ON THE GEOMETRY OF PROJECTIVE TENSOR PRODUCTS

OHAD GILADI, JOSCHA PROCHNO, CARSTEN SCHÜTT, NICOLE TOMCZAK-JAEGERMANN, AND ELISABETH WERNER

Abstract. In this work, we study the volume ratio of the projective tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ with $1 \leq p \leq q \leq r \leq \infty$. We obtain asymptotic formulas that are sharp in almost all cases. As a consequence of our estimates, these spaces allow for a nearly Euclidean decomposition of Kašin type whenever $1 \leq p \leq q \leq r \leq 2$ or $1 \leq p \leq 2 \leq r \leq \infty$ and $q = 2$. Also, from the Bourgain-Milman bound on the volume ratio of Banach spaces in terms of their cotype 2 constant, we obtain information on the cotype of these 3-fold projective tensor products. Our results naturally generalize to k-fold products $\ell_{p_1}^n \otimes_\pi \cdots \otimes_\pi \ell_{p_k}^n$ with $k \in \mathbb{N}$ and $1 \leq p_1 \leq \cdots \leq p_k \leq \infty$.

1. Introduction

In the geometry of Banach spaces the volume ratio $\mathrm{vr}(X)$ of an n-dimensional normed space X is defined as the n-th root of the volume of the unit ball in X divided by the volume of its John ellipsoid. This notion plays an important role in the local theory of Banach spaces and has significant applications in approximation theory. It formally originates in the works [Sza78] and [STJ80], which were influenced by the famous paper of B. Kašin [Kaš77] on nearly Euclidean orthogonal decompositions. Kašin discovered that for arbitrary $n \in \mathbb{N}$, the space ℓ_1^{2n} contains two orthogonal subspaces which are nearly Euclidean, meaning that their Banach-Mazur distance to ℓ_2^n is bounded by an absolute constant. S. Szarek [Sza78] noticed that the proof of this result depends solely on the fact that ℓ_1^n has a bounded volume ratio with respect to ℓ_2^n. In fact, it is essentially contained in the work of Szarek that if X is a $2n$-dimensional Banach space, then there exist two n-dimensional subspaces each having a Banach-Mazur distance to ℓ_2^n bounded by a constant times the volume ratio of X squared. This observation by S. Szarek and N. Tomczak-Jaegermann was further investigated in [STJ80], where the concept of volume ratio was formally introduced, its connection to the cotype 2 constant of Banach spaces was studied, and Kašin type decompositions were proved for some classes of Banach spaces, such as the projective tensor product spaces $\ell_p^n \otimes_\pi \ell_2^n$, $1 \leq p \leq 2$.

Given two vector spaces X and Y, their algebraic tensor product $X \otimes Y$ is the subspace of the dual space of all bilinear maps on $X \times Y$ spanned by elementary tensors $x \otimes y$, $x \in X$, $y \in Y$ (a formal definition is provided below). The theory of tensor products was established by A. Grothendieck in 1953 in his Résumé [Gro53] and has a huge impact on

Date: April 6, 2017.

2010 *Mathematics Subject Classification.* 46A32, 46B28.

O.G. was supported in part by the Australian Research Council.

J.P. was supported in part by the FWF grant FWFM 162800.

A part of this work was done when N.T-J. held the Canada Research Chair in Geometric Analysis; also supported by NSERC Discovery Grant.

E.W. was supported by NSF grant DMS-1504701.

Banach space theory (see, e.g., the survey paper [Pis12]). This impact and the success of the concept of tensor products is to a large extent due to the work [LP68] of J. Lindenstrauss and A. Pełczynski in the late sixties who reformulated Grothendieck's ideas in the context of operator ideals and made this theory accessible to a broader audience. Today, tensor products appear naturally in numerous applications, among others, in the entanglement of qubits in quantum computing, in quantum information theory in terms of (random) quantum channels (e.g., [AS06, ASW10, ASW11, SWZ11]) or in theoretical computer science to represent locally decodable codes [Efr09]. For an interesting and recently discovered connection between the latter and the geometry of Banach spaces we refer the reader to [BNR12].

The geometry of tensor products of Banach spaces is complicated, even if the spaces involved are of simple geometric structure. For example, the 2-fold projective tensor product of Hilbert spaces, $\ell_2 \otimes_\pi \ell_2$, is naturally identified with the Schatten trace class S_1, the space of all compact operators $T : \ell_2 \to \ell_2$ equipped with the norm $\|T\|_{S_1} = \mathrm{trace}\big(\sqrt{T^*T}\big)$. This space does not have local unconditional structure [GL74]. The geometric structure of triple tensor products is even more complicated and therefore it is hardly surprising that very little is known about the geometric properties of these spaces. For instance, regarding the permanence of cotype (see below for the definition) under projective tensor products, it was proved by N. Tomczak-Jaegermann in [TJ74] that the space $\ell_2 \otimes_\pi \ell_2$ has cotype 2, but the corresponding question in the 3-fold case is still open for more than 40 years. G. Pisier proved in [Pis90] and [Pis92] that the space $L_p \otimes_\pi L_q$ has cotype $\max(p,q)$ if $p, q \in [2,\infty)$ and, till the present day, it is unknown whether these spaces have a non-trivial cotype when $p \in (1,2)$ and $q \in (1,2]$. In the recent paper [BNR12] by J. Briët, A. Naor and O. Regev they showed that the spaces $\ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r$ fail to have non-trivial cotype if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. Their proof uses deep results from the theory of locally decodable codes. A direct, rather surprising consequence of their work is that for $p \in (1,\infty)$ the space $\ell_p \otimes_\pi \ell_{2p/(p-1)} \otimes_\pi \ell_{2p/(p-1)}$ fails to have non-trivial cotype, while, by Pisier's result, the 2-fold projective tensor product $\ell_{2p/(p-1)} \otimes_\pi \ell_{2p/(p-1)}$ has finite cotype. Let us also mention that interest in cotype is, to a great deal, due to a famous result of B. Maurey and G. Pisier [MP76], who showed that a Banach space X fails to have finite cotype if and only if it contains ℓ_∞^n's uniformly.

In view of the various open questions and surprising results around the geometry of projective tensor products, with the present paper, we contribute to a better understanding of the geometric structure of 3-fold projective tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ $(1 \leq p \leq q \leq r \leq \infty)$ by studying one of the key notions in local Banach space geometry, the volume ratio of these spaces. This provides new structural insight and allows, on the one hand, to draw conclusions regarding cotype properties of these spaces and, on the other hand, to see which of these spaces allow a nearly Euclidean decomposition of Kašin type.

2. Presentation of the main result

Given two Banach spaces, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the projective tensor product space, denoted $X \otimes_\pi Y$, is the space $X \otimes Y$ equipped with the norm

$$\|A\|_{X\otimes_\pi Y} = \inf\left\{\sum_{i=1}^m \|x_i\|_X \|y_i\|_Y \ : \ A = \sum_{i=1}^m x_i \otimes y_i,\ x_i \in X,\ y_i \in Y\right\}. \tag{2.1}$$

See Section 3.3 below for more information on projective tensor products.

Given an n-dimensional Banach space $(X, \|\cdot\|_X)$, let B_X denote its unit ball, and let $\mathcal{E}_X$ be the ellipsoid of maximal volume contained in B_X. The volume ratio of X is defined by

$$\mathrm{vr}(X) = \left(\frac{\mathrm{vol}_n(B_X)}{\mathrm{vol}_n(\mathcal{E}_X)}\right)^{1/n}, \tag{2.2}$$

where $\mathrm{vol}_n(\cdot)$ denotes the n-dimensional Lebesgue measure.

Considering the importance of tensor products, it is natural to study the geometric properties of $X \otimes_\pi Y$ and, in particular, the volume ratio $\mathrm{vr}(X \otimes_\pi Y)$ of these spaces. As already mentioned in the introduction, when $X = \ell_p^n$ $(1 \le p \le 2)$ and $Y = \ell_2^n$ this was carried out in [STJ80]. The complete answer was given later by C. Schütt in [Sch82], where it was proved that if $1 \le p \le q \le \infty$, then

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n\right) \asymp_{p,q} \begin{cases} 1, & q \le 2, \\ n^{\frac{1}{2}-\frac{1}{q}}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q} \ge 1, \\ n^{\frac{1}{p}-\frac{1}{2}}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q} \le 1, \\ n^{\max\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{q},0\right)}, & p \ge 2. \end{cases}$$

The notation $\asymp_{p,q}$ means equivalence up to constants that depend only on p and q. In [DM05] this was generalized by A. Defant and C. Michels to the setting $E \otimes_\pi F$, where E and F are symmetric Banach sequence spaces, each either 2-convex or 2-concave.

We study tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ $(1 \le p \le q \le r \le \infty)$. Our main result is as follows.

Theorem A. *Let $n \in \mathbb{N}$ and $1 \le p \le q \le r \le \infty$. Then we have*

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_{p,q,r} \begin{cases} 1, & r \le 2, \\ n^{\max\left(\frac{1}{2}-\frac{1}{q}-\frac{1}{r},0\right)}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r} \ge 1, \\ n^{\frac{1}{p}-\frac{1}{2}}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r} \le 1, \\ n^{\max\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{q}-\frac{1}{r},0\right)}, & p \ge 2. \end{cases}$$

In the case $p \le q \le 2 \le r$, we have

$$1 \lesssim_{p,q,r} \mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \lesssim_{p,q,r} n^{\min\left(\frac{1}{2}-\frac{1}{r},\frac{1}{q}-\frac{1}{2}\right)}.$$

Here and in what follows $\lesssim_{p,q,r}$, $\gtrsim_{p,q,r}$ mean inequalities with implied positive constants that depend only on the parameters p, q, r. The asymptotic notation $\asymp_{p,q,r}$ means that we have both $\lesssim_{p,q,r}$ and $\gtrsim_{p,q,r}$.

Remark 2.1. We would like to remark that whenever $1 \le p \le q \le 2 \le r \le \infty$, we are able to improve on the general bound given by Theorem A in the following situations (see Corollary 5.7):

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_{p,q,r} \begin{cases} n^{\min\left(\frac{1}{q}-\frac{1}{2},\frac{1}{2}-\frac{1}{r}\right)}, & p = 1 \le q \le 2 \le r, \\ n^{\frac{1}{q}-\frac{1}{2}}, & p = q \le 2,\ r = \infty, \\ 1, & 1 \le p \le q = 2 \le r. \end{cases}$$

Remark 2.2. Theorem A and Remark 2.1 immediately imply that the spaces $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ allow a nearly Euclidean decomposition of Kašin type, when $1 \le p \le q \le r \le 2$ or $1 \le p \le 2 \le r \le \infty$ and $q = 2$.

Before we proceed, let us comment on the strategy of the proof. Recall first that a Banach space $(X, \|\cdot\|_X)$ is said to have enough symmetries if the only operators that commute with every isometry on X are multiples of the identity. It is known that if $(X, \|\cdot\|_X)$ is n-dimensional and has enough symmetries, then $\mathcal{E}_X$ is given by

$$\mathcal{E}_X = \|\mathrm{id} : \ell_2^n \to X\|^{-1} B_2^n,$$

where B_2^n denotes the n-dimensional Euclidean ball (see, for instance, [TJ89, Sec. 16]). Hence, using formula (2.2), if $(X, \|\cdot\|_X)$ is n-dimensional and has enough symmetries, then

$$\mathrm{vr}(X) = \left(\frac{\mathrm{vol}_n(B_X)}{\mathrm{vol}_n(B_2^n)}\right)^{1/n} \big\|\mathrm{id} : \ell_2^n \to X\big\| \overset{(*)}{\asymp} \sqrt{n}\big(\mathrm{vol}_n(B_X)\big)^{1/n}\big\|\mathrm{id} : \ell_2^n \to X\big\|, \tag{2.3}$$

where in $(*)$ we used Stirling's formula to deduce $\mathrm{vol}(B_2^n)^{1/n} \asymp 1/\sqrt{n}$. It is also known that projective tensor products of ℓ_p spaces are spaces with enough symmetries [Sch82, STJ80] and therefore formula (2.3) holds for the spaces $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$. Thus, in order to prove Theorem A, it is enough to compute the volume of the unit ball of $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$, and the norm of the natural identity between $\ell_2^{n^3}$ and $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$.

The rest of this paper is organized as follows. In Section 3 we collect some basic results which will be used later. In Section 4 we estimate the volume of the unit ball in $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$. In Section 5 we estimate the norm of the natural identity. In Section 6 we discuss the case of k-fold tensor products. Finally, in Section 7, we present some applications of Theorem A.

3. Preliminaries

In this section, we introduce the necessary notions and background material and provide the main ingredients needed to prove the estimates for the volume ratio of 3-fold projective tensor products. These include an extension of Chevet's inequality, a lower bound on the volume of unit balls in Banach spaces due to Schütt, the famous Blaschke-Santaló inequality, and a multilinear version of an inequality of Hardy and Littlewood.

3.1. General notation.

Given a Banach space $(X, \|\cdot\|_X)$, denote its unit ball by B_X and its dual space by X^*. For two Banach spaces X and Y, we write $\mathcal{L}(X, Y)$ for the space of all bounded linear operators from X to Y.

For $1 \le p \le \infty$, ℓ_p^n is the vector space $\mathbb{R}^n$ with the norm

$$\|(x_i)_{i=1}^n\|_p = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, & 1 \le p < \infty \\ \max\limits_{1\le i\le n} |x_i|, & p = \infty. \end{cases}$$

The unit ball in ℓ_p^n is denoted by $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$. The conjugate p^* of p is defined via the relation $\frac{1}{p} + \frac{1}{p^*} = 1$. Unless otherwise stated, $e_1, \ldots, e_n$ will be the standard unit vectors in $\mathbb{R}^n$.

We shall also use the asymptotic notations $\lesssim$ and $\gtrsim$ to indicate the corresponding inequalities up to universal constant factors, and we shall denote equivalence up to universal constant factors by $\asymp$, where $A \asymp B$ is the same as $(A \lesssim B) \wedge (A \gtrsim B)$. If the constants involved depend on a parameter α, we denote this by $\lesssim_\alpha$, $\gtrsim_\alpha$ and $\asymp_\alpha$, respectively.

3.2. **Polar body and Blaschke-Santaló inequality.** A convex body K in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. The n-dimensional Lebesgue measure of a convex body $K \subseteq \mathbb{R}^n$ is denoted by $\mathrm{vol}_n(K)$. If 0 is an interior point of a convex body K in $\mathbb{R}^n$, we define the polar body of K by

$$K^\circ := \big\{ y \in \mathbb{R}^n \,:\, \forall x \in K : \langle x, y \rangle \leq 1 \big\},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product on $\mathbb{R}^n$. Note that the unit ball of any norm on $\mathbb{R}^n$ is a convex body and its polar body is just the unit ball of the corresponding dual norm. Moreover, we have $(K^\circ)^\circ = K$. The famous Blaschke-Santaló inequality provides a sharp upper bound for the volume product of a convex body with its polar (see, for example, [Pis89, Sec. 7] or [Sch14]).

Lemma 3.1 (Blaschke-Santaló inequality)**.** *Let K be an origin symmetric convex body in $\mathbb{R}^n$. Then*

$$\mathrm{vol}_n(K) \cdot \mathrm{vol}_n(K^\circ) \leq \mathrm{vol}_n(B_2^n)^2,$$

with equality if and only if K is an ellipsoid.

3.3. **Tensor products and extended Chevet inequality.** Given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, the algebraic tensor product $X \otimes Y$ can be constructed as the space of linear functionals on the space of all bilinear forms on $X \times Y$. Given $x \in X$, $y \in Y$ and a bilinear form B on $X \times Y$, we define $(x \otimes y)(B) := B(x, y)$. On the tensor product space define the projective tensor product space, denoted by $X \otimes_\pi Y$, as $X \otimes Y$ equipped with the norm

$$\|A\|_{X \otimes_\pi Y} := \inf \left\{ \sum_{i=1}^m \|x_i\|_X \|y_i\|_Y \,:\, A = \sum_{i=1}^m x_i \otimes y_i,\ x_i \in X,\ y_i \in Y \right\}.$$

Also, define the injective tensor norm space, denoted $X \otimes_\epsilon Y$, as the tensor product space $X \otimes Y$ equipped with the norm

$$\|A\|_{X \otimes_\epsilon Y} := \sup \left\{ \Big| \sum_{i=1}^n \varphi(x_i) \psi(y_i) \Big| \,:\, A = \sum_{i=1}^m x_i \otimes y_i,\ \varphi \in B_{X^*},\ \psi \in B_{Y^*} \right\}.$$

Here, we only consider finite-dimensional spaces and therefore the tensor products are always complete. In this case, it can be shown that $X \otimes_\pi Y = \mathcal{N}(X^*, Y)$, the space of all nuclear operators from X^* into Y. We will often use the fact that $(X \otimes_\pi Y)^*$, the dual space of $X \otimes_\pi Y$, is the space of operators from X^* to Y, $\mathcal{L}(X^*, Y)$, equipped with the standard operator norm. It is also known that $\mathcal{L}(X^*, Y)$ can be identified with the injective tensor product $X \otimes_\epsilon Y$. In particular, for $A \in X \otimes_\epsilon Y$, we have

$$\|A\|_{X \otimes_\epsilon Y} = \sup_{x \in B_{X^*}} \|Ax\|_Y.$$

We refer the reader to [Rya02] and [DFS08] for more information about tensor products.

Recall that for a sequence $x = (x_i)_{i=1}^n$ in a Banach space $(X, \|\cdot\|_X)$, the norm $\|x\|_{\omega,2}$ is given by

$$\|x\|_{w,2} := \sup_{\|\varphi\|_{X^*}=1} \left(\sum_{i=1}^n |\varphi(x_i)|^2 \right)^{\frac{1}{2}}.$$

The following inequality is due to Chevet [Che78].

Lemma 3.2 (Chevet's inequality)**.** *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces and consider sequences $x_1, \ldots, x_m \in X$, $y_1, \ldots, y_n \in Y$, and sequences $(g_{i,j})_{i,j=1}^{m,n}$, $(\xi_i)_{i=1}^m$, $(\eta_j)_{j=1}^n$ of independent identically distributed standard Gaussians random variables. Then*

$$\mathbb{E}\left\| \sum_{i,j=1}^{m,n} g_{i,j} x_i \otimes y_j \right\|_{X\otimes_\epsilon Y} \le \|(x_i)_{i=1}^m\|_{\omega,2}\, \mathbb{E}\left\| \sum_{j=1}^n \eta_j y_j \right\|_Y + \left\|(y_j)_{j=1}^n\right\|_{\omega,2} \mathbb{E}\left\| \sum_{i=1}^m \xi_i x_i \right\|_X. \quad (3.1)$$

It is known that $B_{X^*\otimes_\pi Y^*} = \operatorname{conv}(B_{X^*} \otimes B_{Y^*})$ (see, for example, Proposition 2.2 in [Rya02]) and thus

$$\left\|(x_i \otimes y_j)_{i,j}\right\|_{\omega,2} = \left\|(x_i)_i\right\|_{\omega,2} \cdot \left\|(y_j)_j\right\|_{\omega,2}. \quad (3.2)$$

Using inequalities (3.1) and (3.2), we obtain the following 3-fold version of Chevet's inequality.

Lemma 3.3 (3-fold Chevet inequality)**.** *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces. Assume that $x_1, \ldots, x_m \in X$, $y_1, \ldots, y_n \in Y$ and $z_1, \ldots, z_\ell \in Z$. Let $g_{i,j,k}$, ξ_i, η_j, ρ_k, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, \ell$, be independent standard Gaussians random variables. Then*

$$\mathbb{E}\left\| \sum_{i,j,k=1}^{m,n,\ell} g_{ijk}\, x_i \otimes y_j \otimes z_k \right\|_{X\otimes_\epsilon Y\otimes_\epsilon Z} \le \Lambda,$$

where

$$\begin{aligned} \Lambda \;:=\;& \|(x_i)_{i=1}^m\|_{w,2} \left\|(y_j)_{j=1}^n\right\|_{w,2} \mathbb{E}\left\| \sum_{k=1}^\ell \rho_k z_k \right\|_Z + \|(x_i)_{i=1}^m\|_{w,2} \left\|(z_k)_{k=1}^\ell\right\|_{w,2} \mathbb{E}\left\| \sum_{j=1}^n \eta_j y_j \right\|_Y \\ &+ \left\|(y_j)_{j=1}^n\right\|_{w,2} \left\|(z_k)_{k=1}^\ell\right\|_{w,2} \mathbb{E}\left\| \sum_{i=1}^m \xi_i x_i \right\|_X. \end{aligned}$$

We would like to point out that, simply using the triangle inequality, a corresponding lower bound can be obtained up to an absolute constant.

3.4. **Volume ratio and Rademacher cotype.** The concept of Rademacher cotype was introduced to Banach space theory by J. Hoffmann-Jørgensen [HJ74] in the early 1970s. The basic theory was developed by B. Maurey and G. Pisier [MP76].

A Banach space $(X, \|\cdot\|_X)$, is said to have Rademacher cotype α if there exists a constant $C \in (0, \infty)$ such that for all $m \in \mathbb{N}$ and all $x_1, \ldots, x_m \in X$,

$$\left(\sum_{i=1}^m \|x_i\|_X^\alpha \right)^{1/\alpha} \le C\, \mathbb{E}\left\| \sum_{i=1}^m \varepsilon_i x_i \right\|_X. \quad (3.3)$$

In (3.3) and in what follows, $(\varepsilon_i)_{i=1}^\infty$ denotes a sequence of independent symmetric Bernoulli random variables, that is, $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$ for all $i \in \mathbb{N}$. The smallest C

that satisfies (3.3) is denoted by $C_\alpha(X)$ and called the cotype α constant of X. By taking $x_1 = x_2 = \cdots = x_m$ it follows that necessarily $\alpha \geq 2$.

In [BNR12] it was shown that, whenever $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$, then

$$C_\alpha \left(\ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r\right) = \infty,$$

for every $2 \leq \alpha < \infty$. However, it is still an open question whether for example $\ell_2 \otimes_\pi \ell_2 \otimes_\pi \ell_2$ has finite cotype.

One reason to study volume ratio of tensor products is its relation to the cotype constant. More precisely, given an estimate on volume ratio, one can use the following result by Bourgain and Milman [BM87]) which connects volume ratio and cotype of a Banach space.

Theorem 1 ([BM87])**.** *Let X be a Banach space. Then*

$$\mathrm{vr}(X) \lesssim C_2(X) \log\left(2C_2(X)\right).$$

The relation between volume ratio and cotype property is far from being well understood. For example, it was asked in [STJ80] whether bounded volume ratio implies cotype q for every $q > 2$.

In Section 7 below we discuss the cotype property of the space $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$, as well as the extended case of k-fold tensor products for $k > 3$.

3.5. **Volume of unit balls in Banach spaces.** The following result is a special case of Lemma 1.5 in [Sch82]. It provides a lower estimate for the volume of the unit ball of a normed space by the volume of a B_∞^n ball of a certain radius, arising from an average over sign vectors.

Lemma 3.4. *Let $(X, \|\cdot\|_X)$ be an n-dimensional Banach space, and let $e_1, \dots, e_n$ be basis vectors such that $\|e_i\|_X = 1$, and $(\varepsilon_i)_{i=1}^\infty$ a sequence of independent symmetric Bernoulli random variables. Then*

$$2^n \left(\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i e_i\right\|_X\right)^{-n} \leq \mathrm{vol}_n(B_X).$$

We will use this result in combination with the 3-fold version of Chevet's inequality to obtain a lower bound on the volume of the unit ball in $\ell_{p^*}^n \otimes_\epsilon \ell_{q^*}^n \otimes_\epsilon \ell_{r^*}^n$ which gives, using the Blaschke-Santaló inequality, an upper bound on the volume of the unit ball in the dual space $\ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r$, as well as tensor products of more than three ℓ_p spaces.

3.6. **Rademacher versus Gaussian averages.** In order to use Chevet's inequality in combination with Lemma 3.4, we need to pass from a Rademacher average to a Gaussian one. The following result due to Pisier shows that Rademacher averages are dominated by Gaussian averages in arbitrary Banach spaces. Note however that in general these averages are not equivalent.

Lemma 3.5 ([Pis86])**.** *Let $(X, \|\cdot\|_X)$ be a Banach space, $1 \leq p < \infty$ and let $\xi_1, \dots, \xi_n$ be independent, symmetric random variables. Assume that $\mathbb{E}|\xi_i| = \mathbb{E}|\xi_j|$ for all $1 \leq i, j \leq n$. Then, for all $x_1, \dots, x_n \in X$, we have*

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^p \leq \left(\mathbb{E}|\xi_1|\right)^{-p} \mathbb{E}\left\|\sum_{i=1}^n \xi_i x_i\right\|^p.$$

In particular, if we choose $g_1, \ldots, g_n$ to be independent Gaussian random variables, then since $\big(\mathbb{E}|g_1|\big)^{-p} = (\pi/2)^{p/2}$, we have

$$\mathbb{E}\Big\| \sum_{i=1}^n \varepsilon_i x_i \Big\|^p \lesssim_p \mathbb{E}\Big\| \sum_{i=1}^n g_i x_i \Big\|^p.$$

3.7. **A multilinear Hardy-Littlewood type inequality.** An essential tool in proving upper bounds on the norm of the natural identity between $\ell_2^{n^3}$ and $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ is the following inequality, which is a generalization of a classical inequality by Hardy and Littlewood [HL34]. For now, we state it only for the case of 3-fold tensors. In Section 6 we also present the consequences of the general version.

Theorem 2 ([PP81], Thm. B)**.** *Let $n \in \mathbb{N}$ and $1 \le p, q, r \le \infty$ so that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le \frac{1}{2}$. Then, for all $A \in \ell_{p^*}^n \otimes_\epsilon \ell_{q^*}^n \otimes_\epsilon \ell_{r^*}^n$, we have*

$$\|A\|_{\ell_\mu^{n^3}} \lesssim \|A\|_{\ell_{p^*}^n \otimes_\epsilon \ell_{q^*}^n \otimes_\epsilon \ell_{r^*}^n},$$

where μ is given by

$$\mu := \frac{3}{2 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}}. \tag{3.4}$$

4. The volume of the unit ball in $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$

In this section we evaluate the volume of the unit ball of $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ up to constants depending only on the parameters p, q, r. The main ingredients in the proof are the 3-fold version of Chevet's inequality (Lemma 3.3) and the Blaschke-Santaló inequality (Lemma 3.1).

The next theorem will be the consequence of the following two subsections.

Theorem B. *Let $n \in \mathbb{N}$ and $1 \le p \le q \le r \le \infty$. Then*

$$n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1} \le \mathrm{vol}_{n^3}\big(B_{\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n}\big)^{1/n^3} \lesssim_{p,q,r} n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1}.$$

4.1. **Lower bounds on the volume.** Let us first we fix some more notation. For $1 \le p, q, r \le \infty$ and $n \in \mathbb{N}$, let

$$X_\pi^n := \ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n,$$

where we suppress the parameters p, q, r that are clear from the context. For the corresponding norm, we write in short $\|\cdot\|_\pi$ instead of $\|\cdot\|_{X_\pi}$. Similarly, we define $X_\epsilon^n := \ell_{p^*}^n \otimes_\epsilon \ell_{q^*}^n \otimes_\epsilon \ell_{r^*}^n$ and write $\|\cdot\|_\epsilon$ instead of $\|\cdot\|_{X_\epsilon}$. We denote the corresponding unit balls by B_π^n and B_ϵ^n respectively. Any $A \in X_\epsilon^n$, we express in the form

$$A = \sum_{i,j,k=1}^n A_{i,j,k}\, e_i \otimes e_j \otimes e_k.$$

The main tool in proving a lower bound is the following estimate that compares the injective norm $\|\cdot\|_\epsilon$ with the ℓ_1-norm in $\mathbb{R}^{n^3}$.

Proposition 4.1. *Let $n \in \mathbb{N}$, $1 \le p, q, r \le \infty$ and assume $A \in X_\epsilon^n$. Then*

$$\|A\|_\epsilon \ge \frac{1}{2}\, n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1} \sum_{i,j,k=1}^{n} |A_{i,j,k}| .$$

Proof. To prove this, we identify X_ϵ^n with a space of operators. We have

$$\|A\|_\epsilon = \max_{\|x\|_p=\|y\|_q=\|z\|_r=1} \left| \sum_{i,j,k=1}^{n} A_{i,j,k}\, x_i y_j z_k \right| .$$

In what follows, $\varepsilon, \delta, \eta \in \{-1,1\}^n$ denote sign vectors. We divide the proof into three different cases, depending on which side of 2 the parameters p, q, r lie.

Case 1: Assume that $2 \le p \le q$. In this case we choose $x = n^{-\frac{1}{p}}(\varepsilon_1, \dots, \varepsilon_n)$, $y := n^{-\frac{1}{q}}(\delta_1, \dots, \delta_n)$, and $z = n^{-\frac{1}{r}}(\eta_1, \dots, \eta_n)$. Then we obtain

$$\begin{aligned}
\max_{\|x\|_p=\|y\|_q=\|z\|_r=1} \left| \sum_{i,j,k=1}^{n} A_{i,j,k} x_i y_j z_k \right| &\ge n^{-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}} \max_{\varepsilon,\delta,\eta \in \{-1,1\}^n} \left| \sum_{i,j,k=1}^{n} A_{i,j,k}\, \varepsilon_i \delta_j \eta_k \right| \\
&= n^{-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}} \max_{\delta,\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \left| \sum_{j,k=1}^{n} A_{i,j,k}\, \delta_j \eta_k \right| \\
&\ge n^{-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}} \max_{\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \frac{1}{2^n} \sum_{\delta \in \{-1,1\}^n} \left| \sum_{j,k=1}^{n} A_{i,j,k}\, \delta_j \eta_k \right| .
\end{aligned}$$

Applying Khintchine's inequality and then Hölder's inequality, we obtain

$$\begin{aligned}
\max_{\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \frac{1}{2^n} \sum_{\delta \in \{-1,1\}^n} \left| \sum_{j,k=1}^{n} A_{i,j,k}\, \delta_j \eta_k \right| &\ge \frac{1}{\sqrt{2}} \max_{\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{i,j,k}\, \eta_k \right|^2 \right)^{1/2} \\
&\ge \frac{1}{\sqrt{2n}} \max_{\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{i,j,k}\, \eta_k \right| .
\end{aligned}$$

Again, applying Khinchine's inequality and then Hölder's inequality,

$$\begin{aligned}
\max_{\eta \in \{-1,1\}^n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{i,j,k}\, \eta_k \right| &\ge \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2^n} \sum_{\eta \in \{-1,1\}^n} \left| \sum_{k=1}^{n} A_{i,j,k}\, \eta_k \right| \\
&\ge \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} |A_{i,j,k}|^2 \right)^{1/2} \\
&\ge \frac{1}{\sqrt{2n}} \sum_{i,j,k=1}^{n} |A_{i,j,k}|.
\end{aligned}$$

Case 2: Assume that $p \le 2 \le q$. In this case, we choose for $x \in B_p^n$ the standard unit vectors $e_1, \dots, e_n$ and y, z as in the previous case. We get, again by using the inequalities of

Khintchine and Hölder,

$$
\begin{aligned}
\max_{\|x\|_p=\|y\|_q=\|z\|_r=1}\left|\sum_{i,j,k=1}^n A_{i,j,k}\, x_i y_j z_k\right| &\geq n^{-\frac{1}{q}-\frac{1}{r}} \max_{1\leq i\leq n} \max_{\delta,\eta\in\{-1,1\}^n}\left|\sum_{j,k=1}^n A_{i,j,k}\,\delta_j\eta_k\right| \\
&= n^{-\frac{1}{q}-\frac{1}{r}} \max_{1\leq i\leq n} \max_{\eta\in\{-1,1\}^n}\sum_{j=1}^n\left|\sum_{k=1}^n A_{i,j,k}\,\eta_k\right| \\
&\geq n^{-\frac{1}{q}-\frac{1}{r}-1} \max_{\eta\in\{-1,1\}^n}\sum_{i,j=1}^n\left|\sum_{k=1}^n A_{i,j,k}\,\eta_k\right| \\
&\geq n^{-\frac{1}{q}-\frac{1}{r}-1} \sum_{i,j=1}^n \frac{1}{2^n}\sum_{\eta\in\{-1,1\}^n}\left|\sum_{k=1}^n A_{i,j,k}\,\eta_k\right| \\
&\geq \frac{1}{\sqrt{2}} n^{-\frac{1}{q}-\frac{1}{r}-1} \sum_{i,j=1}^n \left(\sum_{k=1}^n |A_{i,j,k}|^2\right)^{1/2} \\
&\geq \frac{1}{\sqrt{2}} n^{-\frac{1}{q}-\frac{1}{r}-\frac{3}{2}} \sum_{i,j,k=1}^n |A_{i,j,k}|.
\end{aligned}
$$

Case 3: Assume that $p \leq q \leq 2$. Choose for $x \in B_p^n$ and $y \in B_q^n$ the standard unit vectors $e_1, \dots, e_n$ and z as in the previous two cases. We have

$$
\begin{aligned}
\max_{\|x\|_p=\|y\|_q=\|z\|_r=1}\left|\sum_{j,k=1}^n A_{i,j,k}\, x_i y_j z_k\right| &\geq n^{-\frac{1}{r}} \max_{1\leq i,j\leq n} \max_{\eta\in\{-1,1\}^n}\left|\sum_{k=1}^n A_{i,j,k}\,\eta_k\right| \\
&= n^{-\frac{1}{r}} \max_{1\leq i,j\leq n} \sum_{k=1}^n |A_{i,j,k}| \\
&\geq n^{-\frac{1}{r}-2} \sum_{i,j,k=1}^n |A_{i,j,k}|.
\end{aligned}
$$

This completes the proof of the proposition. □

As an immediate consequence of Lemma 4.1, we obtain a lower bound on the volume radius of the unit ball B_π^n in X_π^n.

Corollary 4.2. *Let $n \in \mathbb{N}$ and $1 \leq p, q, r \leq \infty$. Then we have*

$$
\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \geq n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1}.
$$

Proof. From Lemma 4.1, we obtain that

$$
B_\epsilon^n \subseteq 2\, n^{\min\left(\frac{1}{p},\frac{1}{2}\right)+\min\left(\frac{1}{q},\frac{1}{2}\right)+\frac{1}{r}+1} B_1^{n^3}.
$$

Switching to the polar bodies implies

$$
B_\pi^n \supseteq \frac{1}{2}\, n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1} B_\infty^{n^3}.
$$

Taking volumes and the n^3-rd root, the previous inclusion immediately gives

$$
\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \geq n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1},
$$

which completes the proof. □

4.2. **Upper bounds on the volume.** To compute the matching upper bound on the volume radius of B_π^n, we will use Lemma 3.4. To be more precise, the idea is as follows. Using our extended version of Chevet's inequality (see Lemma 3.3), we obtain a lower bound for the volume of B_ϵ^n from Lemma 3.4. We then use the Blaschke-Santaló inequality (see Lemma 3.1) to derive an upper bound on the volume of B_π^n.

The next proposition will be a consequence of the 3-fold Chevet inequality, where we apply Lemma 3.3 to the space X_ϵ^n and choose $(x_i)_{i=1}^n, (y_j)_{j=1}^n, (z_k)_{k=1}^n$ to be the standard basis vectors.

Proposition 4.3. *Let $n \in \mathbb{N}$ and $1 \le p \le q \le r \le \infty$. Let $(\varepsilon_{i,j,k})_{i,j,k=1}^\infty$ be a sequence of independent Bernoulli random variables. Then we have*

$$\mathbb{E}\Big\| \sum_{i,j,k=1}^n \varepsilon_{i,j,k}\ e_i \otimes e_j \otimes e_k \Big\|_\epsilon \lesssim_{p,q,r} n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\frac{1}{r^*}-1}. \tag{4.1}$$

Proof. First, recall that the Rademacher average is smaller than the Gaussian average (see Lemma 3.5) and so it is enough to prove inequality (4.1) with Gaussian random variables. In order to do that, recall the well known fact that, for all $1 \le \alpha < \infty$, and standard Gaussian random variables $g_1, \dots, g_n$,

$$\mathbb{E}\Big\| \sum_{i=1}^n g_i e_i \Big\|_{\ell_\alpha^n} \asymp_\alpha n^{1/\alpha}.$$

Also, it is known that if we consider the standard unit vectors $e_1, \dots, e_n$ in ℓ_α^n, then we have

$$\big\| (e_i)_{i=1}^n \big\|_{\omega,2} = n^{\max\left(\frac{1}{\alpha},\frac{1}{2}\right)-\frac{1}{2}}.$$

Then, Lemma 3.3 implies

$$\begin{aligned}\mathbb{E}\Big\| \sum_{i,j,k=1}^n \varepsilon_{i,j,k} e_i \otimes e_j \otimes e_k \Big\|_\epsilon &\lesssim_{p,q,r} n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\frac{1}{r^*}-1} \\ &\quad + n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{r^*},\frac{1}{2}\right)+\frac{1}{q^*}-1} + n^{\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\max\left(\frac{1}{r^*},\frac{1}{2}\right)+\frac{1}{p^*}-1} \\ &\le 3n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\frac{1}{r^*}-1}, \end{aligned} \tag{4.2}$$

where in the last inequality we used the assumption that $p \le q \le r$. □

An upper bound on the volume radius B_π^n is now an immediate consequence of Lemma 3.4, Proposition 4.3, and the Blaschke-Santaló inequality.

Corollary 4.4. *Let $n \in \mathbb{N}$ and $1 \le p \le q \le r \le \infty$. Then we have*

$$\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \lesssim_{p,q,r} n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1}.$$

Proof. Applying Proposition 4.3 to X_ϵ^n and using Lemma 3.4, we get

$$\mathrm{vol}_{n^3}\left(B_\epsilon^n\right)^{1/n^3} \gtrsim_{p,q,r} n^{-\max\left(\frac{1}{p^*},\frac{1}{2}\right)-\max\left(\frac{1}{q^*},\frac{1}{2}\right)-\frac{1}{r^*}+1}. \tag{4.3}$$

Lemma 3.1 implies that

$$\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \cdot \mathrm{vol}_{n^3}\left(B_\epsilon^n\right)^{1/n^3} \le \left(\mathrm{vol}_{n^k}\left(B_2^{n^3}\right)^2\right)^{1/n^3} \asymp n^{-3}. \tag{4.4}$$

Thus, combining (4.3) and (4.4), we obtain

$$\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \lesssim_{p,q,r} n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\frac{1}{r^*}-4}.$$

Since $\max\left(\frac{1}{p^*},\frac{1}{2}\right)-1=-\min\left(\frac{1}{p},\frac{1}{2}\right)$, we have

$$\mathrm{vol}_{n^3}\left(B_\pi^n\right)^{1/n^3} \lesssim_{p,q,r} n^{-\min\left(\frac{1}{p},\frac{1}{2}\right)-\min\left(\frac{1}{q},\frac{1}{2}\right)-\frac{1}{r}-1},$$

and the proof is complete. □

5. The operator norm of the identity operator

In this section we will present upper and lower bounds for $\mathrm{id}\in\mathcal{L}(\ell_2^{n^3},X_\pi^n)$ that are sharp for most choices of p,q,r. The lower bounds are based on Chevet's inequality, on the special structure of the space of diagonal tensors in X_π^n or the choice of particular 3-fold tensors. To obtain upper bounds, we use the results for 2-fold projective tensor products and a generalized version of an inequality that was proved by Hardy and Littlewood to study bilinear forms. The main theorem in this section reads as follows.

Theorem C. *Let $n\in\mathbb{N}$ and $1\le p\le q\le r\le\infty$. Then we have*

$$\|\mathrm{id}:\ell_2^{n^3}\to X_\pi^n\| \asymp_{p,q,r} \begin{cases} n^{\frac{1}{2}+\frac{1}{r}}, & r\le 2,\\ n^{\max\left(\frac{1}{q}+\frac{1}{r},\frac{1}{2}\right)}, & p\le 2\le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\ge 1,\\ n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}, & p\le 2\le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1,\\ n^{\max\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r},\frac{1}{2}\right)-\frac{1}{2}}, & 2\le p.\end{cases}$$

In the case $q\le 2\le r$, we obtain the following bounds,

$$n^{\frac{1}{2}+\frac{1}{r}} \lesssim_{p,q,r} \|\mathrm{id}:\ell_2^{n^3}\to X_\pi^n\| \lesssim n^{\min\left(\frac{1}{q}+\frac{1}{r},1\right)}.$$

The lower bound is proved in Section 5.1 and the upper bound in Section 5.2. Also, in Section 5.3 we show how the lower bound can be improved in some special cases when $q\le 2\le r$ in the previous theorem. In what follows, we will use the shorthand notation $\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}$ for $\|\mathrm{id}:\ell_2^{n^3}\to X_\pi^n\|$.

5.1. **Lower bounds on the operator norm.** The following lower bound can be obtained by considering the space of diagonal tensors in X_π^n and by using the 3-fold version of Chevet's inequality (see Proposition 4.3).

Lemma 5.1. *Let $n\in\mathbb{N}$ and $1\le p,q,r\le\infty$. Then we have*

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \gtrsim_{p,q,r} \max\left(n^{\max\left(\min\left(1,\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)-\frac{1}{2},0\right)}, n^{\min\left(\frac{1}{p},\frac{1}{2}\right)+\min\left(\frac{1}{q},\frac{1}{2}\right)+\frac{1}{r}-\frac{1}{2}}\right). \tag{5.1}$$

Proof. By Theorem 1.3 in [AF96], it is known that the space of diagonal tensors in X_π^n is isometric to $\ell_s^{n^3}$, where s is given by

$$s:=\frac{1}{\min\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r},1\right)}.$$

Hence, we have

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \geq \frac{\left\|\sum_{i,j,k=1}^n e_i\otimes e_i\otimes e_i\right\|_\pi}{\left\|\sum_{i,j,k=1}^n e_i\otimes e_i\otimes e_i\right\|_{\ell_2^{n^3}}} = \frac{\left\|\sum_{i,j,k=1}^n e_i\otimes e_i\otimes e_i\right\|_{\ell_s^{n^3}}}{\left\|\sum_{i,j,k=1}^n e_i\otimes e_i\otimes e_i\right\|_{\ell_2^{n^3}}} = n^{\frac{1}{s}-\frac{1}{2}}.$$

Since $\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \geq 1$, we have in fact

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \geq n^{\max\left(\min\left(1,\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)-\frac{1}{2},0\right)}. \tag{5.2}$$

Next, notice that by Proposition 4.3, there exists a choice of signs $(\varepsilon_{i,j,k})_{i,j,k=1}^n$ such that

$$\left\|\sum_{i,j,k=1}^n \varepsilon_{i,j,k}e_1\otimes e_2\otimes e_3\right\|_\epsilon \lesssim_{p,q,r} n^{\max\left(\frac{1}{p^*},\frac{1}{2}\right)+\max\left(\frac{1}{q^*},\frac{1}{2}\right)+\frac{1}{r^*}-1}.$$

On the other hand,

$$\left\|\sum_{i,j,k=1}^n \varepsilon_{i,j,k}\ e_1\otimes e_2\otimes e_3\right\|_{\ell_2^{n^3}} = n^{\frac{3}{2}}.$$

Thus,

$$\begin{aligned}\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} = \|\mathrm{id}\|_{(X_\pi^n)^*\to\ell_2^{n^3}} &\geq \frac{\left\|\sum_{i,j,k=1}^n \varepsilon_{i,j,k}\ e_1\otimes e_2\otimes e_3\right\|_{\ell_2^{n^3}}}{\left\|\sum_{i,j,k=1}^n \varepsilon_{i,j,k}\ e_1\otimes e_2\otimes e_3\right\|_\epsilon} \\ &\gtrsim_{p,q,r} n^{\frac{5}{2}-\max\left(\frac{1}{p^*},\frac{1}{2}\right)-\max\left(\frac{1}{q^*},\frac{1}{2}\right)-\frac{1}{r^*}} \overset{(*)}{=} n^{\min\left(\frac{1}{p},\frac{1}{2}\right)+\min\left(\frac{1}{q},\frac{1}{2}\right)+\frac{1}{r}-\frac{1}{2}},\end{aligned} \tag{5.3}$$

where in $(*)$ we used the fact that for any $p\geq 1$, $1-\max\left(\frac{1}{p^*},\frac{1}{2}\right) = \min\left(\frac{1}{p},\frac{1}{2}\right)$. Combining (5.2) and (5.3), the proof is complete. □

5.2. **Upper bounds on the operator norm.** For tensor products of two spaces, the norm of the identity was estimated in [Sch82].

Proposition 5.2. *Let $n\in\mathbb{N}$ and $1\leq p\leq q\leq\infty$. Then*

$$\|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n} \asymp_{p,q} \begin{cases} n^{\frac{1}{q}} & q\leq 2, \\ n^{\min\left(\frac{1}{p}+\frac{1}{q},1\right)-\frac{1}{2}} & p\leq 2\leq q, \\ n^{\max\left(\frac{1}{p}+\frac{1}{q},\frac{1}{2}\right)-\frac{1}{2}} & 2\leq p.\end{cases}$$

To obtain an upper bound for 3-fold tensor products of ℓ_p-spaces, we use the following recursive formula.

Proposition 5.3. *Let $n\in\mathbb{N}$ and $1\leq p,q,r\leq\infty$. Then*

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \leq n^{\min\left(\frac{1}{r},\frac{1}{2}\right)}\cdot\|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n}.$$

Proof. First, note that

$$\|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n} = \|\mathrm{id}\|_{(\ell_p^n\otimes_\pi\ell_q^n)^*\to\ell_2^{n^2}}.$$

Let $A \in (X_\pi^n)^*$. By the definition of the injective tensor product norm, we have

$$\|\mathrm{id}\|_{(\ell_p^n\otimes_\pi \ell_q^n)^*\to\ell_2^{n^2}} \cdot \|A\|_{(X_\pi^n)^*} = \|\mathrm{id}\|_{(\ell_p^n\otimes_\pi \ell_q^n)^*\to\ell_2^{n^2}} \cdot \|A\|_{\ell_r^n\to(\ell_p^n\otimes_\pi \ell_q^n)^*} \geq \|A\|_{\ell_r^n\to\ell_2^{n^2}}. \tag{5.4}$$

Also, for $x = (x_1, \dots, x_n) \in \ell_r^n$, we have

$$Ax = \sum_{i,j=1}^n \left[\sum_{k=1}^n A_{i,j,k}x_k\right] e_i \otimes e_j \in \mathbb{R}^{n^2}.$$

Thus, we obtain

$$\|A\|^2_{\ell_r^n\to\ell_2^{n^2}} = \sup_{\|x\|_r=1} \|Ax\|^2_{\ell_2^{n^2}} = \sup_{\|x\|_{\ell_r^n}=1} \sum_{i,j=1}^n \left|\sum_{k=1}^n A_{i,j,k}x_k\right|^2$$

Choosing $x = n^{-1/r}\cdot\varepsilon$, $\varepsilon \in \{-1,1\}^n$, we get

$$\|A\|^2_{\ell_r^n\to\ell_2^{n^2}} \geq n^{-\frac{2}{r}} \sum_{i,j=1}^n \mathbb{E}\left|\sum_{k=1}^n A_{i,j,k}\varepsilon_k\right|^2 = n^{-\frac{2}{r}} \sum_{i,j,k=1}^n |A_{i,j,k}|^2 = \left(n^{-\frac{1}{r}}\|A\|_{\ell_2^{n^3}}\right)^2. \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$n^{\frac{1}{r}} \|\mathrm{id}\|_{(\ell_p^n\otimes_\pi \ell_q^n)^*\to\ell_2^{n^2}} \|A\|_\epsilon \geq \|A\|_{\ell_2^{n^3}}.$$

Equivalently, since $\|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n} = \|\mathrm{id}\|_{(\ell_p^n\otimes_\pi \ell_q^n)^*\to\ell_2^{n^2}}$, we have

$$\frac{\|A\|_{\ell_2^{n^3}}}{\|A\|_\epsilon} \leq n^{\frac{1}{r}} \|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n}.$$

Since $A \in (X_\pi^n)^*$ was arbitrary, this gives

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \leq n^{\frac{1}{r}} \|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n}. \tag{5.6}$$

Also, since for all $x \in \mathbb{R}^n$, $\|x\|_{\ell_\infty^n} \leq \|x\|_{\ell_2^n} \leq \sqrt{n}\|x\|_{\ell_\infty^n}$, we have

$$\|\mathrm{id}\|^2_{(\ell_p^n\otimes_\pi \ell_q^n)^*\to\ell_2^{n^2}} \|A\|^2_\epsilon \geq \sup_{\|x\|_{\ell_r^n}=1} \sum_{i,j=1}^n \left|\sum_{k=1}^n A_{i,j,k}x_k\right|^2 \geq \max_{1\leq k\leq n} \sum_{i,j=1}^n |A_{i,j,k}|^2 \geq \frac{1}{n}\|A\|_{\ell_2^{n^3}}.$$

This implies

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \leq \sqrt{n}\,\|\mathrm{id}\|_{\ell_2^{n^2}\to\ell_p^n\otimes_\pi\ell_q^n}. \tag{5.7}$$

Combining (5.6) and (5.7), the result follows. □

Another useful tool is the following corollary, which follows immediately from Theorem 2 in Section 3.

Corollary 5.4. *Let $n \in \mathbb{N}$ and assume that $1 \leq p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}$. Then*

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n} \lesssim 1.$$

Proof. Since we assume that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le\frac{1}{2}$, if μ is defined as in (3.4), then we get $\mu\le 2$. Hence, we have

$$\|A\|_{\ell_2^{n^3}}\le\|A\|_{\ell_\mu^{n^3}}\lesssim\|A\|_{(X_\pi^n)^*}.$$

Therefore, we have

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}=\|\mathrm{id}\|_{(X_\pi^n)^*\to\ell_2^{n^3}}\lesssim 1,$$

which completes the proof. ☐

Using the above upper bounds, we obtain the following.

Lemma 5.5. *Let $n\in\mathbb{N}$ and $1\le p\le q\le r\le\infty$. Then we have*

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}\lesssim_{p,q,r}\begin{cases} n^{\frac{1}{2}+\frac{1}{r}}, & r\le 2, \quad (5.8a)\\ n^{\min\left(\frac{1}{q}+\frac{1}{r},1\right)}, & q\le 2\le r, \quad (5.8b)\\ n^{\max\left(\frac{1}{q}+\frac{1}{r},\frac{1}{2}\right)}, & p\le 2\le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\ge 1, \quad (5.8c)\\ n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}, & p\le 2\le q,\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1, \quad (5.8d)\\ n^{\max\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r},\frac{1}{2}\right)-\frac{1}{2}}, & 2\le p. \quad (5.8e)\end{cases}$$

Proof. The bounds (5.8a), (5.8b), (5.8c) follow from Proposition 5.2 and Proposition 5.3. Next, assume that p,q,r are such that $\frac{1}{2}\le\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$. Then there exists $\tilde{p}\ge p$, $\tilde{q}\ge q$, $\tilde{r}\ge r$ such that $\frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}+\frac{1}{\tilde{r}}=\frac{1}{2}$. Hence, given $A\in X_\pi^n$, we have

$$\begin{aligned}\|A\|_\pi &\le n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}\|A\|_{\ell_{\tilde{p}}^n\otimes_\pi\ell_{\tilde{q}}^n\otimes_\pi\ell_{\tilde{r}}^n}\\ &\le n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}\|\mathrm{id}\|_{\ell_2^{n^3}\to\ell_{\tilde{p}}^n\otimes_\pi\ell_{\tilde{q}}^n\otimes_\pi\ell_{\tilde{r}}^n}\|A\|_{\ell_2^{n^3}}\\ &\lesssim n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}\|A\|_{\ell_2^{n^3}},\end{aligned}$$

where in the last inequality we used Corollary 5.4. Hence, we have

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}\lesssim n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}. \tag{5.9}$$

Thus, if we assume that $p\le 2\le q$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$, then we must have $\frac{1}{2}\le\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$, in which case (5.9) proves (5.8d). Finally, assume that $p\ge 2$. If $\frac{1}{2}\le\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le 1$, then (5.9) holds. If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\le\frac{1}{2}$, then by Corollary 5.4,

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}\lesssim 1. \tag{5.10}$$

If we have $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\ge 1$, then since $p\ge 2$ we must have $\frac{1}{q}+\frac{1}{r}\ge\frac{1}{2}$. Hence, using Proposition 5.2 and Proposition 5.3, we have

$$\|\mathrm{id}\|_{\ell_2^{n^3}\to X_\pi^n}\lesssim n^{\frac{1}{p}+\max\left(\frac{1}{q}+\frac{1}{r},\frac{1}{2}\right)-\frac{1}{2}}=n^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{1}{2}}. \tag{5.11}$$

Combining (5.9), (5.10), and (5.11), (5.8e) follows. ☐

5.3. **Improved lower bounds for some special cases.** In this short section we show that the lower bound for the case $p \le q \le 2 \le r$ in Theorem A can be improved for some particular choices of p, q and r.

Lemma 5.6. *Let $n \in \mathbb{N}$ and $1 \le p \le q \le 2 \le r \le \infty$. Then we have*

$$\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} \ge \max\left(n^{\frac{1}{2}+\frac{1}{r}}, n^{\min\left(\frac{1}{q}+\frac{1}{r},1\right)+\frac{1}{p}-1}, n^{\frac{1}{q}}\right).$$

Proof. The first bound follows from Lemma 5.1. In order to prove the second bound, we transfer to a 3-fold tensor that is constructed from a 2-fold tensor $B \in \ell_q^n \otimes_\pi \ell_r^n$ and the special vector $x = (1, 1, \dots, 1) \in \ell_p^n$. We have

$$\begin{aligned}
\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} &\ge \sup_{B \in \ell_q^n \otimes_\pi \ell_r^n} \frac{\|x \otimes B\|_\pi}{\|x \otimes B\|_{\ell_2^{n^3}}} \\
&= \sup_{B \in \ell_q^n \otimes_\pi \ell_r^n} \frac{\|B\|_{\ell_q^n \otimes_\pi \ell_r^n} \|x\|_{\ell_p^n}}{\|B\|_{\ell_2^{n^2}} \|x\|_{\ell_2^n}} \\
&= n^{\frac{1}{p}-\frac{1}{2}} \|\mathrm{id}\|_{\ell_2^{n^2} \to \ell_q^n \otimes_\pi \ell_r^n} \\
&= n^{\min\left(\frac{1}{q}+\frac{1}{r},1\right)+\frac{1}{p}-1},
\end{aligned}$$

where in the last equality we used Proposition 5.2. To obtain the third bound, we again transfer to the 2-fold case, choosing $B \in \ell_p^n \otimes_\pi \ell_q^n$ and $x = (1, 0, \dots, 0) \in \ell_r^n$. Then we have

$$\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} \ge \sup_{B \in \ell_p^n \otimes_\pi \ell_q^n} \frac{\|x \otimes B\|_\pi}{\|x \otimes B\|_{\ell_2^{n^3}}} = \sup_{B \in \ell_p^n \otimes_\pi \ell_q^n} \frac{\|B\|_{\ell_p^n \otimes_\pi \ell_q^n} \|x\|_{\ell_r^n}}{\|B\|_{\ell_2^{n^2}} \|x\|_{\ell_2^n}} = \|\mathrm{id}\|_{\ell_2^{n^2} \to \ell_p^n \otimes_\pi \ell_q^n} = n^{\frac{1}{q}},$$

where again in the last equality we used Proposition 5.2. This completes the proof. □

Combining Lemma 5.6 with Lemma 5.5, we obtain the following improvements on the bounds in particular cases.

Corollary 5.7. *(i) If $p = 1 \le q \le 2 \le r$, then*

$$\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} \asymp_{q,r} n^{\min\left(\frac{1}{q}+\frac{1}{r},1\right)}.$$

In particular, in such case we have

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_{q,r} n^{\min\left(\frac{1}{q}-\frac{1}{2},\frac{1}{2}-\frac{1}{r}\right)}.$$

(ii) If $p = q \le 2$ and $r = \infty$, then

$$\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} \asymp_p n^{\frac{1}{q}}.$$

In particular, in such case we have

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_p n^{\frac{1}{q}-\frac{1}{2}}.$$

(iii) If $1 \le p \le q = 2 \le r$, then

$$\|\mathrm{id}\|_{\ell_2^{n^3} \to X_\pi^n} \asymp_{p,r} n^{\frac{1}{2}+\frac{1}{r}}.$$

In particular, in such case we have

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_{p,r} 1.$$

Corollary 5.7 suggests that it is the lower bound that should be improved. See also Corollary 6.1 below for another such indication.

6. The case of k-fold projective tensor products

In this section, we briefly present the generalization of our main result to the case of k-fold projective tensor products. Before stating the generalized main result, we fix some notation. Given $k \in \mathbb{N}$, $1 \le p_1 \le p_2 \le \cdots \le p_k \le \infty$ and $n \in \mathbb{N}$, let

$$X_{\mathbf{p}}^n := \ell_{p_1}^n \otimes_\pi \cdots \otimes_\pi \ell_{p_k}^n. \tag{6.1}$$

Also, let j_0 be the largest $1 \le j \le k$ such that $p_j \le 2$ ($j_0 = 0$ if $p_1 > 2$). The space $X_{\mathbf{p}}^n$ has enough symmetries, and so formula (2.3) gives

$$\mathrm{vr}(X_{\mathbf{p}}^n) \asymp n^{k/2} \left(\mathrm{vol}_{n^k}(B_{X_{\mathbf{p}}^n})\right)^{1/n^k} \|\mathrm{id}\|_{\ell_2^n \to X_{\mathbf{p}}^n}. \tag{6.2}$$

In what follows, $\gtrsim_{\mathbf{p}}$ and $\lesssim_{\mathbf{p}}$ mean inequalities with an implied constant that depends only on $p_1, \ldots, p_k$. All the tools that were used above can be generalized to the case of k-fold tensor products by straightforward induction.

Regarding the volume of $B_{X_{\mathbf{p}}^n}$, note that in the k-fold version of Chevet's inequality (see Lemma 6.2 below for a complete statement), there is an additional k factor since the upper bound is now comprised of k terms, and also note that the lower bound of the volume of $B_{X_{\mathbf{p}}}^n$ now contains a factor of $2^{-\frac{k-j_0}{2}}$, since for each j with $p_j \ge 2$, the use of Khinchine's inequality incurs a factor of $1/\sqrt{2}$. As a result, we have in the general case,

$$2^{-\frac{k-j_0}{2}} n^{-\sum_{j=1}^{k-1} \min\left(\frac{1}{p_j}, \frac12\right) - \frac{1}{p_k} - 1} \lesssim_{\mathbf{p}} \mathrm{vol}_{n^k}(B_{X_{\mathbf{p}}^n}) \lesssim_{\mathbf{p}} k\, n^{-\sum_{j=1}^{k-1} \min\left(\frac{1}{p_j}, \frac12\right) - \frac{1}{p_k} - 1}.$$

Regarding the norm of the identity, an analogue of Theorem C follows by simply using induction. The only exception is Theorem 2, which in the general case still gives $\|\mathrm{id}\|_{\ell_2^{n^k} \to X_{\mathbf{p}}^n} \lesssim 1$ whenever $\sum_{j=1}^k \frac{1}{p_j} \le \frac12$.

Combining those tools, Theorem A can be generalized in the following way:

Theorem D. *Let $k \ge 3$. Let $1 \le p_1 \le p_2 \le \cdots \le p_k \le \infty$ and $n \in \mathbb{N}$. Then the following estimates hold:*

(1) *If $p_k \le 2$, then*

$$1 \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k.$$

(2) *If $\sum_{j=1}^k \frac{1}{p_j} \ge 1$, $p_{k-1} \le 2 < p_k$, then*

$$1 \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k\, n^{\min\left(\frac{1}{p_{k-1}} + \frac{1}{p_k}, 1\right) - \frac{1}{p_k} - \frac12}.$$

(3) *If $\sum_{j=1}^k \frac{1}{p_j} \ge 1$, $p_{k-1} > 2$, then*

$$1 \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k\, n^{\max\left(\frac{1}{p_{k-1}} + \frac{1}{p_k}, \frac12\right) - \frac{1}{p_{k-1}} - \frac{1}{p_k}}$$

(4) *If $\sum_{j=1}^k \frac{1}{p_j} \ge 1$, and If $\sum_{j=2}^k \frac{1}{p_j} \le \frac12$, then*

$$2^{-\frac{k}{2}} n^{\frac12 - \sum_{j=2}^k \frac{1}{p_j}} \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k\, n^{\frac12 - \sum_{j=2}^k \frac{1}{p_j}}$$

(5) *If* $\frac{1}{2} \le \sum_{j=1}^k \frac{1}{p_j} \le 1$, *then*

$$2^{-\frac{k}{2}} n^{\max\left(\frac{1}{p_1}-\frac{1}{2},0\right)} \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k\, n^{\max\left(\frac{1}{p_1}-\frac{1}{2},0\right)}.$$

(6) *If* $\sum_{j=1}^k \frac{1}{p_j} \le \frac{1}{2}$, *then*

$$2^{-\frac{k}{2}} n^{\frac{1}{2}-\sum_{j=1}^k \frac{1}{p_j}} \lesssim_{\mathbf{p}} \mathrm{vr}(X_\pi^n) \lesssim_{\mathbf{p}} k\, n^{\frac{1}{2}-\sum_{j=1}^k \frac{1}{p_j}}.$$

Considering copies of the same ℓ_p^n space, it was shown in [DP09], that if one considers the space

$$\otimes_\pi^k \ell_p^n := \underbrace{\ell_p^n \otimes_\pi \ell_p^n \otimes_\pi \cdots \otimes_\pi \ell_p^n}_{k \text{ times}},$$

then the following holds true:

$$\mathrm{vr}\left(\otimes_\pi^k \ell_p^n\right) \asymp_p \begin{cases} 1 & p \le 2k, \\ n^{\frac{1}{2}-\frac{k}{p}} & p \ge 2k. \end{cases} \tag{6.3}$$

Using Theorem D, we can get a result in the spirit of (6.3), with worse dependence on k, and in some case, with worse dependence on n as well:

Corollary 6.1. *The following holds:*

(1) *If* $p \le 4$ *or* $k \le p \le 2k$ *then* $1 \lesssim_p \mathrm{vr}\left(\otimes_\pi^k \ell_p^n\right) \lesssim_p k$.

(2) *If* $4 \le p \le k$ *then* $1 \lesssim_p \mathrm{vr}\left(\otimes_\pi^k \ell_p^n\right) \lesssim_p k\, n^{\frac{1}{2}-\frac{2}{p}}$.

(3) *If* $p \ge 2k$ *then* $2^{-\frac{k}{2}}\, n^{\frac{1}{2}-\frac{k}{p}} \lesssim_p \mathrm{vr}\left(\otimes_\pi^k \ell_p^n\right) \lesssim_p k\, n^{\frac{1}{2}-\frac{k}{p}}$.

Corollary 6.1 combined with (6.3) suggest that it is the lower bound that should be improved in Theorem D. Finally, for the sake of completeness, let us state the k-fold version of Chevet's inequality that plays a major role in the proof of Theorem D:

Lemma 6.2. *Let* $k \in \mathbb{N}$ *and* $\{x_{i_j}\}_{i_j=1}^{n_j} \subseteq X_j$, $1 \le j \le k$ *be sequences in the Banach spaces* $(X_1, \|\cdot\|_{X_1}), \ldots, (X_k, \|\cdot\|_{X_k})$, *respectively. Then*

$$\left\| \sum_{\substack{1 \le i_m \le n \\ 1 \le m \le k}} g_{i_1,\ldots,i_k} x_{i_1} \otimes \cdots \otimes x_{i_k} \right\|_{X_1 \otimes_\epsilon \cdots \otimes_\epsilon X_k} \le \sum_{j=1}^k \left[\prod_{j' \ne j} \left\|\{x_{i_j}\}_{i_j}\right\|_{\omega,2} \right] \mathbb{E} \left\| \sum_{i_j=1}^n g_{i_j} x_{i_j} \right\|_{X_j}.$$

7. Remarks on the cotype of projective tensor products

As already mentioned in the introduction, from our main result and its k-fold generalization, we obtain information on the cotype of the 3-fold and k-fold projective tensor products.

Consider now the infinite dimensional space

$$X := \ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r,$$

where $1 \le p \le q \le r \le \infty$. Since the space $X_\pi^n = \ell_p^n \otimes_\pi \ell_q \otimes_\pi \ell_r^n$ is a finite dimensional subspace of X, it follows by Theorem 1 that if

$$\lim_{n\to\infty} \mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q \otimes_\pi \ell_r^n\right) = \infty,$$

then X does not have cotype 2, that is $C_2(X) = \infty$. In order to obtain a result about cotype $\alpha > 2$, recall the following result from [TJ79]: if $(X, \|\cdot\|_X)$ is n-dimensional, then there exist $x_1, \dots, x_n \in X$ such that

$$\Big(\sum_{i=1}^{n} \|x_i\|_X^2\Big)^{1/2} \gtrsim C_2(X)\,\mathbb{E}\,\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\|_X.$$

In particular, in the space $X_\pi^n = \ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$, there exist $x_1, \dots, x_{n^3} \in X_\pi^n$ such that

$$C_2(X_\pi^n)\,\mathbb{E}\,\Big\|\sum_{i=1}^{n^3} \varepsilon_i x_i\Big\|_\pi \lesssim \Big(\sum_{i=1}^{n^3} \|x_i\|_\pi^2\Big)^{1/2}$$
$$\overset{(*)}{\leq} n^{3\left(\frac{1}{2}-\frac{1}{\alpha}\right)}\Big(\sum_{i=1}^{n^3} \|x_i\|_\pi^\alpha\Big)^{1/\alpha} \leq n^{3\left(\frac{1}{2}-\frac{1}{\alpha}\right)} C_\alpha(X_\pi^n)\,\mathbb{E}\,\Big\|\sum_{i=1}^{n^3} \varepsilon_i x_i\Big\|_\pi,$$

where in $(*)$ we used Hölder's inequality. Altogether, we have $C_\alpha(X_\pi^n) \gtrsim n^{3\left(\frac{1}{\alpha}-\frac{1}{2}\right)} C_2(X_\pi^n)$. This fact, together with Theorem A, gives the following result.

Corollary 7.1. *Let $X = \ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r$ with $1 \leq p \leq q \leq r \leq \infty$. Then $C_\alpha(X) = \infty$ in the following cases:*

- $p \leq 2$, $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$, *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$, *and for* $\alpha < \frac{3}{1+\frac{1}{q}+\frac{1}{r}}$ *;*
- $p < 2$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$, *and for* $\alpha < \frac{3}{2-\frac{1}{p}}$ *;*
- $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < \frac{1}{2}$, *and for* $\alpha < \frac{3}{1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}$ *.*

Finally, let us comment on the case of k-fold tensor products. Let

$$X := \ell_{p_1} \otimes_\pi \cdots \otimes_\pi \ell_{p_k},$$

where $1 \leq p_1 \leq \cdots \leq p_k \leq \infty$. If $X_{\mathbf{p}}^n$ is defined as in (6.1), then a similar argument as above implies that $C_\alpha(X_{\mathbf{p}}^n) \gtrsim n^{k\left(\frac{1}{\alpha}-\frac{1}{2}\right)} C_2(X_{\mathbf{p}}^n)$. Thus, using Theorem D, a similar result to Corollary 7.1 can be obtained. The result then reads as follows.

Corollary 7.2. *Let $X = \ell_{p_1} \otimes_\pi \cdots \otimes_\pi \ell_{p_k}$ with $1 \leq p_1 \leq \cdots \leq p_k \leq \infty$. Then $C_\alpha(X) = \infty$ in the following cases:*

- $\sum_{j=1}^k \frac{1}{p_j} \geq 1$ *and* $\sum_{j=2}^k \frac{1}{p_j} < \frac{1}{2}$ *and for* $\alpha < \frac{k}{\frac{k-1}{2}+\sum_{j=2}^k \frac{1}{p_j}}$ *;*
- $\sum_{j=1}^k \frac{1}{p_j} \leq 1$ *and* $p_1 < 2$ *and for* $\alpha < \frac{k}{\frac{k+1}{2}-\frac{1}{p_1}}$ *;*
- $\sum_{j=1}^k \frac{1}{p_j} < \frac{1}{2}$ *and for* $\alpha < \frac{k}{\frac{k-1}{2}+\sum_{j=1}^n \frac{1}{p_j}}$ *.*

References

[AS06] G. Aubrun and S. Szarek, *Tensor products of convex sets and the volume of separable states on N qudits*, Phys. Rev. A. **73** (2006), no. 022109.

[ASW11] G. Aubrun, S. Szarek, and E. Werner, *Hastings's Additivity Counterexample via Dvoretzky's Theorem*, Commun. Math. Phys. **305** (2011), no. 1, 85–97.

[ASW10] ______, *Nonadditivity of Rényi entropy and Dvoretzky's theorem*, J. Math. Phys. **51** (2010), no. 2.

[AF96] A. Arias and J. D. Farmer, *On the structure of tensor products of l_p-spaces*, Pacific J. Math. **175** (1996), no. 1, 13–37.

[BM87] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbf{R}^n$, Invent. Math. **88** (1987), no. 2, 319–340.

[BNR12] J. Briët, A. Naor, and O. Regev, *Locally decodable codes and the failure of cotype for projective tensor products*, Electron. Res. Announc. Math. Sci. **19** (2012), 120–130.

[Che78] S. Chevet, *Séries de variables aléatoires gaussiennes à valeurs dans* $E\hat{\otimes}_\varepsilon F$*. Application aux produits d'espaces de Wiener abstraits*, Séminaire sur la Géométrie des Espaces de Banach (1977–1978), École Polytech., Palaiseau, 1978, pp. Exp. No. 19, 15 (French).

[DM05] A. Defant and C. Michels, *Norms of tensor product identities*, Note Mat. **25** (2005/06), no. 1, 129–166.

[DP09] A. Defant and C. Prengel, *Volume estimates in spaces of homogeneous polynomials*, Math. Z. **261** (2009), no. 4, 909–932.

[DFS08] J. Diestel, J. H. Fourie, and J. Swart, *The metric theory of tensor products*, American Mathematical Society, Providence, RI, 2008. Grothendieck's résumé revisited.

[Efr09] K. Efremenko, *3-query locally decodable codes of subexponential length*, STOC'09—Proceedings of the 2009 ACM International Symposium on Theory of Computing, ACM, New York, 2009, pp. 39–44.

[GL74] Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structure*, Acta Mathematica **133** (1974), 24–48.

[Gro53] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1953), 1–79 (French).

[HJ74] J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. **52** (1974), 159–186.

[HL34] G. H. Hardy and J. E. Littlewood, *Bilinear forms bounded in space* $[p, q]$, Q. J. Math. **5** (1934), no. 1, 241–254.

[Kaš77] B. S. Kašin, *The widths of certain finite-dimensional sets and classes of smooth functions*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 2, 334–351, 478 (Russian).

[LP68] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in* L_p*-spaces and their applications*, Studia Math. **29** (1968), 275–326.

[MP76] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), no. 1, 45–90 (French).

[Pis86] G. Pisier, *Probabilistic methods in the geometry of Banach spaces*, Probability and analysis (Varenna, 1985), Lecture Notes in Math., vol. 1206, Springer, Berlin, 1986, pp. 167–241.

[Pis89] ———, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989.

[Pis90] ———, *Random series of trace class operators.*, In Proceedings Cuarto CLAPEM Mexico 1990. Contribuciones en probabilidad y estadistica matematica (1990), 29–42.

[Pis92] ———, *Factorization of operator valued analytic functions*, Adv. Math. **93** (1992), no. 1, 61–125.

[Pis12] ———, *Grothendieck's theorem, past and present*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 2, 237–323.

[PP81] T. Praciano-Pereira, *On bounded multilinear forms on a class of* l^p *spaces*, J. Math. Anal. Appl. **81** (1981), no. 2, 561–568.

[Rya02] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002.

[Sch14] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, 2014.

[Sch82] C. Schütt, *On the volume of unit balls in Banach spaces*, Compositio Math. **47** (1982), no. 3, 393–407.

[Sza78] S. Szarek, *On Kashin's almost Euclidean orthogonal decomposition of* l_n^1, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), no. 8, 691–694 (English, with Russian summary).

[SWZ11] S. Szarek, E. Werner, and K. Zyczkowski, *How often is a random quantum state* k*–entangled?*, J. Phys. A **40** (2011), no. 44.

[STJ80] S. Szarek and N. Tomczak-Jaegermann, *On nearly Euclidean decomposition for some classes of Banach spaces*, Compositio Math. **40** (1980), no. 3, 367–385.

[TJ74] N. Tomczak-Jaegermann, *The moduli of smoothness and convexity and the Rademacher averages of the trace classes* S_p $(1 \leq p < \infty)$, Studia Math. **50** (1974), no. 2, 163–182.

[TJ79] ———, *Computing 2-summing norm with few vectors*, Ark. Mat. **17** (1979), no. 2, 273–277.
[TJ89] ———, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 38, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

School of Mathematical & Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
E-mail address: ohad.giladi@newcastle.edu.au

School of Mathematics & Physical Sciences, University of Hull, Cottingham Road, Hull, HU6 7RX, United Kingdom
E-mail address: j.prochno@hull.ac.uk

Mathematisches Seminar, Christian-Albrechts-University Kiel, Ludewig-Meyn-Strasse 4, 24098 Kiel, Germany
E-mail address: schuett@math.uni-kiel.de

Department of Mathematical and Statistical Sciences, University of Alberta, 632 Central Academic Building, Edmonton, AB T6G 2G1, Canada
E-mail address: nicole.tomczak@ualberta.ca

Department of Mathematics, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, OH 44106, USA
E-mail address: elisabeth.werner@case.edu