The p-affine surface area and geometric interpretations*

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Abstract

We investigate properties of the p-affine surface area for $-\infty \leq p \leq \infty$ and give a geometric interpretation of the p-affine surface area in terms of the weighted floating bodies.

1 Introduction

The affine surface area was originally introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficiently smooth boundary. Its definition involves the Gauss curvature of the boundary points of a convex body. Hence it provides a tool to "measure" the boundary structure of a convex body. Therefore it is not surprising that the affine surface area occurs naturally in problems related to the boundary of a convex body, so for instance in the approximation of convex bodies by polytops. For more information about this subject and the role the affine surface area plays there, we refer to the works by Bárány, [Ba1 , Ba2], Gruber [Gr1, Gr2, Gr3], Schütt [Sch1, Sch2] and Schütt and Werner [SchW2].

Extensions of the affine surface area to higher dimensions and arbitrary convex bodies were only found much later than Blaschke's times by Leichtweiss [L1, L2], Lutwak [Lu1], Schütt and Werner [SchW1], Schmuckenschläger [Schm], Meyer and Werner [MW1] and Werner [W1]. Additional references to the affine surface area as well as further applications can also be found in those papers as well as in Leichtweiss [L3], Ludwig and Reitzner [LudR], Lutwak and Oliker [Lu-O] and [W2].

Here we want to concentrate on the p-affine surface area which, for $p > 0$, was introduced in 1996 by Lutwak [Lu2]. For $p = 1$, the p-affine surface area is just

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the affine surface area. Hug [H] gave new definitions of the p-affine surface area. He also proved that this new definitions give the same p-affine surface area as that defined by Lutwak.

Meyer and Werner [MW2] found a geometric interpretation of the p-affine surface area in terms of the Santaló bodies. They also observed that the definition of Lutwak for the p-affine surface area makes sense for $-n < p \leq 0$ and their geometric interpretation in terms of the Santaló bodies also holds for this range of p. They also gave a definition of the p-affine surface area for $p = -n$ together with its geometric interpretation.

In [SchW2] and [SchW3] it was suggested to extend the p-range even further, namely to $-\infty \leq p \leq \infty$. This extension was motivated in [SchW2] by the fact that there is a geometric interpretation of the p-affine surface area in terms of random polytopes and this interpretation holds for $-\infty \leq p \leq \infty$. In [SchW3] a geometric interpretation of the p-affine surface area for all p is given using the surface bodies.

In this paper we give a new characterization of the p-affine surface area using weighted floating bodies. The paper is organized as follows:

In Section 2 we introduce the p-affine surface area for $-\infty \leq p \leq \infty$ and discuss some of the properties of the p-affine surface area.

In Section 3 we introduce the weighted floating bodies and give a geometric interpretation of the p-affine surface area in terms of the weighted floating bodies.

Throughout the paper we shall use the following notations.
$B_2^n(a,r)$ is the n-dimensional Euclidean ball with radius r centered at a. We put $B_2^n = B_2^n(0,1)$. By $||.||$ we denote the standard Euclidean norm on $\mathbb{R}^n$, by $<\cdot,\cdot>$ the standard inner product on $\mathbb{R}^n$. For two points x and y in $\mathbb{R}^n$ $[x,y] = \{\alpha x + (1-\alpha)y : 0 \leq \alpha \leq 1\}$ denotes the line segment from x to y and $[x,A]$ is the convex hull of x and a subset $A \subset \mathbb{R}^n$.
For a convex body $K \in \mathbb{R}^n$, $\text{int}(K)$ is the interior of K and ∂K is the boundary of K. We also write S^{n-1} for ∂B_2^n. For $x \in \partial K$, $N_{\partial K}(x)$ is the outer unit normal vector to ∂K in x. It may not be unique.

For $u \in S^{n-1}$, $h_K(u) = \max\{< x, u >: x \in K\}$ is the support functional of K at u and $f_\kappa(u)$ is the Gauss curvature function, that is the reciprocal of the Gauss curvature $\kappa(x)$ at this point $x \in \partial K$ that has u as outer normal. m denotes the Lebesgue measure on $\mathbb{R}^n$, $\mu_{\partial K}$ is the usual surface measure on the boundary ∂K of K and σ is the spherical Lebesgue measure.

$H(x,\xi)$ is the hyperplane containing the point x and orthogonal to ξ. $H^-(x,\xi)$ is the closed halfspace containing the point $x+\xi$, $H^+(x,\xi)$ the other halfspace.

2 Definitions and Properties

Throughout the remainder of the paper we will always assume that the convex body K under consideration has the origin in its interior and if K is a symmetric convex body we will assume that the origin is the center of symmetry.

Definition 1 *Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. Let $-\infty \leq p \leq \infty$, $p \neq -n$. We define the p- affine surface area $O_p(K)$ by*

$$O_{\pm\infty} = \int_{\partial K} \frac{\kappa(x)}{< x, N_{\partial K}(x) >^n} d\mu_{\partial K}(x) \tag{1}$$

and

$$O_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N_{\partial K}(x) >^{\frac{n(p-1)}{n+p}}} d\mu_{\partial K}(x), \tag{2}$$

provided the above integrals exist.

The definitions for $O_{\pm\infty}$ are the same as in [SchW2, SchW3].
There is another definition for O_∞ by [Lu2] which differs from this one by a factor of $\text{vol}_n(K)^{\frac{n}{n+1}} \text{vol}_n(K^*)^{-\frac{n}{n+1}}$. K^* is the polar body of K.

Properties of $\mathbf{O_p}$

We now summarize some properties of O_p along with comments about the history and brief remarks about the proofs.

a) We have

$$O_0(K) = \int_{\partial K} < x, N_{\partial K}(x) > d\mu_{\partial K}(x) = n\text{vol}_n(K). \tag{3}$$

Note that if ∂K is C^2 with a.e. strictly positive Gaussian curvature, then

$$\int_{\partial K} \frac{\kappa(x)}{< x, N_{\partial K}(x) >^n} d\mu_{\partial K}(x) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u)$$

and

$$\int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{< x, N_{\partial K}(x) >^{\frac{n(p-1)}{n+p}}} d\mu_{\partial K}(x) = \int_{S^{n-1}} \frac{f_\kappa(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u).$$

Thus for K with ∂K C^2 and a.e. strictly positive Gaussian curvature

$$O_{\pm\infty} = \int_{\partial K} \frac{\kappa(x)}{< x, N_{\partial K}(x) >^n} d\mu_{\partial K}(x) = \int_{S^{n-1}} \frac{1}{h_K(u)^n} d\sigma(u) = n\text{vol}_n(K^*).$$

$$O_p(B_2^n) = \text{vol}_{n-1}(\partial B_2^n) \text{ for all } p \neq -n. \tag{4}$$

b) It is easy to see (see [SchW3]) that O_p is finite for all p with $0 \leq p \leq \infty$. This need not to be so for negative values of p. That was also shown in [SchW3]. More precisely, the following item c) was shown in [SchW3] :

c) Let $1 < r < \infty$. Let $B_r^n = \{x \in \mathbb{R}^n | \sum_{i=1}^n |x_i|^r \leq 1\}$. Then, for all $p \geq 0$ and all r

$$O_p(B_r^n) = \frac{2^n (r-1)^{\frac{p(n-1)}{n+p}}}{r^{n-1}} \frac{\left(\Gamma(\frac{n+rp-p}{r(n+p)})\right)^n}{\Gamma(\frac{n(n+rp-p)}{r(n+p)})}.$$

Moreover, for all p with $p < -n$ and r with $1 < r < 1 + \frac{n}{|p|}$ we have

$$O_p(B_r^n) = \infty.$$

d) The p-affine surface area is linearily invariant, i.e. for all linear maps T with $\det(T) = 1$ we have

$$O_p(K) = O_p(T(K)).$$

This had been shown by Ltuwak [Lu2] and later by another method by Hug [H] for p with $0 < p \leq \infty$. The linear invarinace for $-n < p \leq \infty$ follows from the results in [MW2]. The proof of [H] carries over to negative p. This was shown in [SchW3]. More precisely, the following was shown in [SchW3]:

Let K be a convex body in $\mathbb{R}^n$ such that $0 \in \text{int}(K)$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear, invertible map. Then for all $-\infty \leq p \leq \infty$, $p \neq -n$

$$O_p(T(K)) = \det(T)^{\frac{n-p}{n+p}} O_p(K)$$

e) Let P be a polytope in $\mathbb{R}^n$. Then $O_p(P) = 0$ for $p > 0$. This was proved by Lutwak [Lu2].

For $p < -n$, the exponent $\frac{p}{n+p}$ of the curvature $\kappa^{\frac{p}{n+p}}$ in (2) is positive. As κ is equal to 0 μ-a.e. on ∂P for a polytope P, we also have that $O_p(P) = 0$ in this case.

For $-n < p < 0$, the exponent $\frac{p}{n+p}$ of the curvature $\kappa^{\frac{p}{n+p}}$ in (2) is negative. Again, as κ is equal to 0 μ-a.e. on ∂P for a polytope P, we have that $O_p(P) = \infty$ in this case.

f) The p-affine surface area satisfies the p-affine isoperimetric inequality

$$O_p(K)^{n+p} \leq n^{n+p} \mathrm{vol}_n(K)^{n-p} \mathrm{vol}_n(B_2^n)^{2p} \tag{5}$$

with equality if and only if K is an ellipsoid.

This has also been shown in [Lu2] and later by another method in [H] for $0 < p \leq \infty$. For $p = 0$ we trivially have equality by a).

The p-affine isoperimetric inequality does not hold however in general for $p < 0$. Indeed, it follows from c) that for $r > 2$ there are values p, $-n < p < \frac{-n}{r-1}$ such that $O_p(B_r^n) = \infty$ and hence the inequality cannot hold.

It follows from e) that for $p < -n$ the p-affine isoperimetric inequality does not necessarily hold either. For such p we have for any polytope P that $O_p(P) = 0$ and thus $O_p(P)^{n+p} = \infty$.

g) The p-affine surface area is upper semicontinuous for $p \geq 0$. This was also proved in [Lu2].

Again, for $p < 0$ this need not be the case. Let $-n < p < 0$. Then $O_p(B_2^n) = \mathrm{vol}_{n-1}(\partial B_2^n)$. B_2^n can be approximated by polytopes P but by e) we have for all polytopes in this case that $O_p(P) = \infty$.

3 Geometric Interpretations

As already mentioned in the introduction, geometric interpretations of the p-affine surface area exist in terms of the Santaló bodies for $-n < p \leq \infty$ [MW2], in terms of random polytopes for $-\infty \leq p \leq \infty$, $p \neq -n$, [SchW2] and in terms of the surface bodies for $-\infty \leq p \leq \infty$, $p \neq -n$, [SchW3].

We define now the weighted floating body. Recall that m denotes the Lebesgue measure on $\mathbf{R}^n$.

Definition 2 *Let K be a convex body in $\mathbf{R}^n$.*
Let $0 \leq s$ and let $f : K \to \mathbb{R}$ be an integrable function such that $f > 0$ m-a.e.

The weighted floating body $F(K, f, s)$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $(f\ m)$-measure less than or equal to s from K. More precisely,

$$F(K, f, s) = \bigcap_{\int_{K \cap H^-} f\ dm \leq s} H^+ \tag{6}$$

Proposition 3 *Let K be a convex body in $\mathbf{R}^n$ such that $0 \in int(K)$ and let $f : K \to \mathbb{R}$ be an integrable function such that $f > 0$ m-a.e. Let T be a linear transformation such that $\det T \neq 0$. Then we have*

$$F(TK, f \circ T^{-1}, s) = T\left(F(K, f, \frac{s}{det\ T})\right).$$

Proof

Let $H(x_0, \xi)$ be a hyperplane through x_0 with normal ξ. Then $T(H(x_0, \xi)) = H(y_0, \eta)$ is a hyperplane through $y_0 = Tx_0$ with normal $\eta = T^{-1*}\xi$. Thus

$$\int_{K \cap H^-(x_0,\xi)} f\ dm = \int_{T^{-1}\left(T(K \cap H^-(x_0,\xi))\right)} f\ dm =$$

$$\int_{T^{-1}\left(TK \cap H^-(y_0,\eta)\right)} f\ dm = \frac{1}{\det T} \int_{TK \cap H^-(y_0,\eta)} f \circ T^{-1}\ dm.$$

Hence

$$F(TK, f \circ T^{-1}, s) = \bigcap_{\int_{TK \cap H^-(y_0,\eta)} f \circ T^{-1}\ dm \leq s} H^+(y_0, \eta) =$$

$$\bigcap_{\int_{K \cap H^-(x_0,\xi)} f\ dm \leq \frac{s}{\det\ T}} T(H^+(x_0, \xi)) = T\left(F(K, f, \frac{s}{\det T})\right)$$

Lemma 4 *Let K be a convex body in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be a m-a.e. strictly positive, continuous function.*

(i) For all s such that $F(K, f, s) \neq \emptyset$ and all $x_s \in \partial F(K, f, s) \cap int(K)$ there exists a supporting hyperplane H to $\partial F(K, f, s)$ through x_s such that $\int_{K \cap H^-} f\ dm = s$.

(ii) $F(K, f, 0) = K$

(iii) There is an s_0 so that for all s with $s \leq s_0$ the body $F(K, f, s)$ is nonempty.

Proof

(i) By definition of $F(K, f, s)$ we have for every support hyperplane H to $F(K, f, s)$ through x_s that

$$s \leq \int_{K \cap H^-} f\ dm.$$

Suppose not. Then there exists a support hyperplane H_0 to $F(K, f, s)$ through x_s such that $s > \int_{K \cap H_0^-} f\ dm$. We choose $\tilde{H}_0$ parallel to H_0 such that $s =$

$\int_{K\cap \tilde{H}_0^-} f\ dm$ which we can do as f is continuous. Consequently, $x_s \notin F(K,f,s)$, a contradiction.

On the other hand, there is a sequence of hyperplanes H_i with $F(K,f,s) \subseteq H_i^+$ and $\int_{K\cap H_i^-} f dm \leq s$ such that the distance between x_s and H_i is less than $\frac{1}{i}$. We check this.

Since $x_s \in \partial F(K,f,s)$ there is $z \notin F(K,f,s)$ with $\|x_s - z\| < \frac{1}{i}$. There is a hyperplane H_i separating z from $F(K,f,s)$ satisfying

$$\int_{K\cap H_i^-} f dm \leq s \qquad \text{and} \qquad F(K,f,s) \subseteq H_i^+.$$

We have for the distance d between x_s and H_i

$$d(x_s, H_i) \leq \|x_s - z\| < \frac{1}{i}.$$

By compactness and as for $x_0 \in \partial K$, $\xi \in S^{n-1}$, $t \in \mathbb{R}^+$, $\int_{K\cap H^-(x_0 - t\xi,\xi)} f dm$ is a continuos function in t (where it is defined), there is a subsequence that converges to a hyperplane H with $x_s \in H$ and $\int_{K\cap H^-} f dm \leq s$.

(ii) We can assume w.l.o.g. that $0 \in \text{int}(K)$. We have for all $s \geq 0$ that $F(K,f,s) \subset K$, hence $F(K,f,0) \subset K$. Suppose that $F(K,f,0)$ is a proper subset of K. Then there exists $x \in \partial K$ and $x \notin F(K,f,0)$. Let $x_0 = [0,x] \cap F(K,f,0)$. Then $x_0 \in \text{int}(K)$ and hence by (i) there exists a support hyperplane H to $F(K,f,0)$ through x_0 such that $\int_{K\cap H^-} f dm = 0$. However, as $x_0 \in \text{int}(K)$, we have that $[x_0, x) \subset \text{int}(K \cap H^-)$. Let $x_1 \in (x_0, x)$. Then there exists $\alpha > 0$ such that $B_2^n(x_0,\alpha) \subset K \cap H^-$. Now, as $f > 0$ m-a.e.,

$$m(B_2^n(x_0,\alpha)) = m\bigg(\{y \in B_2^n(x_0,\alpha) : f(y) > 0\}\bigg) =$$

$$m\bigg(\cup_{j=1}^{\infty} \{y \in B_2^n(x_0,\alpha) : f(y) > \frac{1}{j}\}\bigg).$$

Let $\epsilon > 0$ be given, By continuity of m from below, there exits j_ϵ such that

$$m(B_2^n(x_0,\alpha)) \leq (1+\epsilon)\ m\bigg(\{y \in B_2^n(x_0,\alpha) : f(y) > \frac{1}{j_\epsilon}\}\bigg).$$

Therefore

$$0 = \int_{K\cap H^-} f dm \geq \int_{B_2^n(x_0,\alpha)} f dm \geq \int_{\{y\in B_2^n(x_0,\alpha): f(y) > \frac{1}{j_\epsilon}\}} f(y) dm,$$

$$\geq \frac{j_\epsilon}{1+\epsilon} \text{vol}_n(B_2^n(x_0,\alpha)) > 0,$$

a contradiction.

(iii) Suppose (iii) is not true. Then for all s there is $s_0 \leq s$ such that $F(K, f, s_0) = \emptyset$. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} s_n = 0$. Then for all s_n there is $s \leq s_n$ such that $F(K, f, s) = \emptyset$. But by (ii) we have that $F(K, f, 0) = K \neq \emptyset$, which is a contradiction.

Now we come to the geometric interpretation of the p-affine surface area using weighted floating bodies.

Let K be a convex body and $x \in \partial K$. We define $r(x)$ as the maximum of all real numbers ρ so that $B_2^n(x - \rho N_{\partial K}(x), \rho) \subseteq K$. This has been used in [SchW1] to investigate the floating body. It was pointed there that for all α with $0 \leq \alpha < 1$ the integral $\int_{\partial K} r(x)^{-\alpha} d\mu_{\partial(K)}(x)$ is finite. The cube is an example showing that $\int_{\partial K} r(x)^{-1} d\mu_{\partial(K)}(x)$ may be infinite.

Theorem 5 *Let K be a convex body in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$ be a continuous function such that $f \geq c$ on K where $c > 0$ is a constant. Then*

$$c_n \lim_{s\to 0} \frac{vol_n(K) - vol_n(F(K, f, s))}{s^{\frac{2}{n+1}}} = \int_{\partial K} \frac{\left(\kappa(x)\right)^{\frac{1}{n+1}} d\mu_{\partial(K)}(x)}{\left(f(x)\right)^{\frac{2}{n+1}}},$$

where $c_n = 2\left(\frac{vol_{n-1}(B_2^{n-1})}{n+1}\right)^{\frac{2}{n+1}}$.

In the next corollary we give a geometric interpretation of the p-affine surface area. The advantage of the weighted floating body over the surface body of [SchW3] is that no integrability assumptions on the function $\frac{1}{r(x)}$ are needed.

We will also use the following function $f_p = f_{p,K}$ defined for a given convex body K and $-\infty \leq p \leq \infty$, $p \neq -n$ by:

$$f_p(x) = \frac{< x, N_{\partial K}(x) >^{\frac{n(n+1)(p-1)}{2(n+p)}}}{\kappa(x)^{\frac{n(p-1)}{2(n+p)}}}, \quad \text{for } x \in \partial K. \tag{7}$$

If K is such that f_p is continuous on ∂K, we extend f_p to all of K such that f_p is continuous on K.

Corollary 6 *Let $-\infty \leq p \leq \infty$, $p \neq -n$. Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior and such that the function f_p is continuous on K and such that $f_p \geq c$ on K where $c > 0$ is a constant.*

$$c_n \lim_{s\to 0} \frac{vol_n(K) - vol_n(F(K, f_p, s))}{s^{\frac{2}{n+1}}} = O_p(K).$$

Proof The proof follows immediately from Theorem 5.

For the proof of Theorem 5 we need several lemmas.

Lemma 7 *Let K and L be two convex bodies in $\mathbf{R}^n$ such that $0 \in int(L)$ and $L \subseteq K$. Then*

$$vol_n(K) - vol_n(L) = \frac{1}{n} \int_{\partial K} < x, N_{\partial K}(x) > \left(1 - \left(\frac{||x_L||}{||x||}\right)^n\right) d\mu_{\partial K}(x),$$

where $x_L = [0,x] \cap \partial L$.

The proof of Lemma 7 is standard.

We can assume without loss of generality that 0 is an interior point of K and for $x \in \partial K$ and $s > 0$ we put

$$x_s = [0,x] \cap \partial F(K,f,s). \tag{8}$$

Since we want to apply the Lebesgue convergence theorem, we need a dominating function. This function turns out to be a multiple of $(\frac{1}{r(x)})^{\frac{n-1}{n+1}}$. As mentioned before, it was shown in [SchW1] that this function is integrable.

In Lemmas 8 and 9 as well as in the proofs x_s is as in (8).

Lemma 8 *Let K be a convex body in $\mathbb{R}^n$ such that $0 \in int(K)$ and let $f: K \to \mathbb{R}$ be a continuous function such that $f \geq c$ on K where $c > 0$ is a constant.*

Then there is $s_0 > 0$ such that for all s with $0 \leq s \leq s_0$ and for all $x \in \partial K$

$$0 \leq \frac{< x, N_{\partial K}(x) > \left(1 - (\frac{\|x_s\|}{\|x\|})^n\right)}{n \ s^{\frac{2}{n+1}}} \leq C \ r(x)^{-\frac{n-1}{n+1}},$$

where C is an absolute constant.

In the next Lemma we use the indicatrix of Dupin. For a convex body K in $\mathbb{R}^n$ the indicatrix of Dupin exists $\mu_{\partial K}$-a.e. on ∂K and is an ellipsoid or an elliptic cylinder (see [L1]).

Lemma 9 *Let K be a convex body in $\mathbb{R}^n$ such that $0 \in int(K)$. Let $f: K \to \mathbb{R}$ be a continuous function such that $f \geq c$ on K where $c > 0$ is a constant.*

(i) If the indicatrix of Dupin at x is an ellipsoid, then

$$\lim_{s\to 0} \frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n+1}}} = \frac{(n+1)^{\frac{2}{n+1}}\ \kappa(x)^{\frac{1}{n+1}}}{2\left(\ vol_{n-1}(B_2^{n-1})\ f(x)\right)^{\frac{2}{n+1}}},$$

except in the trivial case when $x_s \in \partial K$ and then the limit is 0.

(ii) If the indicatrix of Dupin at x is an elliptic cylinder, then

$$\lim_{s\to 0} \frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n+1}}} = 0.$$

Proof of Theorem 5

We may assume that $0 \in \text{int}(K)$.

By Lemma 7

$$\frac{\text{vol}_n(K) - \text{vol}_n(F(K,f,s))}{s^{\frac{2}{n+1}}} = \frac{1}{n}\int_{\partial K} \frac{< x, N_{\partial K}(x) > (1-(\frac{\|x_s\|}{\|x\|})^n)}{s^{\frac{2}{n+1}}} d\mu_{\partial K}(x)$$

By Lemma 8 the functions under the integral sign are bounded uniformly in s by an L^1-function. By Lemma 9 they are converging pointwise a.e. We apply Lebesgue's convergence theorem.

Proof of Lemma 8

Let $x \in \partial K$.

We first consider the case that $x_s \in \partial K$. Then, by construction of x_s and as $0 \in \text{int}(K)$, $x_s = x$ and therefore

$$\frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{s^{\frac{2}{n-1}}} = 0$$

in this case.

Now we consider the case that $x_s \in \text{int}(K)$. As x and x_s are colinear and $\|x_s\| \le \|x\|$

$$\frac{\|x_s\|}{\|x\|} = 1 - \frac{\|x - x_s\|}{\|x\|}.$$

Hence

$$\frac{1}{n} < x, N_{\partial K}(x) > \left(1 - \left(\frac{\|x_s\|}{\|x\|}\right)^n\right)$$
$$= \frac{1}{n} < x, N_{\partial K}(x) > \left(1 - \left(1 - \frac{\|x - x_s\|}{\|x\|}\right)^n\right)$$
$$\leq \left\langle \frac{x}{\|x\|}, N_{\partial K}(x) \right\rangle \|x - x_s\| \tag{9}$$

The last expression is also denoted by Δ_s:

$$\Delta_s = \left\langle \frac{x}{\|x\|}, N_{\partial K}(x) \right\rangle \|x - x_s\| = \langle x - x_s, N_{\partial K}(x) \rangle .$$

It is the distance of x to the hyperplane through x_s and orthogonal to $N_{\partial K}(x)$. By Lemma 4 (iii) there is an s_0 so that for all s with $s \leq s_0$ the body $F(K, f, s)$ is nonempty.

As $x_s \in \mathrm{int}(K)$, by Lemma 4 (i) there is a hyperplane H with $x_s \in H$ and $\int_{K \cap H^-} f dm = s$. Thus

$$s = \int_{K \cap H^-} f(y) dm(y) \geq c \ \mathrm{vol}_n(K \cap H^-). \tag{10}$$

Since 0 is an interior point of K there is a constant $\alpha > 0$ such that for all $x \in \partial K$ we have

$$\alpha \leq < \frac{x}{\|x\|}, N_{\partial K}(x) > . \tag{11}$$

Now we consider two cases. The first case is

(i) $\|x - x_s\| \leq < \frac{x}{\|x\|}, N_{\partial K}(x) > \ r(x)$.

We have that

$$\mathrm{vol}_n(K \cap H^-) \geq \mathrm{vol}_n(B_2^n(x - r(x) N_{\partial K}(x), r(x)) \cap H^-).$$

The height of the cap $B_2^n(x - r(x)N_{\partial K}(x), r(x)) \cap H^-$ is bigger than or equal to the distance d of x_s to the boundary of $B_2^n(x - r(x)N_{\partial K}(x), r(x))$ and

$$d = r(x) - \left(r(x)^2 + \|x - x_s\|^2 - 2r(x)\|x - x_s\| < \frac{x}{\|x\|}, N_{\partial K}(x) > \right)^{\frac{1}{2}}$$
$$\geq \|x - x_s\| \ < \frac{x}{\|x\|}, N_{\partial K}(x) > \ \left(1 - \frac{\|x - x_s\|}{2r(x) < \frac{x}{\|x\|}, N_{\partial K}(x) >}\right)$$
$$\geq \frac{1}{2}\|x - x_s\| \ < \frac{x}{\|x\|}, N_{\partial K}(x) > = \frac{\Delta_s}{2}.$$

The last inequality holds by assumption (i). Thus (see [SchW1])

$$\mathrm{vol}_n(B_2^n(x-r(x)N_{\partial K}(x),r(x))\cap H^-)\geq \frac{r(x)^{\frac{n-1}{2}}}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})(\frac{\Delta_s}{2})^{\frac{n+1}{2}}$$

and hence in this case, using (9) and (10),

$$\frac{<x,N_{\partial K}(x)>\left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n+1}}}\leq \frac{2}{r(x)^{\frac{n-1}{n+1}}}\left(\frac{(n+1)}{c\ \mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}.$$

Now we consider the case

(ii) $\|x-x_s\|><\frac{x}{\|x\|},N_{\partial K}(x)>\ r(x)$.

Since 0 is an interior point, there is $\beta>0$ such that $B_2^n(0,\beta)\subseteq K$. We consider the convex hull C of x and $B_2^n(0,\beta)\cap H(0,x)$:

$$C=[x,B_2^n(0,\beta)\cap H(0,x)].$$

C is a cone whose base is an $n-1$-dimensional Euclidean ball with radius β.

Then

$$\mathrm{vol}_n(K\cap H^-)\geq \mathrm{vol}_n(C\cap H^-)$$

The expression on the right hand side is smallest, if the hyperplane H that passes through x_s is orthogonal to x. Then the set $C\cap H^-$is a cone whose base is an $n-1$-dimensional Euclidean ball with radius $\frac{\beta\|x-x_s\|}{\|x\|}$.

Thus we get

$$\mathrm{vol}_n(C\cap H^-)\geq \frac{1}{n\ \|x\|^{n-1}}\left(\beta^{n-1}\|x-x_s\|^n\ \mathrm{vol}_{n-1}(B_2^{n-1})\right) \tag{12}$$

Hence (9), (10) and (12) give

$$\frac{<x,N_{\partial K}(x)>\left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n-1}}}\leq$$

$$\left(\frac{n}{\mathrm{c\ vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}\ \frac{<\frac{x}{\|x\|},N_{\partial K}(x)>}{\|x-x_s\|^{\frac{n-1}{n+1}}}\ \left(\frac{\|x\|}{\beta}\right)^{\frac{2(n-1)}{n+1}}$$

which, using the assumption of case(ii) and (11), is less than or equal to

$$\left(\frac{n}{c\ \mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}\ \left(\frac{\mathrm{diam}(K)}{\beta}\right)^{\frac{2(n-1)}{n+1}}\left(\frac{1}{\alpha}\right)^{\frac{n-1}{n+1}}\ \left(\frac{1}{r(x)}\right)^{\frac{n-1}{n+1}}.$$

$\mathrm{diam}(K)$ denotes the diameter of K.

This finishes the proof of Lemma 8.

Proof of Lemma 9

Let $x \in \partial K$.

(9) holds:

$$\frac{1}{n}\langle x, N_{\partial K}(x)\rangle \left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right) \leq \quad \left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle \|x-x_s\|$$

In the same way we obtain the inverse inequality.

$$\frac{1}{n} < x, N_{\partial K}(x) > \left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right) =$$
$$\frac{1}{n} < x, N_{\partial K}(x) > \left((1-\left(1-\frac{\|x-x_s\|}{\|x\|}\right)^n\right)$$

Since $(1-t)^n \leq 1-nt+\frac{n(n-1)}{2}t^2$ for t small enough

$$\frac{1}{n} < x, N_{\partial K}(x) > \left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right)$$
$$\geq \left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle \|x-x_s\| \left(1-\frac{(n-1)}{2}\ \frac{\|x-x_s\|}{\|x\|}\right),$$

for s small enough.

Let $\epsilon > 0$ be given. We choose s_0 such that for all $s \leq s_0$

$$\frac{1}{n} < x, N_{\partial K}(x) > \left(1-\left(\frac{\|x_s\|}{\|x\|}\right)^n\right) \geq \left\langle \frac{x}{\|x\|}, N_{\partial K}(x)\right\rangle \|x-x_s\|\,(1-\epsilon)\,. \tag{13}$$

We first consider the case that $x_s \in \partial K$. Again, by construction of x_s and as $0 \in \text{int}(K)$, $x_s = x$ and therefore

$$\frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{s^{\frac{2}{n-1}}} = 0$$

in this case.

Now we consider the case that $x_s \in \text{int}(K)$. By Lemma 4 (i) there is a hyperplane H_s with $x_s \in H_s$ and $\int_{K\cap H_s^-} f dm = s$. As f is continuous on K, there exists a neighborhood U of x such that for all $y \in U$

$$(1-\epsilon)f(x) \leq f(y) \leq (1+\epsilon)f(x).$$

We choose $s_1 \leq s_0$ so small that for all $s \leq s_1$ we have that $K \cap H_s^- \subset U$. Thus we get for all $s \leq s_1$

$$\begin{aligned}(1-\epsilon)\ f(x)\ \text{vol}_n(K\cap H_s^-) \quad &\leq \quad s \;=\; \int_{K\cap H_s^-} f dm \\ &\leq \quad (1+\epsilon)\ f(x)\ \text{vol}_n(K\cap H_s^-) \end{aligned} \tag{14}$$

We first consider the case

(i) The indicatrix of Dupin at x is an ellipsoid. By Proposition 3, $F(K,f,s)$ is invariant under linear transformations with determinant 1 and hence we can assume that the indicatrix is a Euclidean ball with radius $\sqrt{\rho}=\sqrt{\rho}(x)$.

Let $C(r,\Delta)$ be the cap of height Δ of a Euclidean ball with radius r. By [SchW1], Lemma 11 we have for $s \leq s_1$ sufficiently small that

$$\mathrm{vol}_n(K\cap H_s^-) \leq (1+\epsilon)\ \mathrm{vol}_n\left(C(\rho, \|x-x_s\| < \frac{x}{\|x\|}, N_{\partial K}(x) >)\right)$$

$$\leq (1+2\epsilon)\frac{2^{\frac{n+1}{2}}}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})\rho^{\frac{n-1}{2}}\left(\|x-x_s\| < \frac{x}{\|x\|}, N_{\partial K}(x) >\right)^{\frac{n+1}{2}}. \qquad (15)$$

For the last inequality see for instance [SchW1], Lemma 8. And, again by [SchW1], Lemma 11 we have for $s \leq s_1$ sufficiently small that

$$\mathrm{vol}_n(K\cap H_s^-) \geq (1-\epsilon)\ \mathrm{vol}_n\left(C(\rho, (1-d\epsilon)\|x-x_s\| < \frac{x}{\|x\|}, N_{\partial K}(x) >)\right)$$

$$\geq (1-2\epsilon)\frac{2^{\frac{n+1}{2}}}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})\rho^{\frac{n-1}{2}}\left((1-d\epsilon)\|x-x_s\| < \frac{x}{\|x\|}, N_{\partial K}(x) >\right)^{\frac{n+1}{2}}, \quad (16)$$

where d is a constant. Hence we get with (9), (13), (14), (15), (16) and the fact that $\rho^{-\frac{n-1}{n+1}} = \kappa^{\frac{1}{n+1}}$

$$\frac{1-\epsilon}{2}\left(\frac{n+1}{(1+2\epsilon)^2\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}\ \frac{\kappa(x)^{\frac{1}{n+1}}}{f(x)^{\frac{2}{n+1}}}$$

$$\leq \frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n+1}}} \leq$$

$$\frac{1}{2(1-d\epsilon)}\left(\frac{n+1}{(1-2\epsilon)^2\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}\ \frac{\kappa(x)^{\frac{1}{n+1}}}{f(x)^{\frac{2}{n+1}}}.$$

Now we treat the case

(ii) The indicatrix of Dupin at x is an elliptic cylinder. Again, as above we can assume that it is a spherical cylinder i.e. the product of a k-dimensional plane and an $n-k-1$-dimensional Euclidean ball of radius ρ. Moreover we can assume that ρ is arbitrarily large. Using Lemma 9 of [SchW1] and (9) we get for s sufficiently small that with a (new) absolute constant d

$$\frac{< x, N_{\partial K}(x) > \left(1-(\frac{\|x_s\|}{\|x\|})^n\right)}{n\ s^{\frac{2}{n+1}}} \leq (1+d\epsilon)\ \left(\frac{n+1}{\mathrm{vol}_{n-1}(B_2^{n-1})}\right)^{\frac{2}{n+1}}\ \frac{\rho^{-\frac{n-1}{n+1}}}{f(x)^{\frac{2}{n+1}}},$$

which goes to 0 as ρ is arbitrarily large.

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