leqslant 1\}$ denotes the line segment from x to y.

For a convex body K in $\mathbb{R}^n$, $\mathring{K}$ is the interior of K and ∂K is the boundary of K. We also write S^{n-1} for ∂B_2^n. For $x \in \partial K$, $N_{\partial K}(x)$ is the outer unit normal vector to ∂K in x. It may not be unique.

For $u \in S^{n-1}$, $h_K(u)$ is the support function of K at u. $\mu_{\partial K}$ is the usual surface measure on the boundary ∂K of K and σ is the spherical Lebesgue measure. $\mathrm{vol}_{n-1}(A)$ denotes the surface area measure of a subset A of the boundary of a convex body.

$H(x,\xi)$ is the hyperplane containing the point x and orthogonal to ξ. $H^-(x,\xi)$ is the closed half-space containing the point $x+\xi$, $H^+(x,\xi)$ the other half-space. Let $\mathscr{U}$ be a convex, open subset of $\mathbb{R}^n$ and let $f : \mathscr{U} \to \mathbb{R}$ be a convex function. $df(x) \in \mathbb{R}^n$ is called subdifferential at the point $x_0 \in \mathscr{U}$, if we have for all $x \in \mathscr{U}$

$$f(x_0) + \langle df(x_0), x - x_0 \rangle \leqslant f(x).$$

A convex function has a subdifferential at every point and it is differentiable at a point if and only if the subdifferential is unique. Let $\mathscr{U}$ be an open, convex subset in $\mathbb{R}^n$ and $f : \mathscr{U} \to \mathbb{R}$ a convex function. f is said to be twice differentiable in a generalized sense in $x_0 \in \mathscr{U}$, if there is a linear map $d^2f(x_0)$ and a neighborhood $\mathscr{U}(x_0) \subseteq \mathscr{U}$ such that we have for all $x \in \mathscr{U}(x_0)$ and for all subdifferentials $df(x)$

$$||df(x) - df(x_0) - d^2f(x_0)(x - x_0)|| \leqslant \Theta(||x - x_0||)||x - x_0||,$$

where Θ is a monotone function with $\lim_{t\to 0} \Theta(t) = 0$. $d^2f(x_0)$ is called generalized Hesse-matrix. If $f(0) = 0$ and $df(0) = 0$ then we call the set

$$\{x \in \mathbb{R}^n | x^t d^2f(0)x = 1\}$$

the indicatrix of Dupin at 0. Since f is convex this set is an ellipsoid or a cylinder with a base that is an ellipsoid of lower dimension. The eigenvalues of $d^2f(0)$ are called principal curvatures and their product is called the Gauß–Kronecker curvature κ. Geometrically, the eigenvalues of $d^2f(0)$ that are different from 0 are the lengths of the principal axes of the indicatrix raised to the power -2.

YAIMA : 2269
ARTICLE IN PRESS

For $u \in S^{n-1}$, $f_\kappa(u)$ is the Gauß curvature function, that is the reciprocal of the Gauß–Kronecker curvature $\kappa(x)$ at this point $x \in \partial K$ that has u as outer normal.

2. The surface body

Let K be a convex body and $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function with $\int_{\partial K} f(x)\, d\mu_{\partial(K)}(x) = 1$. The probability measure $\mathbb{P}_f$ is the measure on ∂K with density f.

Definition 1. Let $0 \leqslant s$ and let $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function with $\int_{\partial K} f(x)\, d\mu_{\partial(K)}(x) = 1$.

The surface body $K_{f,s}$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $\mathbb{P}_f$-measure less than or equal to s from ∂K. More precisely,

$$K_{f,s} = \bigcap_{\mathbb{P}_f(\partial K \cap H^-) \leqslant s} H^+. \tag{1}$$

We write usually K_s for $K_{f,s}$ if it is clear which function f we are considering.

Remarks. (i) It follows from the Hahn–Banach theorem that $K_0 \subseteq K$. If in addition f is $\mu_{\partial K}$—almost everywhere nonzero, then $K_0 = K$ as it is shown in Lemma 2(iv) (See Fig. 1).

(ii) For many convex bodies K and functions f the bodies $K_{f,s}$ shrink continuously from $K_{f,0} = K$ to a body that consists of one point only. Usually, this point is an interior point of K. In most cases the volume of $K_{f,s}$ is strictly positive until it is reduced to a point and below we give conditions for K and f for this to happen.

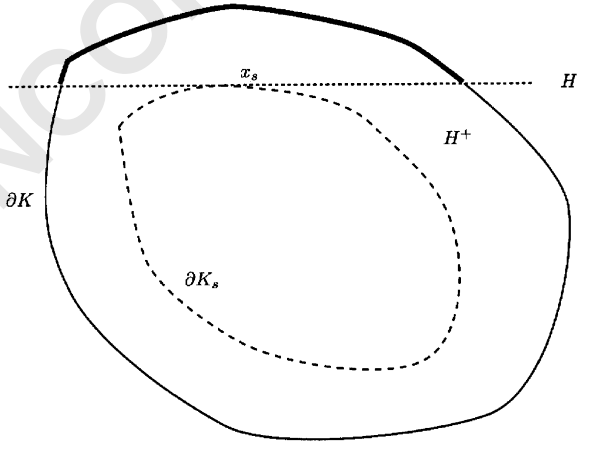

Fig. 1.

In general, however this may not be so. We describe two cases:

1. $K_{f,s}$ shrinks to a convex set of lower dimension that is contained in the boundary of K. Eventually, it shrinks to a point in the boundary of K.

2. There is a constant $c>0$ and s_0 such that for all s with $0\leqslant s<s_0$ the volume of $K_{f,s}$ is larger than c and $K_{f,s_0}=\emptyset$ (see Example (ii) in Remarks 6).

(iii) Through a similar construction we obtain a "weighted floating body":

Let $0\leqslant s$ and let $f:K\to\mathbb{R}$ be a nonnegative, integrable function.

The weighted floating body $F(K,f,s)$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of measure less than or equal to s from K. More precisely,

$$F(K,f,s)=\bigcap_{\int_{K\cap H^-} f\,dx\leqslant s} H^+. \tag{2}$$

These bodies are investigated in [W3].

We say that a sequence of hyperplanes H_i, $i\in\mathbb{N}$, in $\mathbb{R}^n$ converges to a hyperplane H if we have for all $x\in H$ that

$$\lim_{i\to\infty} d(x,H_i)=0,$$

where $d(x,H)=\inf\{\|x-y\|\ |y\in H\}$. This is equivalent to: The sequence of the normals of H_i converges to the normal of H and there is a point $x\in H$ such that

$$\lim_{i\to\infty} d(x,H_i)=0.$$

Recall that for a hyperplane $H(x,\xi)$ through x, with normal ξ, $H^-(x,\xi)$ is the half-space containing $x+\xi$.

Lemma 2. *Let K be a convex body in $\mathbb{R}^n$ and let $f:\partial K\to\mathbb{R}$ be an a.e. positive, integrable function with $\int_{\partial K} f(x)\,d\mu_{\partial(K)}(x)=1$. Let $\xi\in S^{n-1}$.*

(i) *Let $x_0\in\mathbb{R}^n$. Then*

$$\mathbb{P}_f(\partial K\cap H^-(x_0-t\xi,\xi))$$

is a continuous function of t on

$$\left[\min_{y\in K}\langle x_0-y,\xi\rangle,\max_{y\in K}\langle x_0-y,\xi\rangle\right).$$

($\mathbb{P}_f(\partial K\cap H^-(x_0-t\xi,\xi))$ is not necessarily a continuous function on the closed interval.)

(ii) *Let $x_0\in\mathbb{R}^n$. Then*

$$\mathbb{P}_f(\partial K\cap H^-(x_0-t\xi,\xi))$$

is strictly increasing function of t on

$$\left[\min_{y\in K}\langle x_0-y,\xi\rangle,\max_{y\in K}\langle x_0-y,\xi\rangle\right].$$

(iii) *Let H_i, $i\in\mathbb{N}$, be a sequence of hyperplanes that converge to the hyperplane H_0. Assume that the hyperplane H_0 intersects the interior of K. Then we have*

$$\lim_{i\to\infty}\mathbb{P}_f(\partial K\cap H_i^-)=\mathbb{P}_f(\partial K\cap H_0^-).$$

(*If H_0 does not intersect the interior of K the equality does not hold necessarily.*)
(iv)

$$\overset{\circ}{K}\subseteq\bigcup_{0<s}K_s.$$

In particular, $K=K_0$.

Proof. (i)

$$\mathrm{vol}_{n-1}(\partial K\cap H^-(x_0-t\xi,\xi))$$

is a continuous function of t on

$$\left[\min_{y\in K}\langle x_0-y,\xi\rangle,\max_{y\in K}\langle x_0-y,\xi\rangle\right).$$

Since f is an integrable function (i) follows.
(ii) Since $H^-(x_0,\xi)$ is the half-space containing $x_0+\xi$ we have for t_1 and t_2 with $t_1<t_2$

$$H^-(x_0-t_1\xi,\xi)\subsetneq H^-(x_0-t_2\xi,\xi).$$

Thus

$$\partial K\cap H^-(x_0-t_2\xi,\xi)\cap H^+(x_0-t_1\xi,\xi)$$

has positive $n-1$-dimensional Hausdorff-measure. If

$$\mathbb{P}_f(\partial K\cap H^-(x_0-t_1\xi,\xi))=\mathbb{P}_f(\partial K\cap H^-(x_0-t_2\xi,\xi))$$

then f is a.e. 0 on $\partial K\cap H^-(x_0-t_2\xi,\xi)\cap H^+(x_0-t_1\xi,\xi)$. This is not true.
(iii) Let $x_0\in H_0\cap\overset{\circ}{K}$ and x_i, $i\in\mathbb{N}$, the nearest point to x_0 in H_i. Let ξ_i be the normal to H_i. Thus $H_i=H(x_i,\xi_i)$, $i=0,1,\ldots$ and we have that

$$\lim_{i\to\infty}x_i=x_0,\quad\lim_{i\to\infty}\xi_i=\xi_0,$$

where x_0 is an interior point of K. Therefore for all $\varepsilon>0$ there exists i_0 such that for all $i>i_0$

$$\partial K \cap H^-(x_0+\varepsilon\xi_0,\xi_0) \subseteq \partial K \cap H^-(x_i,\xi_i) \subseteq \partial K \cap H^-(x_0-\varepsilon\xi_0,\xi_0).$$

This implies

$$\mathbb{P}_f(\partial K \cap H^-(x_0+\varepsilon\xi_0,\xi_0)) \leqslant \mathbb{P}_f(\partial K \cap H^-(x_i,\xi_i)) \leqslant \mathbb{P}_f(\partial K \cap H^-(x_0-\varepsilon\xi_0,\xi_0)).$$

Since x_0 is an interior point of K, for ε small enough $x_0-\varepsilon\xi_0$ and $x_0+\varepsilon\xi_0$ are interior points of K. Therefore,

$$H(x_0-\varepsilon\xi_0,\xi_0) \quad \text{and} \quad H(x_0+\varepsilon\xi_0,\xi_0)$$

intersect the interior of K. The claim now follows from (i).

(iv) Suppose the inclusion is not true. Then there is $x \in \mathring{K}$ with $x \notin \bigcup_{0<s} K_s$. Therefore, for every $i \in \mathbb{N}$ there is a hyperplane H_i with $x \in H_i$ and

$$\mathbb{P}_f(\partial K \cap H_i^-) \leqslant \frac{1}{i}.$$

By compactness there is a subsequence H_{i_j}, $j \in \mathbb{N}$, that converges to a hyperplane H with $x \in H$. By choosing another subsequence we make sure that the limit

$$\lim_{j\to\infty} \mathbb{P}_f(\partial K \cap H_{i_j}^-)$$

exists. Clearly,

$$\lim_{j\to\infty} \mathbb{P}_f(\partial K \cap H_{i_j}^-) \leqslant 0.$$

Since $x \in H$ the hyperplane H intersects the interior of K. Thus, by (iii)

$$\mathbb{P}_f(\partial K \cap H^-) \leqslant 0.$$

On the other hand, $\mathrm{vol}_{n-1}(\partial K \cap H^-) > 0$ which implies

$$\mathbb{P}_f(\partial K \cap H^-) > 0$$

since f is a.e. positive.

We have $K = K_0$ because K_0 is a closed set and

$$\mathring{K} \subseteq \bigcup_{s>0} K_s \subseteq K_0.$$

Thus $K \subseteq K_0$. The opposite inclusion follows from the theorem of Hahn–Banach. □

Lemma 3. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be a a.e. positive, integrable function with $\int_{\partial K} f(x)\, d\mu_{\partial(K)}(x) = 1$.*

(i) *For all s such that $K_s \neq \emptyset$, and all $x \in \partial K_s \cap \overset{\circ}{K}$ there exists a supporting hyperplane H to ∂K_s through x such that $\mathbb{P}_f(\partial K \cap H^-) = s$.*

(ii) *Suppose that for all $x \in \partial K$ and all supporting hyperplanes H of K at x the $n-1$-dimensional Hausdorff measure of the set $H \cap K$ is 0. Then we have for all s with $0 < s$ that $K_s \subset \overset{\circ}{K}$.*

The assertion of Lemma 3(i) is not true if $x \in \partial K$. As an example consider the square S with sidelength 1 in $\mathbb{R}^2$ and $f(x) = \frac{1}{4}$ for all $x \in \partial S$. For $s = \frac{1}{16}$ the midpoints of the sides of the square are elements of $S_{\frac{1}{16}}$, but the tangent hyperplanes through these points contain one side and therefore cut off a set of $\mathbb{P}_f$-volume $\frac{1}{4}$ (cf. Fig. 2). The construction in higher dimensions for the cube is done in the same way.

This example also shows that the surface body is not necessarily strictly convex and it shows that the assertion of Lemma 3(ii) does not hold without additional assumptions.

Proof of Lemma 3. (i) By the theorem of Hahn–Banach there is a sequence of hyperplanes H_i with $K_s \subseteq H_i^+$ and $\mathbb{P}_f(\partial K \cap H_i^-) \leqslant s$ such that the distance between x and H_i is less than $\frac{1}{i}$. We check this.

Since $x \in \partial K_s$ there is $z \notin K_s$ with $\|x - z\| < \frac{1}{i}$. There is a hyperplane H_i separating z from K_s satisfying

$$\mathbb{P}_f(\partial K \cap H_i^-) \leqslant s \quad \text{and} \quad K_s \subseteq H_i^+.$$

We have

$$d(x, H_i) \leqslant \|x - z\| < \frac{1}{i}.$$

By compactness there is a subsequence of hyperplanes H_{i_j}, $j \in \mathbb{N}$, that converges to a hyperplane H with $x \in H$. Since x is an element of the interior and $x \in H$, the

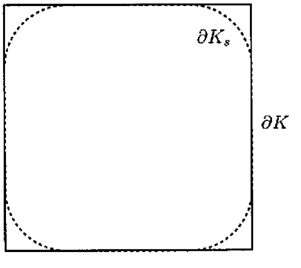

Fig. 2.

hyperplane H intersects the interior of K. Therefore we can apply Lemma 2(iii)

$$s \geqslant \lim_{j\to\infty} \mathbb{P}_f(\partial K \cap H_{i_j}^-) = \mathbb{P}_f(\partial K \cap H^-).$$

If $\mathbb{P}_f(\partial K \cap H^-) < s$ then we choose a hyperplane $\widetilde{H}$ parallel to H such that $\mathbb{P}_f(\partial K \cap \widetilde{H}^-) = s$. This is possible because by Lemma 2(i)

$$\mathbb{P}_f(\partial K \cap H^-(x_0 - t\xi, \xi))$$

is a continuous function of t on $[\min_{y\in K}\langle x_0 - y, \xi\rangle, \max_{y\in K}\langle x_0 - y, \xi\rangle)$. Consequently, x is not an element of K_s. This is a contradiction.

(ii) Suppose there is $x \in \partial K$ with $x \in K_s$ and $0 < s$. By assumption

$$\mathrm{vol}_{n-1}(\partial K \cap H(x, N_{\partial K}(x))) = 0.$$

By Lemma 2(i) we can choose a hyperplane H parallel to $H(x, N_{\partial K}(x))$ that cuts off a set with $\mathbb{P}_f(\partial K \cap \widetilde{H}^-) = s$. This means that $x \notin K_s$. $\square$

Lemma 4. *Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}$ be an a.e. positive, integrable function with $\int_{\partial K} f(x)\, d\mu_{\partial(K)}(x) = 1$.*

(i) *Let s_i, $i \in \mathbb{N}$, be a strictly increasing sequence of positive numbers with $\lim_{i\to\infty} s_i = s_0$. Then we have*

$$K_{s_0} = \bigcap_{i=1}^{\infty} K_{s_i}.$$

(ii) *There exists T with $0 < T \leqslant \frac{1}{2}$ such that K_T is nonempty*, $\mathrm{vol}_n(K_T) = 0$ *and* $\mathrm{vol}_n(K_t) > 0$ *for all $t < T$.*

(iii) *For all s with $0 \leqslant s < T$*

$$K_s = \overline{\bigcup_{\delta>0} K_{s+\delta}}.$$

Clearly, if K is centrally symmetric with respect to the origin and f satisfies $f(x) = f(-x)$, then $T = 1/2$ and K_T contains only one element, namely the center of symmetry. The assumption that f is a.e. positive is necessary.

Proof. (i) Since we have for all $i \in \mathbb{N}$ that $K_{s_0} \subseteq K_{s_i}$, we get trivially

$$K_{s_0} \subseteq \bigcap_{i=1}^{\infty} K_{s_i}.$$

We may assume that we have $K_{s_i} \neq \emptyset$ for all $i \in \mathbb{N}$. Otherwise the equation is obviously true. Since all K_{s_i} are compact and non-empty the intersection is also nonempty. Suppose there is $x \in \bigcap_{i=1}^{\infty} K_{s_i}$ with $x \notin K_{s_0}$. Then there is a hyperplane H_0 such that

$\mathbb{P}_f(\partial K \cap H_0^-) \leqslant s_0$ and $x \in \overset{\circ}{H_0^-}$. We consider the supporting hyperplanes to K_{s_i} that are parallel to H_0 and that are contained in H^-. Moreover, we may assume that $H_{i+1}^- \subseteq H_i^-$. We have $\mathbb{P}_f(\partial K \cap H_i^-) \geqslant s_i$. Since the distances of H_i to H_0 are monotonely decreasing the sequence of hyperplanes H_i converge to a hyperplane $\widetilde{H}_0$. Since for all $i \in \mathbb{N}$ we have $H_i \cap K \neq \emptyset$ it follows by the compactness of K that $\widetilde{H}_0 \cap K \neq \emptyset$. By Lemma 2(ii) we find that

$$\mathbb{P}_f(\partial K \cap \widetilde{H}_0^-) \geqslant s_0.$$

(it is enough to use monotonicity here). We consider two cases now. First, suppose that $H_0 \cap K \neq \emptyset$. If $H_0 \neq \widetilde{H}_0$ we get a contradiction to the strict monotonicity of the function $\mathbb{P}_f(\partial K \cap H^-)$. Thus H_i converge to H_0 and therefore there is i such that $x \notin H_i^-$. It follows that $x \notin K_{s_i}$ which is not true.

The second case is $H_0 \cap K = \emptyset$. Then $\partial K \cap H_0^- = \partial K$ and consequently $s_0 \geqslant 1$. Since $\lim_{i\to\infty} s_i \geqslant 1$ we find an i such that $K_{s_i} = \emptyset$. To check this it is enough to consider two parallel hyperplanes both of which intersect the interior of K.

(ii) We put

$$T = \sup\{s | \mathrm{vol}_n(K_s) > 0\}.$$

Since the sets K_s are compact, convex, nonempty sets,

$$\bigcap_{\mathrm{vol}_n(K_s)>0} K_s$$

is a compact, convex, nonempty set. On the other hand, by (i) we have

$$K_T = \bigcap_{s<T} K_s = \bigcap_{\mathrm{vol}_n(K_s)>0} K_s.$$

Now we show that $\mathrm{vol}_n(K_T) = 0$. Suppose that $\mathrm{vol}_n(K_T) > 0$. Then there is $x_0 \in \overset{\circ}{K_T}$. Let

$$t_0 = \inf\{\mathbb{P}_f(\partial K \cap H^-) | x_0 \in H\}.$$

Since we require that $x_0 \in H$ we have that $\mathbb{P}_f(\partial K \cap H^-)$ is only a function of the normal of H. Since x_0 is an element of the interior of K_T it is also an element of the interior of K. Thus H intersects the interior of K and we can apply Lemma 2(iii). Therefore $\mathbb{P}_f(\partial K \cap H^-)$ is a continuous function of H: $\lim_{i\to\infty} H_i = H$ implies

$$\lim_{i\to\infty} \mathbb{P}_f(\partial K \cap H_i^-) = \mathbb{P}_f(\partial K \cap H^-).$$

Since $\lim_{i\to\infty} H_i = H$ holds if and only if the normals ξ_i of H_i converge to the normal ξ of H in the Euclidean norm, we conclude that $\mathbb{P}_f(\partial K \cap H^-)$ is a continuous function of the normal ξ of H. By compactness this infimum is attained and there is H_0 with $x_0 \in H_0$ and

$$\mathbb{P}_f(\partial K \cap H_0^-) = t_0.$$

Since x_0 is an interior point of K_T we get by Lemma 2(ii) that $T < t_0$. If not, then $t_0 = T$. Therefore $K_T \subseteq H_0^+$ and $x_0 \in H_0$, which means that $x_0 \in \partial K_T$, contradicting the assumption that $x_0 \in \overset{\circ}{K}_T$.

Now we consider $K_{\frac{1}{2}(T+t_0)}$. We claim that x_0 is an interior point of this set and therefore

$$\mathrm{vol}_n(K_{\frac{1}{2}(T+t_0)}) > 0,$$

contradicting the fact that T is the supremum of all t with $\mathrm{vol}_n(K_t) > 0$. We verify now that x_0 is an interior point of $K_{\frac{1}{2}(T+t_0)}$. Suppose x_0 is not an interior point of this set. Then for every $i \in \mathbb{N}$ there x_i with $\|x_i - x_0\| < \frac{1}{i}$ and $x_i \notin K_{\frac{1}{2}(T+t_0)}$. Therefore for every $i \in \mathbb{N}$ there is a hyperplane H_i such that

$$\mathbb{P}_f(\partial K \cap H_i^-) \leqslant \frac{1}{2}(T + t_0), \quad x_i \in H_i \quad \text{and} \quad \|x_i - x_0\| < \frac{1}{i}.$$

We can pass to a convergent subsequence of hyperplanes. By Lemma 2(iii) we conclude that there is a hyperplane H with $x_0 \in H$ and

$$\mathbb{P}_f(\partial K \cap H^-) \leqslant \tfrac{1}{2}(T + t_0).$$

Since $t_0 > \frac{1}{2}(T + t_0)$ this contradicts the definition of t_0.

(iii) Suppose that this is not true. Then there are $x \in K_s$ and $r > 0$ with

$$B_2^n(x, r) \cap \bigcup_{\delta > 0} K_{s+\delta} = \emptyset.$$

Since $\mathrm{vol}_n(K_s) > 0$ the set $B_2^n(x, r) \cap K_s$ contains an interior point. Therefore, there is an interior point y of K_s (which is in particular an interior point of K) such that $y \notin \bigcup_{\delta>0} K_{s+\delta}$. Therefore, for every $n \in \mathbb{N}$ there is a hyperplane H_n with $y \in H_n$ and

$$\mathbb{P}_f(\partial K \cap H_n^-) \leqslant s + \frac{1}{n}.$$

Let n_0 be so big that $s + \frac{1}{n_0} < T$. By compactness there is a convergent subsequence of hyperplanes H_{n_j}, $j \in \mathbb{N}$ with limit H_0 such that $y \in H_0$. The hyperplane H_0 intersects the interior of K because y is an interior point of K.

Therefore, we can apply Lemma 2(iii).

$$s \geqslant \lim_{j\to\infty} \mathbb{P}_f(\partial K \cap H_{n_j}^-) = \mathbb{P}_f(\partial K \cap H_0^-).$$

This implies that y is not an interior point of K_s which is not true. □

In the next proposition we need the Hausdorff distance d_H which for two convex bodies K and L in $\mathbb{R}^n$ is

$$d_H(K,L) = \max\left\{ \max_{x\in L} \min_{y\in K} \|x-y\|, \ \max_{y\in K} \min_{x\in L} \|x-y\| \right\}.$$

Proposition 5. *Let K be a convex body in $\mathbb{R}^n$ and let $f:\partial K \to \mathbb{R}$ be a positive, continuous function with $\int_{\partial K} f(x)\, d\mu_{\partial K}(x) = 1$.*

(i) *Suppose that K has a C^1-boundary. Let $x \in \partial K_s \cap \overset{\circ}{K}$ such that $K_s \neq \emptyset$. Let H be a supporting hyperplane of K_s at x such that $\mathbb{P}_f(\partial K \cap H^-) = s$ (By Lemma 3 there is always such a hyperplane). Then x is the center of gravity of $\partial K \cap H$ with respect to the measure*

$$\frac{f(y)\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y), N_{\partial K}(y)\rangle},$$

i.e.

$$x = \frac{\int_{\partial K\cap H} \frac{y f(y)\, d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y), N_{\partial K}(y)\rangle}}{\int_{\partial K\cap H} \frac{f(y)\, d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y), N_{\partial K}(y)\rangle}},$$

where $N_{\partial K}(y)$ is the unit outer normal to ∂K at y and $N_{\partial K\cap H}(y)$ is the unit outer normal to $\partial K \cap H$ at y in the plane H.

(ii) *If K has a C^1-boundary and $K_s \subset \overset{\circ}{K}$, then K_s is strictly convex.*

(iii) *Suppose that K has a C^1-boundary and $K_T \subset \overset{\circ}{K}$. Then K_T consists of one point $\{x_T\}$ only. This holds in particular, if for every $x \in \partial K$ there are $r(x) > 0$ and $R(x) < \infty$ such that $B_2^n(x - r(x)N_{\partial K}(x), r(x)) \subseteq K \subseteq B_2^n(x - R(x)N_{\partial K}(x), R(x))$.*

(iv) *For all s with $0 \leqslant s < T$ and $\varepsilon > 0$ there is $\delta > 0$ such that $d_H(K_s, K_{s+\delta}) < \varepsilon$.*

Remark 6. (i) We call the point x_T of Proposition 5 the surface point. In general, K_T does not consist of one point only (see the example in 6(ii)). If K_T does not consist of one point only, then we define x_T to be the centroid of K_T.

(ii) In Proposition 5 we have shown that under certain assumptions the surface body reduces to a point. In general this is not the case. We give an example. Let K be the Euclidean ball B_2^n and

$$f = \frac{\chi_C + \chi_{-C}}{2\,\mathrm{vol}_{n-1}(C)},$$

where C is a cap of the Euclidean ball with surface area equal to $\frac{1}{4}\mathrm{vol}_{n-1}(\partial B_2^n)$. Then we get that for all s with $s<\frac{1}{2}$ that K_s contains a Euclidean ball with positive radius. On the other hand $K_{1/2} = \emptyset$.

(iii) If K is a convex body that is centrally symmetric with respect to the point x_0 and f is symmetric (i.e. $f(x_0+x) = f(x_0-x)$), then the surface point x_T coincides with the center of symmetry x_0.

If K is not symmetric then $T<\frac{1}{2}$ is possible. An example for this is a regular triangle C in $\mathbb{R}^2$. If the sidelength is 1 and $f=\frac{1}{3}$, then $T=\frac{4}{9}$ and $C_{\frac{4}{9}}$ consists of the barycenter of C.

Proof of Proposition 5. (i) Let $\widetilde{H}$ be another hyperplane passing through x and ε the angle between the two hyperplanes. Then we have

$$s = \mathbb{P}_f(\partial K \cap H^-) \leqslant \mathbb{P}_f(\partial K \cap \widetilde{H}^-).$$

Thus

$$\begin{aligned} 0 &\leqslant \mathbb{P}_f(\partial K \cap \widetilde{H}^-) - \mathbb{P}_f(\partial K \cap H^-) \\ &= \int_{\partial K \cap \widetilde{H}^- \cap H^+} d\mathbb{P}_f - \int_{\partial K \cap \widetilde{H}^+ \cap H^-} d\mathbb{P}_f. \end{aligned}$$

Let ξ be the vector in H with $\|\xi\| = 1$ that is orthogonal to $H \cap \widetilde{H}$ and that points into the direction of the wedge $\partial K \cap \widetilde{H}^- \cap H^+$ (see Fig. 3). Then the last expression equals

$$\int_{\partial K \cap H} \frac{\langle y-x, \xi \rangle f(y) \tan \varepsilon}{\langle N_{\partial K \cap H}(y), N_{\partial K}(y) \rangle}\, d\mu_{\partial K \cap H}(y) + o(\varepsilon).$$

We verify the latter equality. The distance of $y \in \partial K \cap H$ from $H \cap \widetilde{H}$ is $\langle y-x, \xi \rangle$. Next observe that the "height" of the wedge at y is $\langle y-x, \xi\rangle \tan\varepsilon$. This follows from Figs. 3 and 4.

A surface element of ∂K at y equals, up to an error of order $o(\varepsilon)$, the product of a volume element at y in $\partial K \cap H$ and the length of the tangential line segment between H and $\widetilde{H}$ at y. The length of this tangential line segment is, up to an error of order $o(\varepsilon)$,

$$\frac{\langle y-x, \xi \rangle \tan \varepsilon}{\langle N_{\partial K \cap H}(y), N_{\partial K}(y) \rangle}.$$

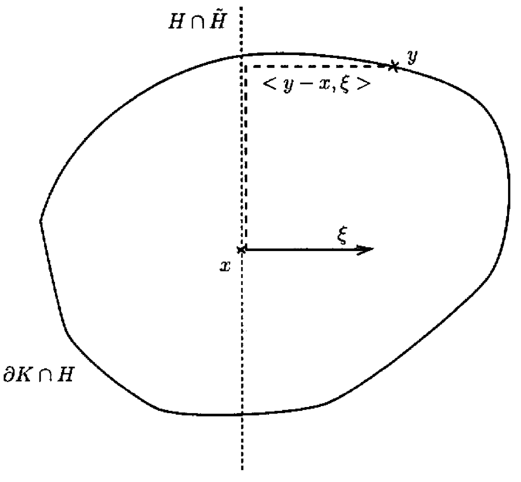

Fig. 3.

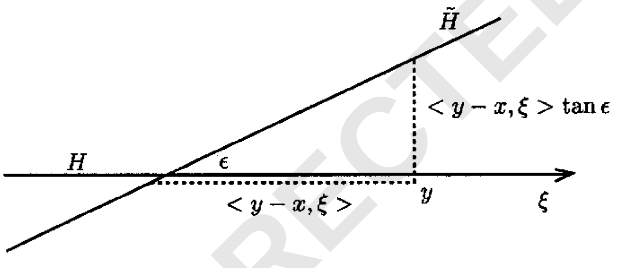

Fig. 4.

See Fig. 5. (For ε small the line passing through y and orthogonal to H is almost orthogonal to $\widetilde{H}$.)

Therefore

$$0 \leqslant \int_{\partial K \cap H} \frac{\langle y - x, \xi \rangle f(y) \tan \varepsilon}{\langle N_{\partial K \cap H}(y), N_{\partial K}(y) \rangle} \, d\mu_{\partial K \cap H}(y) + o(\varepsilon).$$

We divide both sides by ε and pass to the limit for ε to 0. Thus we get for all ξ

$$0 \leqslant \int_{\partial K \cap H} \frac{\langle y - x, \xi \rangle f(y)}{\langle N_{\partial K \cap H}(y), N_{\partial K}(y) \rangle} \, d\mu_{\partial K \cap H}(y).$$

Since this inequality holds for ξ as well as $-\xi$. (Consider another hyperplane $\widetilde{H}$ tilted in the opposite direction.) we get for all ξ

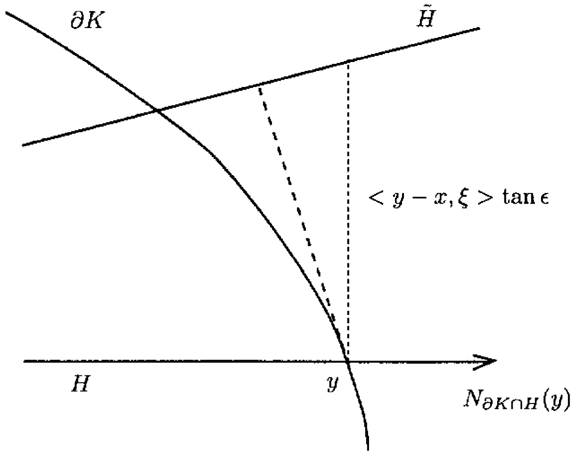

Fig. 5.

$$0=\int_{\partial K\cap H}\frac{\langle y-x,\xi\rangle f(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}\,d\mu_{\partial K\cap H}(y)$$

or

$$0=\left\langle\int_{\partial K\cap H}\frac{(y-x)f(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}\,d\mu_{\partial K\cap H}(y),\xi\right\rangle.$$

This implies

$$0=\int_{\partial K\cap H}\frac{(y-x)f(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}\,d\mu_{\partial K\cap H}(y)$$

and therefore

$$x=\frac{\int_{\partial K\cap H}\frac{yf(y)\,d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}}{\int_{\partial K\cap H}\frac{f(y)\,d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}}.$$

(ii) Suppose that K_s is not strictly convex. Then ∂K_s contains a line-segment $[u,v]$. Let $x\in(u,v)$. As $K_s\subseteq \overset{\circ}{K}$ it follows from Lemma 3(i) that there exists a support-hyperplane $H=H(x,N_{K_s}(x))$ of K_s such that $\mathbb{P}_f(\partial K\cap H^-)=s$. Moreover, we have that $u,v\in H$. By (i)

$$x=u=v=\frac{\int_{\partial K\cap H}\frac{yf(y)\,d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}}{\int_{\partial K\cap H}\frac{f(y)\,d\mu_{\partial K\cap H}(y)}{\langle N_{\partial K\cap H}(y),N_{\partial K}(y)\rangle}}.$$

(iii) Suppose that K_T consists of more than one point. All these points are elements of the boundary of K_T since the volume of K_T is 0 and thus has no interior points. Therefore ∂K_T contains a line-segment $[u, v]$ and cannot be strictly convex, contradicting (ii).

The condition: For every $x \in \partial K$ there is $r(x) < \infty$ such that $K \supseteq B_2^n(x - r(x)N_{\partial K}(x), r(x))$, implies that K has everywhere unique normals. This is equivalent to differentiability of ∂K. By Corollary 25.5.1 of [Ro] ∂K is continuously differentiable. The remaining assertion of (iii) now follows from Lemmas 3(ii) and 4(ii).

(iv) Suppose this is not the case. Then there are s and $\varepsilon > 0$ such that for all δ with $s + \delta < T$

$$d_H(K_s, K_{s+\delta}) \geqslant \varepsilon.$$

Let n_0 be so big that $s + \frac{1}{n_0} < T$. For each n with $n \geqslant n_0$ we choose $x_n \in \partial K_s$ with $d(x_n, K_{s+\frac{1}{n}}) \geqslant \varepsilon$. The sequence x_n, $n \in \mathbb{N}$ has a convergent subsequence whose limit we denote by x_0. Thus for all $n \geqslant n_0$

$$d(x_0, K_{s+\frac{1}{n}}) \geqslant \varepsilon.$$

It follows that

$$d\left(x_0, \overline{\bigcup_{n \in \mathbb{N}} Ks + \tfrac{1}{n}}\right) \geqslant \varepsilon$$

and thus

$$K_s \neq \overline{\bigcup_{n \in \mathbb{N}} K_{s+\frac{1}{n}}}$$

which contradicts Lemma 4(iii) as $x_0 \in K_s$. $\square$

3. The p-affine surface area

Definition 7. Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Let $-\infty \leqslant p \leqslant \infty$, $p \neq -n$. We define the p-affine surface area $O_p(K)$ by

$$O_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa(x)}{\langle x, N_{\partial K}(x) \rangle^n} d\mu_{\partial K}(x) \tag{3}$$

and

$$O_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N_{\partial K}(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_{\partial K}(x) \tag{4}$$

provided the above integrals exist.

In particular, for $p = 0$

$$O_0(K) = \int_{\partial K} \langle x, N_{\partial K}(x) \rangle d\mu_{\partial K}(x) = n \operatorname{vol}_n(K). \tag{5}$$

If the boundary of K is sufficiently smooth then

$$O_{\pm\infty}(K) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n \operatorname{vol}_n(K^*) \tag{6}$$

and

$$O_p(K) = \int_{S^{n-1}} \frac{f_\kappa(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u),$$

where h_K is the support function and f_κ the curvature function, i.e. the reciprocal of the Gauss curvature $\kappa(x)$ at this point $x \in \partial K$ that has u as outer normal.

Blaschke [B] introduced the affine surface area for convex bodies which are sufficiently smooth. This is the case $p = 1$ in the above definition, i.e. O_1. Several authors showed independently that the affine surface area O_1 can be extended to arbitrary convex bodies [L1,Lu1,Schm,SchW1,MW1,W1]. Schütt and Werner [SchW1] showed specifically that the above formula for O_1 extends naturally to arbitrary convex bodies.

Lutwak [Lu2] introduced the p-affine surface area for $1 \leqslant p \leqslant \infty$ and arbitrary convex bodies. He used for the definition expressions that are equivalent to (3) and (4) and showed in the case of smooth convex bodies that both expressions coincide. Hug [H] proved that the expressions coincide for all convex bodies. Meyer and Werner [MW2] introduced a definition for O_{-n} and gave geometric characterizations of the p-affine surface area for $-n \leqslant p \leqslant \infty$.

Let us note that the definition of O_∞ here is different from the definition in [Lu2]. The definitions differ by the factor $\operatorname{vol}_n(K)^{\frac{n}{n+1}} \operatorname{vol}_n(K^*)^{-\frac{n}{n+1}}$.

We have for all convex bodies and all p with $0 \leqslant p \leqslant \infty$ that the quantities $O_p(K)$ are uniformly bounded. For $p = 0$ this follows from (5) and for $p = \pm\infty$ this follows from (6) in the smooth case. For $0 < p < \infty$, it follows from Hölder's inequality. Indeed,

$$O_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N_{\partial K}(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_{\partial K}(x)$$

$$\leqslant \mathrm{vol}_{n-1}(\partial K)^{\frac{n}{n+p}} \left(\int_{\partial K} \frac{\kappa(x)}{\langle x, N_{\partial K}(x)\rangle^{\frac{n(p-1)}{p}}} d\mu_{\partial K}(x) \right)^{\frac{p}{n+p}}.$$

Since 0 is an interior point of K there is a constant $c>0$ such that we have for all $x\in\partial K$ the inequality $c\leqslant\langle x, N_{\partial K}(x)\rangle$. Thus we get

$$O_p(K)\leqslant \mathrm{vol}_{n-1}(\partial K)^{\frac{n}{n+p}} \frac{1}{c^{\frac{n(p-1)}{n+p}}} \left(\int_{\partial K} \kappa(x)\, d\mu_{\partial K}(x)\right)^{\frac{p}{n+p}}$$

$$\leqslant \mathrm{vol}_{n-1}(\partial K)^{\frac{n}{n+p}} \frac{1}{c^{\frac{n(p-1)}{n+p}}} \left(\int_{S^{n-1}} d\sigma(u)\right)^{\frac{p}{n+p}}$$

$$= \mathrm{vol}_{n-1}(\partial K)^{\frac{n}{n+p}} \frac{1}{c^{\frac{n(p-1)}{n+p}}} \left(n\, \mathrm{vol}_n(B_2^n)\right)^{\frac{p}{n+p}}.$$

Similarly, we get for not necessarily smooth K that

$$O_{\pm\infty}(K)\leqslant \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n\, \mathrm{vol}_n(K^*).$$

Thus O_p is finite for all p with $0\leqslant p\leqslant\infty$. This need not to be so for negative values of p. We show that in the following example.

In this example we also compute the p-affine surface areas for the unit balls of the l_n^r-spaces, $1<r<\infty$. Note also that for all p with $0<p\leqslant\infty$ and for all p with $p<-n$

$$O_p(B_1^n) = 0 \quad \text{and} \quad O_p(B_\infty^n) = 0 \tag{7}$$

as the Gaussian curvature is 0 a.e. and that for all p with $-n<p<0$

$$O_p(B_1^n) = \infty \quad \text{and} \quad O_p(B_\infty^n) = \infty. \tag{8}$$

Example 8. Let $1<r<\infty$ and $B_r^n = \{x\in\mathbb{R}^n | \sum_{i=1}^n |x_i|^r \leqslant 1\}$. Then we have

(i) For $1<r<2$ and $-\frac{n}{r-1}\leqslant p<-n$ and for $2<r<\infty$ and $-n<p\leqslant -\frac{n}{r-1}$

$$O_p(B_r^n) = \infty.$$

(ii) For all other cases with $p\neq -n, \pm\infty$ we have

$$O_p(B_r^n) = \frac{2^n (r-1)^{\frac{p(n-1)}{n+p}}}{r^{n-1}} \frac{\left(\Gamma\left(\frac{n+rp-p}{r(n+p)}\right)\right)^n}{\Gamma\left(\frac{n(n+rp-p)}{r(n+p)}\right)}.$$

Moreover, for all $p \neq -n$

$$O_p(B_2^n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} = \mathrm{vol}_{n-1}(\partial B_2^n).$$

Proof. By definition

$$O_p(B_r^n) = \int_{\partial B_r^n} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, N_{\partial B_r^n}(x) \rangle^{\frac{n(p-1)}{n+p}}} \, d\mu_{\partial B_r^n}(x).$$

The curvature is

$$\kappa(x) = \frac{(r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{\frac{n+1}{2}}}$$

and the normal is

$$N_{\partial B_r^n}(x) = \frac{(\mathrm{sgn}(x_1)|x_1|^{r-1}, \ldots, \mathrm{sgn}(x_n)|x_n|^{r-1})}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{\frac{1}{2}}}.$$

Thus we get

$$O_p(B_r^n) = \int_{\partial B_r^n} \frac{\left((r-1)^{n-1} \prod_{i=1}^n |x_i|^{r-2}\right)^{\frac{p}{n+p}}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{\frac{1}{2}}} \, d\mu_{\partial B_r^n}(x).$$

Now we integrate with respect to the variables $x_1, \ldots, x_{n-1}$. The volume of a surface element in the plane of the first $n-1$ coordinates equals the volume of the corresponding surface element on ∂B_r^n times

$$|\langle e_n, N_{\partial B_r^n}(x) \rangle| = \frac{|x_n|^{r-1}}{\left(\sum_{i=1}^n |x_i|^{2r-2}\right)^{\frac{1}{2}}}.$$

Thus, with $(B_r^{n-1})^+$ being the set of all vectors in B_r^{n-1} having nonnegative coordinates.

$$O_p(B_r^n) = 2^n (r-1)^{\frac{p(n-1)}{n+p}} \int_{(B_r^{n-1})^+} \left(\prod_{i=1}^{n} x_i^{r-2} \right)^{\frac{p}{n+p}} x_n^{1-r}\, dx_1 \dots dx_{n-1}$$

$$= 2^n (r-1)^{\frac{p(n-1)}{n+p}} \int_{(B_r^{n-1})^+} \left(\prod_{i=1}^{n-1} x_i^{r-2} \right)^{\frac{p}{n+p}} x_n^{\frac{n-rn-p}{n+p}}\, dx_1 \dots dx_{n-1}. \qquad (9)$$

We show now (i). Let us first assume that $1<r<2$ and $-\frac{n}{r-1} \leqslant p < -n$. We observe that

$$\frac{n-rn-p}{n+p} < 0.$$

Indeed, we have $n+p<0$ and $n-rn-p > n-rn+n = n(2-r) > 0$. Thus

$$x_n^{\frac{n-rn-p}{n+p}} \geqslant 1$$

and

$$O_p(B_r^n) \geqslant 2^n (r-1)^{\frac{p(n-1)}{n+p}} \int_{(B_r^{n-1})^+} \left(\prod_{i=1}^{n-1} x_i^{r-2} \right)^{\frac{p}{n+p}} dx_1 \dots dx_{n-1}.$$

Since $(n-1)^{-\frac{1}{r}} B_\infty^{n-1} \subseteq B_r^{n-1}$

$$O_p(B_r^n) \geqslant 2^n (r-1)^{\frac{p(n-1)}{n+p}} \left(\int_0^{(n-1)^{-\frac{1}{r}}} t^{\frac{p(r-2)}{n+p}}\, dt \right)^{n-1}.$$

As $-\frac{n}{r-1} \leqslant p$ it follows that $\frac{p(r-2)}{n+p} \leqslant -1$ and thus $O_p(B_r^n) = \infty$. In the case $2<r<\infty$ and $-n<p\leqslant -n/(r-1)$ we proceed in the same way. We have $n+p>0$ and $n-rn-p<n(2-r)<0$. From $p\leqslant -n/(r-1)$ we get $(p(r-2))/(n+p) \leqslant -1$.

Now we show (ii). We have to evaluate (9). We use formula 4.635.4 in [GR]. The formula can also be found in volume III of [Fi, p. 392]:

$$O_p(B_r^n) = \frac{2^n (r-1)^{\frac{p(n-1)}{n+p}}}{r^{n-1}} \frac{\left(\Gamma\left(\frac{n+rp-p}{r(n+p)}\right)\right)^n}{\Gamma\left(\frac{n(n+rp-p)}{r(n+p)}\right)}. \qquad \square$$

Remark. Eqs. (5), (7) and (8) also follow from formula (ii) in the above Example if we let $r \to 1$. If we let $r \to \infty$, this holds only for $p \geqslant 0$.

The p-affine surface area is invariant under all linear maps T with $\det(T) = 1$, i.e. we have $O_p(K) = O_p(T(K))$. This had been shown by [Lu2] and later by another method by Hug [H] for p with $0<p\leqslant\infty$. The affine invariance for $-n<p\leqslant\infty$

follows from the results in [MW2]. The proof of [H] seems to carry over to negative p also. We include a proof here for the sake of completeness.

Proposition 9. *Let* $-\infty \leqslant p \leqslant \infty$ *and* $p \neq -n$. *Let* K *be a convex body in* $\mathbb{R}^n$ *such that* $0 \in \overset{\circ}{K}$. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *a linear, invertible map. Then*

$$O_p(T(K)) = \det(T)^{\frac{n-p}{n+p}} O_p(K).$$

(*For* $p = \pm\infty$ *we put* $\frac{n-p}{n+p} = -1$.)

For the proof of Proposition 9 we need some lemmas.

Lemma 10. *Let* K *be a convex body in* $\mathbb{R}^n$ *such that* $0 \in \overset{\circ}{K}$, $\mu_{\partial K}$ *the surface measure on* ∂K, $f : \partial K \to \mathbb{R}$ *an integrable function, and* $T : \mathbb{R}^n \to \mathbb{R}^n$ *an invertible, linear map. Then*

$$\int_{\partial K} f(x)\, d\mu_{\partial K}(x) = \det(T)^{-1} \int_{\partial T(K)} ||T^{-1t}(N_{\partial K}(T^{-1}(y)))||^{-1} f(T^{-1}(y))\, d\mu_{\partial T(K)}(y).$$

Proof. A surface element of ∂K is mapped onto one of $\partial T(K)$ whose volume is bigger by the factor $\det(T)||T^{-1t}(N_{\partial K}(x))||$. We check this. Let $A \subset \partial K$ be a small, open neighborhood of a point $x \in \partial K$ at which ∂K is differentiable. Then

$$\mathrm{vol}_n([0, T(A)]) = \mathrm{vol}_n(T[0, A]) = \det(T)\, \mathrm{vol}_n([0, A]).$$

Since ∂K is differentiable at $x \in A$, the expression $\mathrm{vol}_n([0, A])$ equals up to a small error

$$\frac{1}{n} \langle x, N_{\partial K}(x) \rangle \mathrm{vol}_{n-1}(A)$$

and $\mathrm{vol}_n([0, T(A)])$ equals up to a small error

$$\begin{aligned} &\frac{1}{n} \langle T(x), N_{\partial T(K)}(T(x)) \rangle \mathrm{vol}_{n-1}(T(A)) \\ &= \frac{1}{n} \left\langle T(x), \frac{T^{-1t}(N_{\partial K}(x))}{||T^{-1t}(N_{\partial K}(x))||} \right\rangle \mathrm{vol}_{n-1}(T(A)) \\ &= \frac{1}{n} \left\langle x, \frac{N_{\partial K}(x)}{||T^{-1t}(N_{\partial K}(x))||} \right\rangle \mathrm{vol}_{n-1}(T(A)). \end{aligned}$$

Therefore $\mathrm{vol}_{n-1}(T(A))$ equals up to a small error $\det(T)||T^{-1t}(N_{\partial K}(x))||\mathrm{vol}_{n-1}(A)$. Since ∂K is a.e. differentiable the result follows. □

Lemma 11 (Leichtweiss [L1], Schütt and Werner [SchW1]). *Let* K *be a convex body in* $\mathbb{R}^n$ *and suppose that the generalized Gauss–Kronecker curvature* κ *exists in* $x \in \partial K$.

Let $\Delta(x,t)$ be the height of the cap with volume t, i.e.

$$\mathrm{vol}_n(K \cap H^-(x - \Delta(x,t)N_{\partial K}(x), N_{\partial K}(x))) = t.$$

Then

$$c_n \lim_{t\to 0} \frac{\Delta(x,t)}{t^{\frac{2}{n+1}}} = \kappa^{\frac{1}{n+1}},$$

where $c_n = 2\left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n+1}\right)^{\frac{2}{n+1}}$.

Lemma 12. *Let K be a convex body in $\mathbb{R}^n$ and suppose that the generalized Gauss–Kronecker curvature κ exists in $x \in \partial K$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear, invertible map. Then the generalized Gauss–Kronecker curvature κ exists in $T(x) \in \partial T(K)$ and*

$$\kappa(x) = ||T^{-1t}(N_{\partial K}(x))||^{n+1} \det(T)^2 \kappa(T(x)).$$

Proof. We only show the formula. By Lemma 11

$$c_n \lim_{t\to 0} \frac{\Delta(x,t)}{t^{\frac{2}{n+1}}} = \kappa(x)^{\frac{1}{n+1}} \quad c_n \lim_{s\to 0} \frac{\Delta(T(x),s)}{s^{\frac{2}{n+1}}} = \kappa(T(x))^{\frac{1}{n+1}}.$$

Let $H = H(x - \Delta(x,t)N_{\partial K}(x), N_{\partial K}(x))$. Then we have $\mathrm{vol}_n(K \cap H^-) = t$, $\mathrm{vol}_n(T(K \cap H^-)) = t \det(T)$ and $T(K) \cap T(H^-)$ is a cap of $T(K)$ at $T(x)$. The normal at $T(x)$ is $T^{-1t}(N_{\partial K}(x))||T^{-1t}(N_{\partial K}(x))||^{-1}$. The height of the cap $T(K) \cap T(H^-)$ equals the height of the cap $K \cap H^-$ multiplied by the factor $||T^{-1t}(N_{\partial K}(x))||^{-1}$. We check this. The height of the cap $T(K) \cap T(H^-)$ equals

$$\begin{aligned} &\langle T(x) - T(x - \Delta(x,t)N_{\partial K}(x)), N_{\partial T(K)}(T(x)) \rangle \\ &= \left\langle T(\Delta(x,t)N_{\partial K}(x)), \frac{T^{-1t}(N_{\partial K}(x)))}{||T^{-1t}(N_{\partial K}(x)))||} \right\rangle = \frac{\Delta(x,t)}{||T^{-1t}(N_{\partial K}(x)))||}. \end{aligned}$$

Thus we get

$$\Delta(T(x), t\det(T)) = \Delta(x,t)||T^{-1t}(N_{\partial K}(x))||^{-1}$$

and

$$\frac{\Delta(x,t)}{t^{\frac{2}{n+1}}} = \det(T)^{\frac{2}{n+1}}||T^{-1t}(N_{\partial K}(x))||\frac{\Delta(T(x), t\det(T))}{(t\det(T))^{\frac{2}{n+1}}}.$$

It is left to pass to the limits. □

Proof of Proposition 9. Let $\alpha = p/(n+p)$ and $\beta = n(p-1)/(n+p)$. In the case $p = \pm\infty$ we have $\alpha = 1$ and $\beta = n$. By Lemma 10

$$\begin{aligned}
&\int_{\partial K} \frac{\kappa(x)^{\alpha}}{\langle x, N_{\partial K}(x)\rangle^{\beta}} d\mu_{\partial K}(x) \\
&\quad = \det(T)^{-1} \int_{\partial T(K)} ||T^{-1t}(N_{\partial K}(T^{-1}(y)))||^{-1} \frac{\kappa(T^{-1}(y))^{\alpha}}{\langle T^{-1}(y), N_{\partial K}(T^{-1}(y))\rangle^{\beta}} d\mu_{\partial T(K)}(y) \\
&\quad = \det(T)^{-1} \int_{\partial T(K)} ||T^{-1t}(N_{\partial K}(T^{-1}(y)))||^{-1-\beta} \frac{\kappa(T^{-1}(y))^{\alpha}}{\langle y, N_{\partial T(K)}(y)\rangle^{\beta}} d\mu_{\partial T(K)}(y).
\end{aligned}$$

By Lemma 12 the last expression equals

$$\det(T)^{2\alpha-1} \int_{\partial T(K)} ||T^{-1t}(N_{\partial K}(T^{-1}(y)))||^{\alpha(n+1)-1-\beta} \frac{\kappa(y)^{\alpha}}{\langle y, N_{\partial T(K)}(y)\rangle^{\beta}} d\mu_{\partial T(K)}(y).$$

Notice that $\alpha(n+1) - 1 - \beta = 0$ and $2\alpha - 1 = (p-n)/(n+p)$. $\square$

Now we want to present a geometric characterization of the p-affine surface area for all p similar in spirit to the one given in [SchW2,W3]. A geometric interpretation for $-n \leqslant p \leqslant \infty$ exists already in [MW2].

We will briefly mention the results of [SchW2] as some of the concepts introduced there will also be useful here.

A random polytope is the convex hull of finitely many points that are chosen from K with respect to a probability measure $\mathbb{P}$ on K. The expected volume of a random polytope of N points is

$$\mathbb{E}(\mathbb{P}, N) = \int_K \cdots \int_K \mathrm{vol}_n([x_1, \ldots, x_N])\, d\mathbb{P}(x_1) \ldots d\mathbb{P}(x_N),$$

where $[x_1, \ldots, x_N]$ is the convex hull of the points $x_1, \ldots, x_N$.

For a integrable, nonnegative function $f : \partial K \to \mathbb{R}$ with $\int_{\partial K} f(x)\, d\mu = 1$ we denote by $\mathbb{P}_f$ the probability measure with $d\mathbb{P}_f = f d\mu_{\partial K}$.

In [SchW2] random polytopes are considered where the points are chosen from the boundary of K with respect to $\mathbb{P}_f$ and then the expected volume is

$$\mathbb{E}(f, N) = \mathbb{E}(\mathbb{P}_f, N) = \int_{\partial K} \cdots \int_{\partial K} \mathrm{vol}_n([x_1, \ldots, x_N])\; d\mathbb{P}_f(x_1) \ldots d\mathbb{P}_f(x_N).$$

For q, $-\infty \leqslant q \leqslant \infty$, $q \neq -n$, let the functions $f_q : \partial K \to \mathbb{R}$ be given as follows: For $q = \pm\infty$, put

$$f_{\pm\infty}(x) = \frac{\kappa(x)}{O_{\pm\infty}(K) \langle x, N_{\partial K}(x)\rangle^{n}} \tag{10}$$

and for all other values of q

$$f_q(x) = \frac{\kappa(x)^{\frac{q}{n+q}}}{O_q(K)\langle x, N_{\partial K}(x)\rangle^{\frac{n(q-1)}{n+q}}}. \tag{11}$$

The following theorem is a consequence of the result in [SchW2]. For the proof see [SchW2].

Theorem 13. *Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Assume also that there are r and R in $\mathbb{R}$ with $0<r\leqslant R<\infty$ so that we have for all $x\in\partial K$*

$$B_2^n(x - rN_{\partial K}(x), r) \subseteq K \subseteq B_2^n(x - RN_{\partial K}(x), R).$$

Let $-\infty\leqslant p\leqslant\infty$, $p\neq -n$. For $p\neq -1$ let $q=\frac{n-p(n-2)}{p+1}$. Then

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_q, N)}{\left(\frac{O_q(K)}{N}\right)^{\frac{2}{n-1}}} = c_n O_p(K) \tag{12}$$

and

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f_{\pm\infty}, N)}{\left(\frac{O_{\pm\infty}(K)}{N}\right)^{\frac{2}{n-1}}} = c_n O_{-1}(K), \tag{13}$$

where $c_n = \frac{(n-1)^{\frac{n+1}{n-1}}\Gamma\left(n+1+\frac{2}{n-1}\right)}{2(n+1)!(\mathrm{vol}_{n-2}(\partial B_2^{n-1}))^{\frac{2}{n-1}}}$.

Now we come to the geometric interpretation of the p-affine surface area using surface bodies.

Let K be a convex body and $x\in\partial K$. We define $r(x)$ as the maximum of all real numbers ρ so that $B_2^n(x-\rho N_{\partial K}(x),\rho)\subseteq K$. This has been used in [SchW1] to investigate the floating body. It was pointed out there that for all α with $0\leqslant\alpha<1$ the integral $\int_{\partial K} r(x)^{-\alpha} d\mu_{\partial(K)}(x)$ is finite. The cube is an example showing that $\int_{\partial K} r(x)^{-1} d\mu_{\partial(K)}(x)$ may be infinite.

From now on we assume without loss of generality that 0 is an interior point of K and for $x\in\partial K$ and $s>0$ we put

$$x_s = [0,x]\cap\partial K_{f,s}.$$

We call the function $M_f:\partial K\to\mathbb{R}$

$$M_f(x_0) = \inf_{0<s} \frac{1}{\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s, N_{\partial K_s}(x_s)))} \int_{\partial K\cap H^-(x_s, N_{\partial K_s}(x_s))} f\, d\mu_{\partial K} \tag{14}$$

the minimal function.

Theorem 14. *Let K be a convex body in $\mathbb{R}^n$. Suppose that $f: \partial K \to \mathbb{R}$ is an integrable, almost everywhere strictly positive function such that $\int f \, d\mu_{\partial K} = 1$. Assume that*

$$\int_{\partial K} \frac{1}{((M_f(x))^{\frac{2}{n-1}} r(x)} d\mu_{\partial K}(x) < \infty.$$

Then

$$d_n \lim_{s \to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{f,s})}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} d\mu_{\partial K},$$

where $d_n = 2(\mathrm{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}$.

One cannot expect that the asymptotic formula of Theorem 14 holds for all integrable function. We give an example.

It makes most sense to define

$$\frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} = 0$$

if $\kappa(x) = 0$ and $f(x) = 0$. Consider the convex body K (see Fig. 6) which consists of a half-circle and a triangle attached to it. We define the function f to be equal to 0 on the lines of the triangle and constant on the half-circle such that the integral of f equals 1. Then, since $K_{f,0}$ does not contain the triangular part of K

$$\lim_{s \to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{f,s})}{s^{\frac{2}{n-1}}} = \infty$$

while

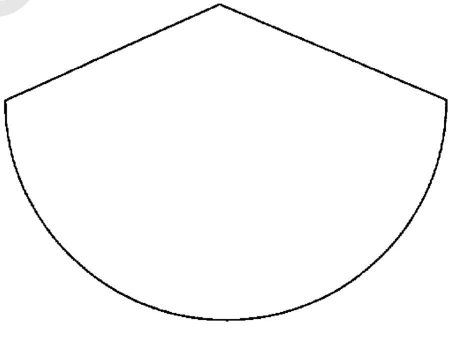

Fig. 6.

$$\int_{\partial K} \frac{\kappa^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} \, d\mu_{\partial K}$$

is clearly finite.

Corollary 15. *Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Let $-\infty \leqslant p \leqslant \infty$, $p \neq -n$. For $p \neq -1$ let $q = \frac{n-p(n-2)}{p+1}$ and for $p = -1$ let $q = \infty$. Let f_q be as in* (10) *and* (11) *and assume that it is almost everywhere strictly positive. Assume that*

$$\int_{\partial K} \frac{1}{(M_{f_q}(x))^{\frac{2}{n-1}} r(x)} \, d\mu_{\partial K}(x) < \infty.$$

Then

$$d_n \lim_{s \to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{f_q,s})}{(sO_q(K))^{\frac{2}{n-1}}} = O_p(K). \tag{15}$$

Thus for every p-affine surface area O_p there is a density f_q with $q = \frac{n-p(n-2)}{p+1}$ so that (15) holds. Conversely, for each density f_q there is an affine surface area O_p with $p = \frac{n-q}{q+n-2}$ such that (15) holds.

For the proof of Theorem 14 we need several lemmas.

Lemma 16. *Let K and L be two convex bodies in $\mathbf{R}^n$ such that $0 \in \mathring{L}$ and $L \subseteq K$. Then*

$$\mathrm{vol}_n(K) - \mathrm{vol}_n(L) = \frac{1}{n} \int_{\partial K} \langle x, N_{\partial K}(x) \rangle \left(1 - \left(\frac{\|x_L\|}{\|x\|}\right)^n\right) d\mu_{\partial K}(x),$$

where $x_L = [0, x] \cap \partial L$ and $\mu_{\partial K}$ is the usual surface measure on ∂K.

The proof of Lemma 16 is standard.

Since we want to apply the Lebesgue convergence theorem, we need a dominating function. This function turns out to have $1/r(x)$ as a factor. In [SchW1,Sch1], dealing with related problems, the dominating function is a multiple of $r(x)^{-\frac{n-1}{n+1}}$ which is integrable. In fact, as mentioned above, $r(x)^{-\alpha}$ is integrable provided that $\alpha < 1$ and there is an example in [SchW1] for which $1/r(x)$ is not integrable.

Lemma 17. *Let K be a convex body in $\mathbb{R}^n$ such that 0 is an interior point of K and let $f : \partial K \to \mathbb{R}$ be an integrable function with $\int_{\partial K} f(x) \, d\mu_{\partial K}(x) = 1$ and such that $f \geqslant 0$ a.e.*

Then there is $s_0>0$ such that for all s with $0\leqslant s\leqslant s_0$ and for almost all $x\in\partial K$

$$0\leqslant\frac{\langle x,N_{\partial K}(x)\rangle\left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right)}{s^{\frac{2}{n-1}}}\leqslant\frac{C}{(M_f(x))^{\frac{2}{n-1}}r(x)},$$

where $x_s=[0,x]\cap\partial K_{f,s}$ and C is an absolute constant. If the normal is not unique we take any normal to a supporting hyperplane at this point.

Proof. By Proposition 5(iv) there is s_0 such that for all s with $0\leqslant s\leqslant s_0$ the point 0 is an interior point of $K_{f,s}$. Thus x_s is well defined.

Let $x\in\partial K$. If the normal $N_{\partial K}(x)$ is not unique then $r(x)=0$ and the estimate is satisfied. We first consider the case that $x_s\in\partial K$. Then, by construction of x_s, $x_s=x$ and therefore

$$\frac{\langle x,N_{\partial K}(x)\rangle\left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{s^{\frac{2}{n-1}}}=0.$$

Thus we may assume that $N_{\partial K}(x)$ is unique and x_s is an interior point of K.

As x and x_s are collinear and $\|x_s\|\leqslant\|x\|$

$$\frac{\|x_s\|}{\|x\|}=1-\frac{\|x-x_s\|}{\|x\|}.$$

Hence

$$\begin{aligned}&\frac{1}{n}\langle x,N_{\partial K}(x)\rangle\left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right)\\&=\frac{1}{n}\langle x,N_{\partial K}(x)\rangle\left(1-\left(1-\frac{\|x-x_s\|}{\|x\|}\right)^n\right)\\&\leqslant\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle\|x-x_s\|.\end{aligned}\tag{16}$$

The last expression is also denoted by Δ_s:

$$\Delta_s=\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle\|x-x_s\|=\langle x-x_s,N_{\partial K}(x)\rangle.$$

It is the distance of x to the hyperplane through x_s and orthogonal to $N_{\partial K}(x)$. As x_s is an interior point of K, by Lemma 3(i) there is a hyperplane H with $x_s\in H$ and $\mathbb{P}_f(\partial K\cap H^-)=s$.

$$s=\mathbb{P}_f(\partial K\cap H^-)=\int_{\partial K\cap H^-}f(y)\,d\mu_{\partial K}(y)\geqslant M_f(x)\mathrm{vol}_{n-1}(\partial K\cap H^-).\tag{17}$$

We show now that there is a constant c such that we have for all $x\in\partial K$

$$c\,\mathrm{vol}_{n-1}(\partial K\cap H^-)\geqslant\begin{cases}(\Delta_s r(x))^{\frac{n-1}{2}} & \text{if } \Delta_s\leqslant\min\{\frac{r(x)}{2}, r(x)\langle\frac{x}{\|x\|},N_{\partial K}(x)\rangle^2\},\\ \Delta_s^{n-1} & \text{if } \Delta_s>\min\{\frac{r(x)}{2}, r(x)\langle\frac{x}{\|x\|},N_{\partial K}(x)\rangle^2\}.\end{cases}\tag{18}$$

This inequality is an analogue of an inequality in [Sch1] (see [Sch1, Lemma 5]). We consider first the case $\Delta_s>\min\{\frac{r(x)}{2}, r(x)\langle\frac{x}{\|x\|},N_{\partial K}(x)\rangle^2\}$. Since 0 is an interior point, there is $\rho>0$ such that $B_2^n(0,\rho)\subseteq K$. We consider the convex hull of x and $B_2^n(0,\rho)$. Then

$$\mathrm{vol}_{n-1}(\partial K\cap H^-)\geqslant\mathrm{vol}_{n-1}(H\cap[x,B_2^n(0,\rho)]).\tag{19}$$

The set $[x,B_2^n(0,\rho)]$ contains a Euclidean ball with center x_s and radius $\rho\frac{\|x-x_s\|}{\|x\|}$. Therefore $H\cap[x,B_2^n(0,\rho)]$ contains a $n-1$-dimensional Euclidean ball whose radius is greater than $\rho\frac{\|x-x_s\|}{\|x\|}$. Thus we get

$$\mathrm{vol}_{n-1}(\partial K\cap H^-)\geqslant\left(\frac{\rho}{\mathrm{diam}(K)}\|x-x_s\|\right)^{n-1}\mathrm{vol}_{n-1}(B_2^{n-1}).\tag{20}$$

Since $\Delta_s\leqslant\|x-x_s\|$ we have established (18) for the case

$$\Delta_s>\min\left\{\frac{r(x)}{2}, r(x)\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle^2\right\}$$

(actually we did not use $\Delta_s>\min\{\frac{r(x)}{2}, r(x)\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle^2\}$).

Now we consider the other case:

$$\Delta_s\leqslant\min\left\{\frac{r(x)}{2}, r(x)\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle^2\right\}.\tag{21}$$

For all s with $0<s\leqslant s_0$

$$\frac{1}{3}\left\langle\frac{x}{\|x\|},N_{\partial K}(x)\right\rangle\|x-x_s\|\leqslant d(x_s,\partial B_2^n(x-r(x)N_{\partial K}(x),r(x))),$$

where $d(x_s,\partial B_2^n(x-r(x)N_{\partial K}(x),r(x)))$ denotes the distance of x_s to the boundary of the Euclidean ball. We show this. In Fig. 7 this distance equals $\|x_s-y_s\|$.

As can be seen from Fig. 7 we have $\|x_s-y_s\|\leqslant\Delta_s\leqslant\|x_s-z_s\|$. We claim that $\|x_s-z_s\|\leqslant3\|x_s-y_s\|$. The ratio between $\|x_s-z_s\|$ and $\|x_s-y_s\|$ is monotone. Indeed, let γ be the angle at $x-r(x)N_{\partial K}(x)$. Then

$$\frac{\|x_s-z_s\|}{\|x_s-y_s\|}=\frac{r(x)}{\|x_s-y_s\|}\left(\frac{1}{\cos\gamma}-1\right)+1,$$

which is decreasing as $s\to0$, for γ with $0\leqslant\gamma\leqslant\frac{\pi}{2}$. Therefore it suffices to consider the

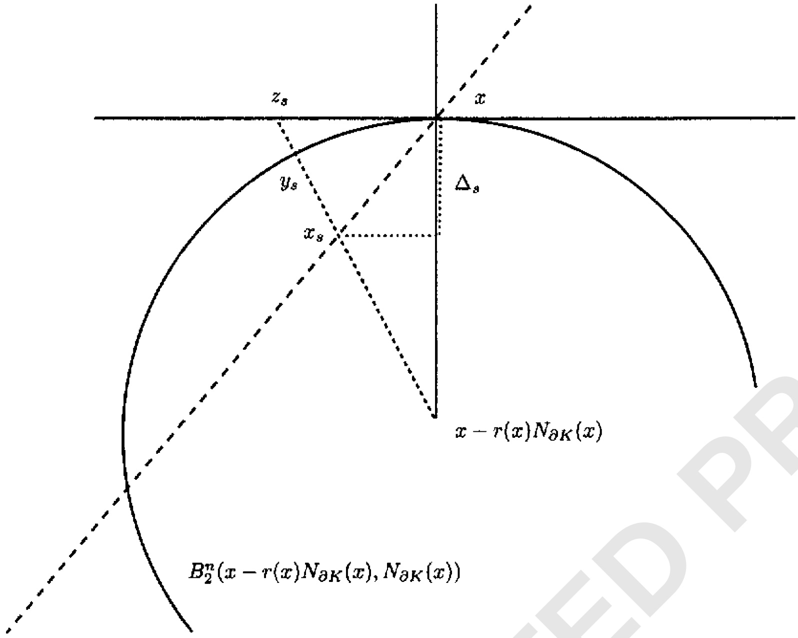

Fig. 7.

case when the line through x_s and y_s is orthogonal to the line through x and x_s. Then we have

$$\begin{aligned}||x_s - z_s|| &= r(x)\left(\frac{1}{\cos\gamma} - 1\right) + ||x_s - y_s|| \\ &= r(x)\left(\frac{1}{r(x) - ||x_s - y_s||} - 1\right) + ||x_s - y_s|| \\ &= \frac{r(x)||x_s - y_s||}{r(x) - ||x_s - y_s||} + ||x_s - y_s|| \leqslant 3||x_s - y_s||.\end{aligned}$$

The last inequality follows because $r(x) - ||x_s - y_s|| \geqslant r(x) - \Delta_s \geqslant \frac{1}{2}r(x)$. Therefore, $\partial B_2^n(x - r(x)N_{\partial K}(x), r(x)) \cap H^-$ is a cap of a Euclidean ball with radius $r(x)$ whose height is greater than $\frac{1}{3}\Delta_s = \frac{1}{3}\left\langle \frac{x}{||x||}, N_{\partial K}(x)\right\rangle ||x - x_s||$.

The surface area of such a cap is greater than (see [SchW2, Lemma 1.3])

$$\mathrm{vol}_{n-1}(B_2^{n-1})r(x)^{\frac{n-1}{2}}\left(\frac{2}{3}\Delta_s - \frac{\Delta_s^2}{9r(x)}\right)^{\frac{n-1}{2}}.$$

As $\mathrm{vol}_{n-1}(\partial K \cap H^-) \geqslant \mathrm{vol}_{n-1}(\partial B_2^n(x - r(x)N_{\partial K}(x), r(x)) \cap H^-)$, this gives the other case of (18). Therefore (17) and (18) give (with a new constant c)

$$s \geqslant \begin{cases} cM_f(x)(\Delta_s r(x))^{\frac{n-1}{2}} & \text{if } \Delta_s \leqslant \min\left\{\frac{r(x)}{2}, r(x)\left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle^2\right\}, \\ cM_f(x)\Delta_s^{n-1} & \text{if } \Delta_s > \min\left\{\frac{r(x)}{2}, r(x)\left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle^2\right\}. \end{cases} \tag{22}$$

It follows

$$s^{\frac{2}{n-1}} \geqslant \begin{cases} (cM_f(x))^{\frac{2}{n-1}}\Delta_s r(x) & \text{if } \Delta_s \leqslant \min\left\{\frac{r(x)}{2}, r(x)\left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle^2\right\}, \\ (cM_f(x))^{\frac{2}{n-1}}\Delta_s^2 & \text{if } \Delta_s > \min\left\{\frac{r(x)}{2}, r(x)\left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle^2\right\}. \end{cases} \tag{23}$$

Therefore, we get for all s with $0 < s \leqslant T$ with a new constant c

$$s^{\frac{2}{n-1}} \geqslant (cM_f(x))^{\frac{2}{n-1}}\Delta_s r(x) \tag{24}$$

and thus with (16) and (24)

$$\frac{\langle x, N_{\partial K}(x)\rangle \left(1 - \left(\frac{\|x_s\|}{\|x\|}\right)^n\right)}{ns^{\frac{2}{n-1}}} \leqslant \frac{\left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle \|x - x_s\|}{(cM_f(x))^{\frac{2}{n-1}} r(x)\Delta_s} \leqslant \frac{1}{(cM_f(x))^{\frac{2}{n-1}} r(x)}. \qquad \square$$

Lemma 18. *Let K be a convex body in $\mathbb{R}^n$ and let $x_0 \in \partial K$ such that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder). Let $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function with $\int f\, d\mu = 1$. Assume that $f(x_0) > 0$ and that*

$$\lim_{\Delta \to 0} \frac{1}{\mathrm{vol}_{n-1}(\partial K \cap H_\Delta^-)} \int_{\partial K \cap H_\Delta^-} |f(x) - f(x_0)|\, d\mu_{\partial K}(x) = 0, \tag{25}$$

where $H_\Delta = H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$. Then there is s_0 so that for all s with $0 < s \leqslant s_0$

$$\frac{1}{2} f(x_0) \leqslant \frac{1}{\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))} \int_{\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))} f(x)\, d\mu_{\partial K}(x). \tag{26}$$

Proof. Let

$$A_\Delta = \left\{ x \in \partial K \cap H_\Delta^- \,\middle|\, f(x) > \frac{9}{10} f(x_0) \right\}. \tag{27}$$

By (25)

$$\lim_{\Delta \to 0} \frac{\mu_{\partial K}(A_\Delta)}{\mu_{\partial K}(\partial K \cap H_\Delta^-)} = 1. \tag{28}$$

Let p be the metric projection from ∂K to $H(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$. For every $\delta > 0$ there is Δ such that for all measurable $A \subseteq \partial K \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$

$$\mathrm{vol}_{n-1}(p(A)) \leqslant \mathrm{vol}_{n-1}(A) \leqslant (1+\delta)\mathrm{vol}_{n-1}(p(A)). \tag{29}$$

This is easily seen since for Δ sufficiently small the normals $N_{\partial K}(x_0)$ and $N_{\partial K}(x)$ differ only by a small angle. Compare the proof of Lemma 2.7 in [SchW2].

We apply an affine transform $T: \mathbb{R}^n \to \mathbb{R}^n$ to K so that the indicatrix of Dupin is transformed into an $n-1$-dimensional Euclidean ball (see formula (5) in [SchW2]). T has the following properties:

$$T(x_0) = x_0 \quad T(N_{\partial K}(x_0)) = N_{\partial K}(x_0) \quad \det(T) = 1$$

and T maps a measurable subset of a hyperplane orthogonal to $N_{\partial K}(x_0)$ onto a subset of the same $n-1$-dimensional measure. By (29) it follows that for all $\varepsilon > 0$ there is $\Delta > 0$ such that for all measurable subsets A of $\partial K \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$

$$(1-\varepsilon)\mathrm{vol}_{n-1}(A) \leqslant \mathrm{vol}_{n-1}(T(A)) \leqslant (1+\varepsilon)\mathrm{vol}_{n-1}(A). \tag{30}$$

Indeed, by (29) the sets A and $p(A)$ have up to a small error the same volume. $T(p(A))$ has the same volume as $p(A)$. Now we compare this to $p^{-1}(T(A))$.

$T(K)$ can be approximated at $x_0 = T(x_0)$ by a n-dimensional Euclidean ball, i.e. for all $\varepsilon > 0$ there are Δ and r, R with $r \leqslant R \leqslant (1+\varepsilon) r$ such that

$$\begin{aligned} &B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ &\subseteq T(K) \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0)) \\ &\subseteq B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0)). \end{aligned} \tag{31}$$

For any $\Delta > 0$ there is s_0 so that for all s with $0 < s \leqslant s_0$

$$K \cap H^-(x_s, N_{\partial K_s}(x_s)) \subseteq K \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0)).$$

This holds since $N_{\partial K_s}(x_s)$ converges to $N_{\partial K}(x_0)$ for $s \to 0$. See Lemma 2.5 in [SchW2]. Thus we can apply (30) to $A = \partial K \cap H^-(x_s, N_{\partial K_s}(x_s))$ and obtain for sufficiently small s

$$\begin{aligned}(1+\varepsilon)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))\\ \geqslant \mathrm{vol}_{n-1}(\partial T(K) \cap T(H^-(x_s, N_{\partial K_s}(x_s))))\\ \geqslant \mathrm{vol}_{n-1}(\partial B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap T(H^-(x_s, N_{\partial K_s}(x_s)))).\end{aligned} \tag{32}$$

For the last inequality we consider the metric projection. Now

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap T(H^-(x_s, N_{\partial K_s}(x_s))))\\ \geqslant \mathrm{vol}_{n-1}(B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap T(H(x_s, N_{\partial K_s}(x_s))))\end{aligned}$$

and the set

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap T(H(x_s, N_{\partial K_s}(x_s))) \tag{33}$$

is a ball whose radius is larger than $\frac{1}{2}$ times the radius of the ball

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0)), \tag{34}$$

where

$$\Delta_0 = \max\{\|x_0 - x\| \; | x \in B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H(x_s, N_{\partial K}(x_s))\}.$$

This follows from Fig. 8.

Hence, by (32)

$$\begin{aligned}(1+\varepsilon)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))\\ \geqslant 2^{-n+1}\mathrm{vol}_{n-1}(B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0)))\end{aligned}$$

By (31), for sufficiently small Δ_0

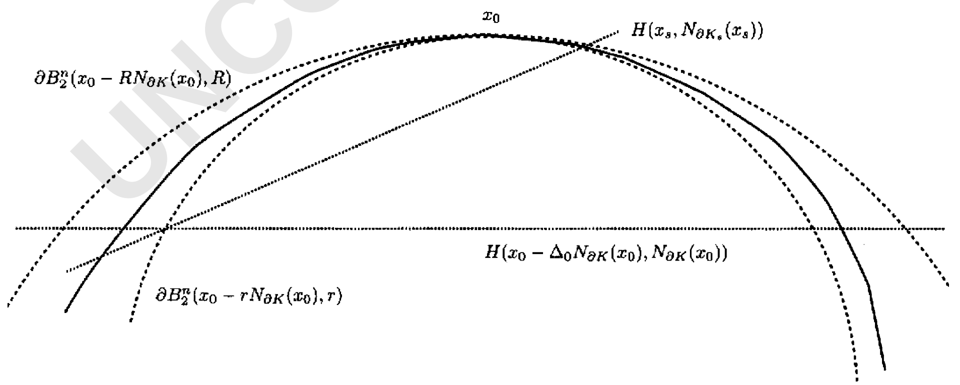

Fig. 8.

$$
\begin{aligned}
&(1+\varepsilon)\mathrm{vol}_{n-1}(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s)))\\
&\geqslant 2^{-n+1}\mathrm{vol}_{n-1}\left(B_2^n\left(x_0 - \frac{R}{1+\varepsilon}N_{\partial K}(x_0), \frac{R}{1+\varepsilon}\right) \cap H(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0))\right)\\
&\geqslant 2^{-n+1}\frac{1}{(1+\varepsilon)^{n-1}}\mathrm{vol}_{n-1}(T(K)\cap H(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0)))\\
&= 2^{-n+1}\frac{1}{(1+\varepsilon)^{n-1}}\mathrm{vol}_{n-1}(K\cap H(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0)))\\
&\geqslant (1+\delta)^{-1}2^{-n+1}\frac{1}{(1+\varepsilon)^{n-1}}\mathrm{vol}_{n-1}(\partial K\cap H^-(x_0 - \Delta_0 N_{\partial K}(x_0), N_{\partial K}(x_0))). \qquad (35)
\end{aligned}
$$

The last inequality follows from (29). By (28) we get that for Δ sufficiently small on a subset of $\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))$ whose measure is at least $\frac{3}{4}$ of this cap we have $(\frac{9}{10})f(x_0) \leqslant f(x)$. This proves (26). □

We say that a family of sets $A_s \subseteq \partial K$, $0 < s \leqslant s_0$, shrinks nicely to a point $x_0 \in \partial K$ if
(i) $\lim_{s\to 0}\mathrm{diam}(A_s) = 0$
and if
(ii) there is $c>0$ such that for every s there is t with

$$\partial K \cap B_2^n(x_0, t) \subseteq A_s \subseteq \partial K \cap B_2^n(x_0, ct).$$

See e.g. [Fo, pp.96–98] in the case of $\mathbb{R}^n$. The results carry over to the case of a boundary of a convex body. In particular, the result that we are using here, that the limit

$$\lim_{r\to 0}\frac{1}{\mathrm{vol}_{n-1}(\partial K\cap B_2^n(x_0, r))}\int_{\partial K\cap B_2^n(x_0,r)} |f(x) - f(x_0)|\, d\mu_{\partial K}(x) = 0 \qquad (36)$$

exists almost everywhere.

If a family A_s, $0<s$, shrinks nicely to a point x_0 then we have

$$\lim_{s\to 0}\frac{1}{\mathrm{vol}_{n-1}(A_s)}\int_{A_s} |f(x) - f(x_0)|\, d\mu_{\partial K}(x) = 0 \qquad (37)$$

provided that (36) holds.

Lemma 19. *Let K be a convex body in $\mathbb{R}^n$ and $x_0 \in \partial K$. Suppose that the indicatrix at x_0 exists and is an ellipsoid (and not a cylinder).*

(i) *Then the family of sets*

$$\partial K \cap H^-(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0)) \quad 0<\Delta$$

shrinks nicely to x_0.

(ii) *Suppose that* $f:\partial K\to\mathbb{R}$ *is an integrable, a.e. strictly positive function and that* $f(x_0)>0$. *Moreover, suppose that*

$$\lim_{r\to 0}\frac{1}{\mathrm{vol}_{n-1}(\partial K\cap B_2^n(x_0,r))}\int_{\partial K\cap B_2^n(x_0,r)}|f(x)-f(x_0)|\,d\mu_{\partial K}(x)=0. \tag{38}$$

Then the family

$$\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))\quad 0<s$$

shrinks nicely to x_0.

Proof. (i) Since the indicatrix at x_0 is an ellipsoid we can approximate ∂K at x_0 by an ellipsoid. Therefore, there are Δ_0, r and R such that $\Delta_0\leqslant r$,

$$B_2^n(x_0-rN_{\partial K}(x_0),r)\subseteq \mathring{K}\cup\{x_0\} \tag{39}$$

and

$$\begin{aligned}&K\cap H^-(x_0-\Delta_0N_{\partial K}(x_0),N_{\partial K}(x_0))\\&\quad\subseteq B_2^n(x_0-RN_{\partial K}(x_0),R)\cap H^-(x_0-\Delta_0N_{\partial K}(x_0),N_{\partial K}(x_0)).\end{aligned} \tag{40}$$

Since we have for all Δ with $0<\Delta\leqslant\Delta_0$

$$\begin{aligned}&B_2^n(x_0-RN_{\partial K}(x_0),R)\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))\\&\quad\subseteq B_2^n(x_0,\sqrt{2R\Delta})\end{aligned}$$

it follows from (40) that

$$K\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))\subseteq B_2^n(x_0,\sqrt{2R\Delta}) \tag{41}$$

which implies

$$\partial K\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))\subseteq \partial K\cap B_2^n(x_0,\sqrt{2R\Delta}). \tag{42}$$

On the other hand, with $H=H(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))$

$$\begin{aligned}&\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\\&\quad=(\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\cap H^-)\cup(\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\cap H^+)\\&\quad\subseteq(\partial K\cap H^-)\cup(\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\cap H^+).\end{aligned}$$

We have

$$B_2^n(x_0,\sqrt{2r\Delta})\cap H^+\subseteq B_2^n(x_0-rN_{\partial K}(x_0),r)\cap H^+.$$

By (39)

$$\begin{aligned}&\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\cap H^+\\ &\quad\subseteq \partial K\cap B_2^n(x_0-rN_{\partial K}(x_0),r)\cap H^+=\emptyset.\end{aligned}$$

Therefore we get

$$\begin{aligned}&\partial K\cap B_2^n(x_0,\sqrt{2r\Delta})\\ &\quad\subseteq \partial K\cap H^-(x_0-\Delta N_{\partial K}(x_0),N_{\partial K}(x_0))\\ &\quad\subseteq \partial K\cap B_2^n(x_0,\sqrt{2R\Delta}).\end{aligned}$$

(ii) Let r, R and Δ_0 as above. We denote the height of the cap $K\cap H^-(x_s,N_{\partial K}(x_0))$ by $\Delta_s=\langle x_0-x_s,N_{\partial K}(x_0)\rangle$. We require that $\Delta_s\leqslant\Delta_0$. We have

$$H(x_s,N_{\partial K}(x_0))=H(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0)).$$

As in the proof of Lemma 3(ii) we show that x_s is an interior point. We have by Lemma 3(i)

$$s=\int_{\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))} f(x)\,d\mu_{\partial K}(x). \tag{43}$$

If the normal is not unique we choose an appropriate one. By (i) the family $\partial K\cap H^-(x_0-\Delta_s N_{\partial K}(x_0),N_{\partial K}(x_0))$, $0<\Delta$, shrinks nicely to x_0. Therefore, by assumption (38)

$$\lim_{\Delta\to 0}\frac{1}{\mathrm{vol}_{n-1}(\partial K\cap H_\Delta^-)}\int_{\partial K\cap H_\Delta^-}|f(x)-f(x_0)|\,d\mu_{\partial K}(x)=0. \tag{44}$$

Thus hypothesis (25) of Lemma 18 is fulfilled and we have (26). Therefore

$$\begin{aligned}&\tfrac{1}{2}f(x_0)\,\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\\ &\quad\leqslant s\leqslant 2f(x_0)\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K}(x_0))).\end{aligned} \tag{45}$$

The left-hand inequality follows from (26) and (43). The right-hand inequality follows by (44) and

$$s\leqslant\int_{\partial K\cap H^-(x_s,N_{\partial K}(x_0))} f(x)\,d\mu_{\partial K}(x).$$

Since $f(x_0)>0$ inequality (45) implies

$$\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K_s}(x_s)))\leqslant 4\,\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s,N_{\partial K}(x_0))). \tag{46}$$

From this we get for sufficiently small Δ_0

$$\begin{aligned}&\mathrm{vol}_{n-1}(\partial B_2^n(x_0 - rN_{\partial K}(x_0), r)) \cap H^-(x_s, N_{\partial K_s}(x_s)))\\&\quad\leqslant 4\,\mathrm{vol}_{n-1}(\partial B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K}(x_0)))\end{aligned} \tag{47}$$

because the metric projection maps a set onto a set of smaller volume. Let h_s be the height of the cap

$$B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_s, N_{\partial K_s}(x_s)).$$

For Δ_0 sufficiently small we have $h_s \leqslant r$. Indeed, suppose $h_s > r$, then by (47)

$$\frac{1}{2} r^{n-1} \mathrm{vol}_{n-1}(\partial B_2^n) \leqslant 8\,\mathrm{vol}_{n-1}(B_2^{n-1}) R^{\frac{n-1}{2}} \left(2\Delta_s - \frac{\Delta_s^2}{R}\right)^{\frac{n-1}{2}}.$$

For sufficiently small Δ_0 this is impossible. Again, by (47)

$$\mathrm{vol}_{n-1}(B_2^{n-1}) r^{\frac{n-1}{2}} \left(2h_s - \frac{h_s^2}{r}\right)^{\frac{n-1}{2}} \leqslant 8\,\mathrm{vol}_{n-1}(B_2^{n-1}) R^{\frac{n-1}{2}} \left(2\Delta_s - \frac{\Delta_s^2}{R}\right)^{\frac{n-1}{2}}.$$

Since $h_s \leqslant r$

$$r^{\frac{n-1}{2}} h_s^{\frac{n-1}{2}} \leqslant 8 R^{\frac{n-1}{2}} (2\Delta_s)^{\frac{n-1}{2}}.$$

This implies

$$h_s \leqslant 128 \frac{R}{r} \Delta_s. \tag{48}$$

In Fig. 9 we see the two-dimensional plane that contains the points x_0 and $x_0 - rN_{\partial K}(x_0)$ and that is orthogonal to the $n-2$-dimensional plane $H(x_s, N_{\partial K_s}(x_s)) \cap H(x_s, N_{\partial K}(x_0))$. The point x_s is not necessarily in the plane seen in Fig. 9. Therefore, the angle γ may appear smaller than it is. We denote the orthogonal projection of the point x_s onto the two-dimensional plane seen in Fig. 9 by $x_{s'}$. Thus both points x_s and $x_{s'}$ appear in the same position in Figs. 9 and 10.

Also, please note that in Figs. 9 and 10 there is only shown the case where $x_0 - \Delta_s N_{\partial K}(x_0) \in H^+(x_s, N_{\partial K_s}(x_s))$. The other case, $x_0 - \Delta_s N_{\partial K}(x_0) \in H^-(x_s, N_{\partial K_s}(x_s))$ is treated in the same way.

Now we want to estimate the radius of the largest cap $B_2^n(x_0 - rN_{\partial K}(x_0), r) \cap H^-(x_0 - \Delta_m N_{\partial K}(x_0), N_{\partial K}(x_0))$ that is contained in $H^-(x_s, N_{\partial K_s}(x_s))$. We do this by examining Fig. 10.

We compute the point in Fig. 10 where the line segments $[x_0, z]$ and $[x_{s'}, v]$ intersect.

In Fig. 10 we introduce the (u, w)-coordinate system. The origin in the (u, w)-plane is at $x_0 - \Delta_s N_{\partial K}(x_0)$. In this coordinate system the line through x_0 and z has the equation

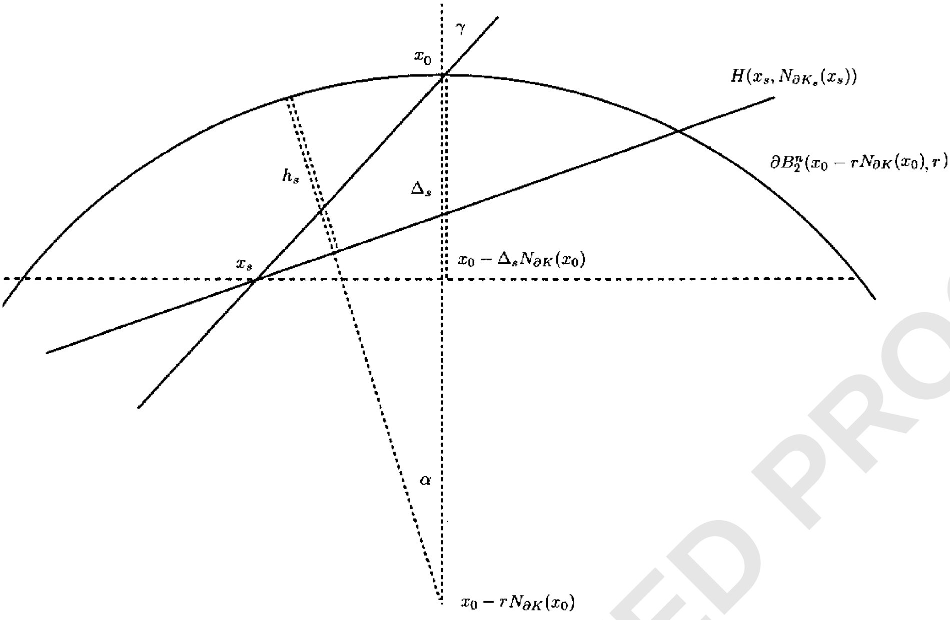

Fig. 9.

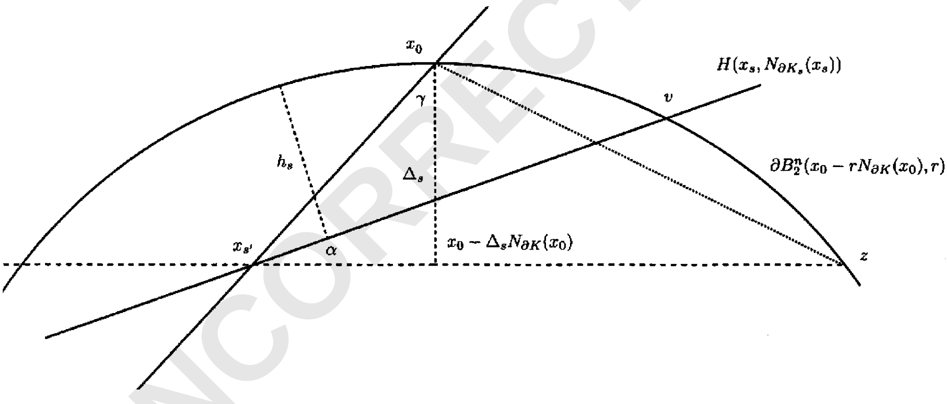

Fig. 10.

$$u = -\frac{\Delta_s}{\sqrt{2r\Delta_s - \Delta_s^2}} w + \Delta_s$$

and the line through $x_{s'}$ and v

$$u = (\tan\alpha)w + \tan\alpha \| x_{s'} - (x_0 - \Delta_s N_{\partial K}(x_0)) \|.$$

Solving for w

$$w = \frac{\Delta_s - \tan\alpha \|x_{s'} - (x_0 - \Delta_s N_{\partial K}(x_0))\|}{\frac{\Delta_s}{\sqrt{2r\Delta_s - \Delta_s^2}} + \tan\alpha}, \tag{49}$$

where α is as in Fig. 10. w is smaller than the radius of the largest cap. We have

$$\Delta_s \tan\gamma \geqslant \|x_{s'} - (x_0 - \Delta_s N_{\partial K}(x_0))\|. \tag{50}$$

Since $x_0 \in H^-(x_s, N_{\partial K_s}(x_s))$ (see Fig. 9)

$$h_s \geqslant r(1 - \cos\alpha).$$

By (48)

$$1 - \cos\alpha \leqslant 128 \frac{R}{r^2} \Delta_s.$$

Therefore, for Δ_0 sufficiently small

$$\alpha^2 \leqslant 528 \frac{R}{r^2} \Delta_s. \tag{51}$$

Together with (49) and (50) we get $w \geqslant C\sqrt{\Delta_s}$ for some constant C. Thus there is a constant C such that for all $\Delta_s \leqslant \Delta_0$

$$\partial K \cap B_2^n(x_0, C\sqrt{\Delta_s}) \subseteq \partial K \cap H^-(x_s, N_{\partial K_s}(x_s)). \tag{52}$$

Now we show the inverse inclusion to (52).

The angle between $N_{\partial K}(x_0)$ and $N_{\partial K_s}(x_s)$ is α. Therefore, the radius of the $n-1$-dimensional Euclidean ball (see Fig. 11)

$$B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H(x_0, N_{\partial K_s}(x_s))$$

equals $R\sin\alpha$ and the height of the associated cap is $R(1 - \cos\alpha)$. By (51) for small Δ_0 this is of the order

$$\frac{1}{2} R\alpha^2 \leqslant 128 \frac{R^2}{r^2} \Delta_s.$$

The height of the cap

$$B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_s, N_{\partial K_s}(x_s)) \tag{53}$$

is less than the height of the cap

$$B_2^n(x_0 - RN_{\partial K}(x_0), R) \cap H^-(x_0, N_{\partial K_s}(x_s))$$

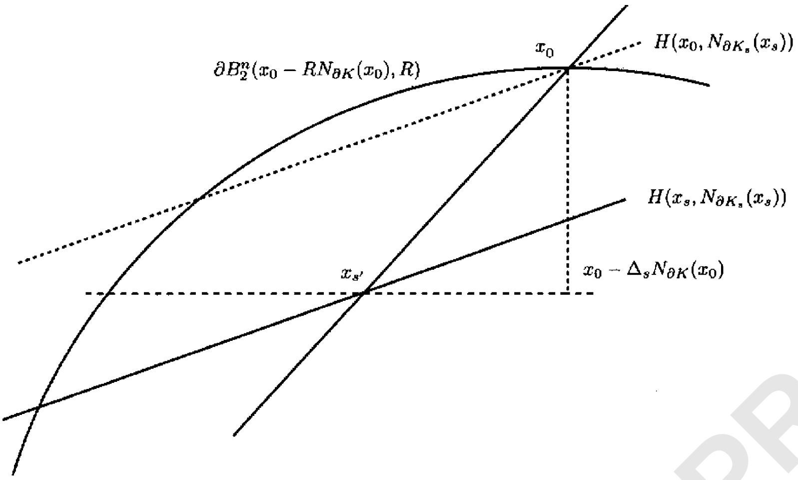

Fig. 11.

plus the distance of x_s to x_0. This in turn is less than $C'\Delta_s$. Therefore the radius of cap (53) is less than $C''\sqrt{\Delta_s}$. $\square$

Lemma 20 (Schütt and Werner [SchW2, Lemma 2.7]). *Let K be a convex body in $\mathbb{R}^n$ such that 0 is an interior point of K. Let $x_0 \in \partial K$. Let $f : \partial K \to \mathbb{R}$ be an integrable, a.e. strictly positive function with $\int_{\partial K} f(x)\, d\mu_{\partial K}(x) = 1$ and such that $f(x_0) > 0$ and*

$$\lim_{\Delta \to 0} \frac{1}{\mathrm{vol}_{n-1}(\partial K \cap H_\Delta^-)} \int_{\partial K \cap H_\Delta^-} |f(x) - f(x_0)|\, d\mu_{\partial K}(x) = 0, \tag{54}$$

where $H_\Delta = H(x - \Delta N_{\partial K}(x), N_{\partial K}(x))$.

Suppose that the indicatrix of Dupin exists at x_0 and is an ellipsoid (and not a cylinder). For all s such that $K_s \neq \emptyset$ and $0 \in K_s$, let x_s be defined by $\{x_s\} = [0, x_0] \cap \partial K_s$. Then for every $\varepsilon > 0$ there is s_ε so that for all s with $0 < s \leqslant s_\varepsilon$ the points x_s are interior points of K and

$$s \leqslant \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K}(x_0))) \leqslant (1 + \varepsilon)s.$$

In [SchW2] this lemma has a stronger assumption. We assume there that the function f is continuous at x_0. It is not difficult to check that the arguments in the proof hold also with assumption (54).

Lemma 21. *Let K be a convex body in $\mathbb{R}^n$. Let $f : \partial K \to \mathbb{R}$ be an integrable function with respect to the surface measure. Then for almost all $x_0 \in \partial K$ where the generalized Gauss curvature exists and is different from 0 the following limit exists and satisfies the equation:*

$$\lim_{\Delta\to 0} \frac{1}{\mathrm{vol}_{n-1}(\partial K \cap H_\Delta^-)} \int_{\partial K \cap H_\Delta^-} |f(x) - f(x_0)| \, d\mu_{\partial K}(x) = 0,$$

where $H_\Delta = H(x_0 - \Delta N_{\partial K}(x_0), N_{\partial K}(x_0))$.

Proof. As in the case of the Euclidean space $\mathbb{R}^n$ it is shown (see e.g. [Fo, pp. 96–98]) that for almost all $x_0 \in \partial K$

$$\lim_{\rho\to 0} \frac{1}{\mathrm{vol}_{n-1}(\partial K \cap B_2^n(x_0,\rho))} \int_{\partial K \cap B_2^n(x_0,\rho)} |f(x) - f(x_0)| \, d\mu_{\partial K}(x) = 0.$$

By Lemma 19 the family $\partial K \cap H_\Delta^-$, $0 < \Delta$, shrinks nicely to x_0 provided that the curvature is not equal to 0. The rest follows from the consideration just above Lemma 19. □

Lemma 22. (i) *Let* $x \in \partial B_2^n$ *and let* H *be a hyperplane with* $x \in H$. *Let* Δ *be the minimal height of a cap* $B_2^n \cap H^-((1-\Delta)x, x)$ *such that*

$$B_2^n \cap H^- \subset B_2^n \cap H^-((1-\Delta)x, x)$$

and assume that $\Delta \leqslant \frac{1}{2}$. *Then*

$$\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-((1-\Delta)x, x)) \leqslant 2^n \, \mathrm{vol}_{n-1}(\partial B_2^n \cap H^-).$$

(ii) *Let* $\mathscr{E}$ *be an ellipsoid in* $\mathbb{R}^n$ *centered at* 0 *with principal axes* $a_1 e_1, \ldots, a_n e_n$ *and let* $H = H(a_n e_n, \xi)$. *Let* Δ *be the minimal height of a cap* $\mathscr{E} \cap H^-((a_n - \Delta)e_n, e_n)$ *such that*

$$\mathscr{E} \cap H^- \subset B_2^n \cap H^-((a_n - \Delta)e_n, e_n)$$

and assume that $\Delta \leqslant \min\{\frac{a_n}{2}, 1\}$. *Then*

$$\mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-((a_n - \Delta)e_n, e_n)) \leqslant 2^{n-1} \left(1 + \frac{8a_n}{\min_{1\leqslant i \leqslant n-1} a_i^2}\right) \mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-).$$

Proof. (i) $\sqrt{\frac{\Delta}{2}}$ is the radius of the cap $B_2^n \cap H^-$. Therefore

$$\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-) \geqslant \left(\frac{\Delta}{2}\right)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}(B_2^{n-1}).$$

Moreover,

$$\mathrm{vol}_{n-1}(\partial B_2^n \cap H^-((1-\Delta)x,x)) \leqslant 2(2\Delta)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}(B_2^{n-1}).$$

From this (i) follows.

(ii) We apply the transform $S:\mathbb{R}^n \to \mathbb{R}^n$

$$S(x) = \left(\frac{x_i}{a_i}\right)_{i=1}^n.$$

Then $S(\mathscr{E}) = B_2^n$. The new Δ is smaller than $\frac{1}{2}$ as required in (i). By Lemma 1.3 of [SchW2] and $\Delta \leqslant 1$

$$\begin{aligned} &\mathrm{vol}_{n-1}(\partial\mathscr{E} \cap H^-((a_n-\Delta)e_n,e_n)) \\ &\quad\leqslant \left(1+\frac{8a_n}{\min_{1\leqslant i\leqslant n-1} a_i^2}\right)^{\frac{1}{2}} \mathrm{vol}_{n-1}(\mathscr{E}\cap H((a_n-\Delta)e_n,e_n)). \end{aligned} \tag{55}$$

Now

$$\begin{aligned} \mathrm{vol}_{n-1}(B_2^n \cap S(H((a_n-\Delta)e_n,e_n))) &= \mathrm{vol}_{n-1}(S(\mathscr{E}\cap H((a_n-\Delta)e_n,e_n))) \\ &= \frac{1}{\prod_{i=1}^{n-1} a_i} \mathrm{vol}_{n-1}(\mathscr{E}\cap H((a_n-\Delta)e_n,e_n)) \end{aligned} \tag{56}$$

and

$$\mathrm{vol}_{n-1}(B_2^n \cap S(H)) = \mathrm{vol}_{n-1}(S(\mathscr{E}\cap H)) = \frac{1}{\prod_{i=1}^n a_i} \frac{1}{||S(\xi)||} \mathrm{vol}_{n-1}(\mathscr{E}\cap H), \tag{57}$$

where ξ is the normal to H. As in the proof of (i)

$$\mathrm{vol}_{n-1}(B_2^n \cap S(H((a_n-\Delta)e_n,e_n))) \leqslant 2^{n-1}\, \mathrm{vol}_{n-1}(B_2^n \cap S(H)).$$

Therefore, using (55)–(57)

$$\begin{aligned} \mathrm{vol}_{n-1}(\partial\mathscr{E}\cap H^-) \geqslant \mathrm{vol}_{n-1}(\mathscr{E}\cap H) &= \prod_{i=1}^n a_i ||S(\xi)|| \mathrm{vol}_{n-1}(B_2^n \cap S(H)). \\ &\geqslant \frac{1}{2^{n-1}} \prod_{i=1}^n a_i ||S(\xi)|| \mathrm{vol}_{n-1}(B_2^n \cap S(H((a_n-\Delta)e_n,e_n))) \\ &= \frac{1}{2^{n-1}} a_n ||S(\xi)|| \mathrm{vol}_{n-1}(\mathscr{E}\cap H((a_n-\Delta)e_n,e_n)) \\ &\geqslant \frac{1}{2^{n-1}} \frac{a_n ||S(\xi)||}{\left(1+\frac{8a_n}{\min_{1\leqslant i\leqslant n-1} a_i^2}\right)^{\frac{1}{2}}} \mathrm{vol}_{n-1}(\partial\mathscr{E}\cap H^-((a_n-\Delta)e_n,e_n)). \end{aligned}$$

Now note that

$$\begin{aligned}||S(\xi)|| &= \left(\sum_{i=1}^{n}\left(\frac{\xi_i}{a_i}\right)^2\right)^{\frac{1}{2}} \geqslant \frac{\xi_n}{a_n} = \frac{1}{a_n}\langle \xi, e_n\rangle \\ &\geqslant \frac{1}{a_n}\min_{x\in\partial\mathscr{E}\cap H((a_n-\Delta)e_n,e_n)}\langle N_{\partial\mathscr{E}}, e_n\rangle \\ &\geqslant \frac{1}{a_n}\left(1+\frac{8a_n}{\min_{1\leqslant i\leqslant n-1}a_i^2}\right)^{-\frac{1}{2}}.\end{aligned}$$

For the last inequality see the proof of Lemma 1.3 of [SchW2]. We use also that $\Delta \leqslant 1$. □

Lemma 23. *Let K be a convex body in $\mathbb{R}^n$ such that 0 is an interior point of K and let $f:\partial K\to\mathbb{R}$ be an integrable function with $\int_{\partial K} f(x)\,d\mu_{\partial K}(x)=1$ and such that $f\geqslant 0$ a.e.*

(i) *For almost all $x\in\partial K$ at which the indicatrix of Dupin is an ellipsoid*

$$\lim_{s\to 0}\frac{\langle x, N_{\partial K}(x)\rangle\left(1-\left(\frac{||x_s||}{||x||}\right)^n\right)}{ns^{\frac{2}{n-1}}} = \frac{\kappa(x)^{\frac{1}{n-1}}}{2(\mathrm{vol}_{n-1}(B_2^{n-1})f(x))^{\frac{2}{n-1}}}.$$

(ii) *For almost all $x\in\partial K$ at which the indicatrix of Dupin is an elliptic cylinder*

$$\lim_{s\to 0}\frac{\langle x, N_{\partial K}(x)\rangle\left(1-\left(\frac{||x_s||}{||x||}\right)^n\right)}{ns^{\frac{2}{n-1}}} = 0.$$

Proof. Let $x_0\in\partial K$. Since f is a.e. strictly greater than 0 we may assume that $f(x_0)>0$. (16) holds for all s with $0<s\leqslant T$, that is

$$\frac{1}{n}\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right) \leqslant \left\langle \frac{x_0}{||x_0||}, N_{\partial K}(x_0)\right\rangle ||x_0 - x_s||.$$

In the same way we obtain the inverse inequality.

$$\begin{aligned}&\frac{1}{n}\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right) \\ &\quad = \frac{1}{n}\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(1-\frac{||x_0-x_s||}{||x_0||}\right)^n\right).\end{aligned}$$

Since $(1-t)^n \leqslant 1-nt+\frac{n(n-1)}{2}t^2$ for all t with $0\leqslant t\leqslant 1$

$$\begin{aligned}&\frac{1}{n}\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right)\\&\quad\geqslant\left\langle\frac{x_0}{||x_0||}, N_{\partial K}(x_0)\right\rangle||x_0-x_s||\left(1-\frac{n-1}{2}\frac{||x_0-x_s||}{||x_0||}\right).\end{aligned}\tag{58}$$

(i) We now assume that the indicatrix of Dupin at x_0 is an ellipsoid. By Lemma 3(ii) x_s is then an interior point of K. By (16) and (58) we can choose s_ε so small that we have for all $s\leqslant s_\varepsilon$

$$1-\varepsilon\leqslant\left|\frac{\frac{1}{n}\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right)}{\langle\frac{x_0}{||x_0||}, N_{\partial K}(x_0)\rangle||x_0-x_s||}\right|\leqslant 1.$$

By this and Lemma 20 we can choose s_ε so small that we have for all $s\leqslant s_\varepsilon$

$$1-\varepsilon\leqslant\left|\frac{\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right)}{ns^{\frac{2}{n-1}}}\frac{\mathbb{P}_f(\partial K\cap H^-(x_s, N_{\partial K}(x_0)))^{\frac{2}{n-1}}}{\left\langle\frac{x_0}{||x_0||}, N_{\partial K}(x_0)\right\rangle||x_0-x_s||}\right|\leqslant 1+\varepsilon.$$

The assumptions of Lemma 20 are satisfied because of Lemma 21. From this and Lemma 21 we conclude that we can choose s_ε so small that we have for all $s\leqslant s_\varepsilon$

$$\begin{aligned}&1-2\varepsilon\\&\leqslant\left|\frac{\langle x_0, N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x_0||}\right)^n\right)}{ns^{\frac{2}{n-1}}}\frac{(f(x_0)\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s, N_{\partial K}(x_0))))^{\frac{2}{n-1}}}{\left\langle\frac{x_0}{||x_0||}, N_{\partial K}(x_0)\right\rangle||x_0-x_s||}\right|\\&\leqslant 1+2\varepsilon.\end{aligned}$$

Let Δ_s denote the height of the cap $\partial K\cap H^-(x_s, N_{\partial K}(x_0)))$, i.e.

$$\Delta_s=\left\langle\frac{x_0}{||x_0||}, N_{\partial K}(x_0)\right\rangle||x_0-x_s||.$$

For the surface area of the cap we have (see [SchW2]) for $s\leqslant s_\varepsilon$

$$\begin{aligned}&(1-\varepsilon)\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\sqrt{\kappa(x_0)}}(2\Delta_s)^{\frac{n-1}{2}}\\&\leqslant\mathrm{vol}_{n-1}(\partial K\cap H^-(x_s, N_{\partial K}(x_0)))\\&\leqslant(1+\varepsilon)\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{\sqrt{\kappa(x_0)}}(2\Delta_s)^{\frac{n-1}{2}}.\end{aligned}$$

Therefore we get

$$1 - 3\varepsilon \leqslant \left| \frac{\langle x_0, N_{\partial K}(x_0)\rangle \left(1 - \left(\frac{\|x_s\|}{\|x_0\|}\right)^n\right)}{ns^{\frac{2}{n-1}}} \frac{2(f(x_0)\mathrm{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}{\kappa(x_0)^{\frac{1}{n-1}}} \right| \leqslant 1 + 3\varepsilon.$$

From this it follows that

$$\lim_{s\to 0} \frac{\langle x_0, N_{\partial K}(x_0)\rangle \left(1 - \left(\frac{\|x_s\|}{\|x_0\|}\right)^n\right)}{ns^{\frac{2}{n-1}}} = \frac{\kappa(x_0)^{\frac{1}{n-1}}}{2(f(x_0)\mathrm{vol}_{n-1}(B_2^{n-1}))^{\frac{2}{n-1}}}.$$

This finishes the proof of Lemma 23(i).

(ii) Recall that, since f is a.e. strictly greater than 0 we may assume that $f(x_0) > 0$. We first consider the case that there is $s_0 > 0$ such that $x_{s_0} \in \partial K$. Then for all s with $0 \leqslant s \leqslant s_0$ we have $x_s \in \partial K$. Hence, by construction of x_s, $x_s = x_0$ and therefore

$$\frac{\langle x_0, N_{\partial K}(x_0)\rangle \left(1 - (\frac{\|x_s\|}{\|x_0\|})^n\right)}{ns^{\frac{2}{n-1}}} = 0.$$

Now we treat the case that for all $s > 0$ the point x_s is an interior point of K. The indicatrix of Dupin at x_0 is an elliptic cylinder and we may assume that the first k axes have infinite lengths and the others not. Then, for every $\varepsilon > 0$, there is an ellipsoid $\mathscr{E}$ and $s_\varepsilon > 0$ such that for all $s \leqslant s_\varepsilon$ we have that

$$\mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)) \subset K \cap H^-(x_s, N_{\partial K}(x_0)) \tag{59}$$

and such that the lengths of the first k principal axes $a_1, \dots, a_k$ are larger than $\frac{1}{\varepsilon}$ (see [SchW1]). As x_s is an interior point of K, by Lemma 3(i) there exists a hyperplane $H(x_s, N_{\partial K_s}(x_s))$ such that

$$s = \mathbb{P}_f(\partial K \cap H^-(x_s, N_{\partial K_s}(x_s))).$$

If the normal $N_{\partial K_s}(x_s)$ is not unique, one of the normals satisfies the equation.

We consider the metric projection

$$p : \partial\mathscr{E} \to H(x_0, N_{\partial K}(x_0)),$$

which in this case is equal to the orthogonal projection. We also consider

$$q : \partial\mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)) \to \partial K$$

with $q(x) = [x, p(x)] \cap \partial K$. The family $q(\partial\mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)))$, $0 < s \leqslant s_\varepsilon$, shrinks nicely to x_0 as $s \to 0$. This is proved in the same way as Lemma 19(i). Therefore we get

$$\lim_{s\to 0} \frac{1}{\mathrm{vol}_{n-1}(q(\partial\mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0))))} \int_{q(\partial\mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)))} |f(x) - f(x_0)|\, d\mu_{\partial K}(x) = 0.$$

This implies that for all $\delta > 0$ there is s_ε such that for all $0 < s \leqslant s_\varepsilon$,

$$\mu_{\partial K}\left(\left\{x \in q(\partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0))) \colon f(x) \geqslant \tfrac{1}{2} f(x_0)\right\}\right)$$
$$\geqslant (1-\delta)\mu_{\partial K}(\{x \in q(\partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)))\}). \tag{60}$$

We choose

$$\frac{1}{10}\left(2^{n+1}\left(1+\frac{8}{\min_{1 \leqslant i \leqslant n-1} b_i^2}\right)\right)^{-1},$$

where b_i, $1 \leqslant i \leqslant n-1$, are the lengths of the principal axes of the indicatrix of Dupin. The lengths a_i, $1 \leqslant i \leqslant n$, of the axes of the ellipsoid $\mathscr{E}$ and the lengths b_i are related in the following way (see [SchW2, p. 258])

$$a_n = \left(\prod_{i=1}^{n-1} b_i\right)^{\frac{2}{n-1}}$$

and

$$a_j = b_j \left(\prod_{i=1}^{n-1} b_i\right)^{\frac{1}{n-1}}, \quad j = 1, \ldots, n-1. \tag{61}$$

By Lemma 22(ii) for all hyperplanes H with $x_0 \in H$, $\partial \mathscr{E} \cap H^- \subset \partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0))$

$$\mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0))) \leqslant 2^{n-1}\left(1+\frac{8a_n}{\min_{1 \leqslant i \leqslant n-1} a_i^2}\right)\mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-).$$

Since we can choose H such that $x_0 \in H$ and

$$\partial \mathscr{E} \cap H^- \subset \partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)) \cap H^-(x_s, N_{\partial K_s}(x_s)),$$

we get for sufficiently small s_ε

$$\mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-)$$
$$\leqslant \mathrm{vol}_{n-1}(\partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)) \cap H^-(x_s, N_{\partial K_s}(x_s)))$$
$$\leqslant 2\mathrm{vol}_{n-1}(q(\partial \mathscr{E} \cap H^-(x_s, N_{\partial K}(x_0)) \cap H^-(x_s, N_{\partial K_s}(x_s)))).$$

Therefore, using (61),

$$\begin{aligned}
&\mathrm{vol}_{n-1}(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))))\\
&\quad\leqslant 2\,\mathrm{vol}_{n-1}(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0)))\\
&\quad\leqslant 2^{n+1}\left(1+\frac{8}{\min_{1\leqslant i\leqslant n-1} b_i^2}\right)\mathrm{vol}_{n-1}(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0)))\cap H^-\\
&\quad\times(x_s,N_{\partial K_s}(x_s)))).
\end{aligned}\tag{62}$$

Now

$$\begin{aligned}
&\{x\in q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s)))|f(x)\geqslant\tfrac{1}{2}f(x_0)\}\\
&\quad=\{x\in q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))|f(x)\geqslant\tfrac{1}{2}f(x_0)\}\\
&\qquad\cap q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s))).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
&\mathrm{vol}_{n-1}\left(\{x\in q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s))):\ f(x)\geqslant\tfrac{1}{2}f(x_0)\}\right)\\
&\quad\geqslant\mathrm{vol}_{n-1}\left(\{x\in q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0)):\ f(x)\geqslant\tfrac{1}{2}f(x_0)\}\right)\\
&\qquad+\ \mathrm{vol}_{n-1}(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s))))\\
&\qquad-\ \mathrm{vol}_{n-1}\Big(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0)))\Big)\\
&\quad\geqslant\frac{9}{10}\frac{1}{2^{n+1}(1+\frac{8}{\min_{1\leqslant i\leqslant n-1} b_i^2})}\mathrm{vol}_{n-1}\Big(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0)))\Big).
\end{aligned}$$

For the last inequality we have used (60). Hence

$$\begin{aligned}
s&=\int_{\partial K\cap H^-(x_s,N_{\partial K_s}(x_s))} f(x)\,d\mu_{\partial K}(x)\\
&\geqslant\int_{q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s)))} f(x)\,d\mu_{\partial K}(x)\\
&\geqslant\int_{\{x\in q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))\cap H^-(x_s,N_{\partial K_s}(x_s))):\ f(x)\geqslant\frac{1}{2}f(x_0)\}} f(x)\,d\mu_{\partial K}(x)\\
&\geqslant\frac{1}{2}f(x_0)\frac{9}{10}\frac{1}{2^{n+1}(1+\frac{8}{\min_{1\leqslant i\leqslant n-1}b_i^2})}\mathrm{vol}_{n-1}(q(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))))\\
&\geqslant\frac{9}{40}f(x_0)\,\frac{1}{2^{n+1}(1+\frac{8}{\min_{1\leqslant i\leqslant n-1}b_i^2})}\mathrm{vol}_{n-1}(\partial\mathscr{E}\cap H^-(x_s,N_{\partial K}(x_0))).
\end{aligned}$$

By Lemma 1.3 of [SchW2] this last expression is bigger or equal than

$$\frac{9}{40}\frac{f(x_0)}{2^{n+1}\left(1+\frac{8}{\min_{1\leqslant i\leqslant n-1}b_i^2}\right)}\mathrm{vol}_{n-1}(B_2^{n-1})\prod_{i=1}^{n-1}a_i\left(\frac{2\Delta_s}{a_n}\right)^{\frac{n-1}{2}}\left(1-\frac{\Delta_s}{2a_n}\right)^{\frac{n-1}{2}},$$

where $\Delta_s=||x_0-x_s||\left\langle\frac{x_0}{||x_0||},N_{\partial K}(x_0)\right\rangle$. Hence we get, using (16)

$$\begin{aligned}\frac{\langle x_0,N_{\partial K}(x_0)\rangle\left(1-\left(\frac{||x_s||}{||x||}\right)^n\right)}{ns^{\frac{2}{n-1}}}&\leqslant\frac{\Delta_s}{s^{\frac{2}{n-1}}}\\ &\leqslant 4\frac{a_n^2\left(\frac{160}{9}\right)^{\frac{2}{n-1}}\left(1+\frac{8}{\min_{1\leqslant i\leqslant n-1}b_i^2}\right)^{\frac{2}{n-1}\left(\mathrm{vol}_{n-1}(B_2^{n-1})\right)^{-\frac{2}{n-1}}}}{f(x_0)^{\frac{2}{n-1}}(2a_n-\Delta_s)(\prod_{i=1}^{n-1}a_i)^{\frac{2}{n-1}}}\\ &\leqslant 4\frac{a_n^2\left(\frac{160}{9}\right)^{\frac{2}{n-1}}\left(1+\frac{8}{\min_{1\leqslant i\leqslant n-1}b_i^2}\right)^{\frac{2}{n-1}}(\mathrm{vol}_{n-1}(B_2^{n-1}))^{-\frac{2}{n-1}}}{f(x_0)^{\frac{2}{n-1}}(2a_n-\Delta_s)(\prod_{i=k+1}^{n-1}a_i)^{\frac{2}{n-1}}}\varepsilon^{\frac{2k}{n-1}},\end{aligned}$$

where for the last inequality we have used that the lengths of the first k principal axes $a_1,\ldots,a_k$ are larger than $\frac{1}{\varepsilon}$. This finishes the proof of Lemma 23(ii). □

Proof of Theorem 14. We may assume that $0\in\mathring{K}$. By Lemma 16

$$\frac{\mathrm{vol}_n(K)-\mathrm{vol}_n(K_{f,s})}{s^{\frac{2}{n-1}}}=\frac{1}{n}\int_{\partial K}\frac{\langle x,N_{\partial K}(x)\rangle(1-\left(\frac{||x_s||}{||x||}\right)^n)}{s^{\frac{2}{n-1}}}d\mu_{\partial K}(x).$$

By Lemma 23 the functions under the integral are converging pointwise a.e. to

$$\frac{\kappa(x)^{\frac{1}{n-1}}}{2(\mathrm{vol}_{n-1}(B_2^{n-1})f(x))^{\frac{2}{n-1}}}.$$

By Lemma 17 the functions under the integral sign are bounded uniformly in s by the function

$$\frac{C}{(M_f(x))^{\frac{2}{n-1}}r(x)}.$$

One of the assumptions of the theorem is that this function has a finite integral. We apply Lebesgue's convergence theorem. □

References

[Ba1] I. Bárány, Random polytopes in smooth convex bodies, Mathematika 39 (1992) 81–92.

[Ba2] I. Bárány, Affine perimeter and limit shape, J. Reine Angew. Math. (1997) 71–84.

[B] W. Blaschke, Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie, Springer, Berlin, 1923.

[Fi] G.M. Fichtenholz, Differential- und Integralrechnung III, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1992.

[Fo] G.B. Folland, Real Analysis, Wiley, New York, 1999.

[GR] I.S. Gradsteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1965.

[Gr1] P.M. Gruber, Approximation of Convex Bodies, Convexity and its Applications, Birkhäuser, Basel, 1983, pp. 131–162.

[Gr2] P. Gruber, Aspects of Approximation of Convex Bodies, Handbook of Convex Geometry, Vol. A, North-Holland, Amsterdam, 1993, pp. 321–345.

[Gr3] P.M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies II, Forum Math. 5 (1993) 521–538.

[H] D. Hug, Contributions to affine surface area, Manuscripta Math. 91 (3) (1996) 283–301.

[L1] K. Leichtweiss, Über ein Formel Blaschkes zur Affinoberfläche, Studia Scient. Math. Hung. 21 (1986) 453–474.

[L2] K. Leichtweiss, Zur Affinoberfläche konvexer Körper, Manuscripta Math. 56 (1986) 429–464.

[L3] K. Leichtweiss, Affine geometry of convex bodies, Johann Ambrosius Barth Verlag, Heidelberg, 1998.

[LudR] M. Ludwig, M. Reitzner, A characterization of affine surface area, Adv. in Math. 147 (1) (1999) 138–172.

[Lu1] E. Lutwak, Extended affine surface area, Adv. in Math. 85 (1991) 39–68.

[Lu2] E. Lutwak, The Brunn–Minkowski–Firey Theory II: affine and geominimal surface areas, Adv. in Math. 118 (1996) 244–294.

[LuO] E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem, J. Differential Geom. 41 (1995) 227–246.

[MW1] M. Meyer, E. Werner, The Santaló-regions of a convex body, Trans. AMS 350 (11) (1998) 4569–4591.

[MW2] M. Meyer, E. Werner, On the p-affine surface area, Adv. in Math. 152 (2000) 288–313.

[Ro] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.

[Schm] M. Schmuckenschläger, The distribution function of the convolution square of a convex symmetric body in $\mathbf{R}^n$, Israel J. Math. 78 (1992) 309–334.

[Sch1] C. Schütt, Random polytopes and affine surface area, Math. Nachr. 170 (1994) 227–249.

[Sch2] C. Schütt, Floating body, illumination body, and polytopal approximation, Convex Geometric Analysis (Berkeley, CA, 1996), 203–229, Mathematical Science Research Institute Publication, Vol. 34, Cambridge University Press, Cambridge, 1999.

[SchW1] C. Schütt, E. Werner, The convex floating body, Math. Scand. 66 (1990) 275–290.

[SchW2] C. Schütt, E. Werner, Polytopes with vertices chosen randomly from the boundary of a convex body, GAFA 2001–2002, Lecture Notes in Mathematics, Vol. 1807, Springer, Berlin, pp. 241–422.

[W1] E. Werner, Illumination bodies and affine surface area, Studia Math. 110 (1994) 257–269.

[W2] E. Werner, A general geometric construction of affine surface area, Studia Math. 132 (3) (1999) 227–238.

[W3] E. Werner, The p-affine surface area and geometric interpretations, Rend. Circ. Math. Palermo, to appear.